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The Cardy limit of the topologically twisted index and black strings in AdS$_5$

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Abstract: We evaluate the topologically twisted index of a general four-dimensional $\mathcal{N}=1$ gauge theory in the "high-temperature" limit. The index is the partition function for $\mathcal{N}=1$ theories on $S^2 \times T^2$, with a partial topological twist along S^2, in the presence of background magnetic fluxes and fugacities for the global symmetries. We show that the logarithm of the index is proportional to the conformal anomaly coefficient of the two-dimensional $\mathcal{N}=(0,2)$ SCFTs obtained from the compactification on S^2. We also present a universal formula for extracting the index from the four-dimensional conformal anomaly coefficient and its derivatives. We give examples based on theories whose holographic duals are black strings in type IIB backgrounds AdS$_5 \times$ SE$_5$, where SE$_5$ are five-dimensional Sasaki-Einstein spaces.

Contents

1 Introduction

The topologically twisted index introduced in [1] is the partition function for three- and four-dimensional gauge theories with at least four supercharges on $\Sigma_g \times T^d$, where $d = 1, 2$, with a topological A-twist on Σ_g. When it is refined with chemical potentials and background magnetic charges for the flavor symmetries, it becomes an efficient tool for studying the nonperturbative properties of supersymmetric gauge theories [1–6]. The large N limit of the index contains interesting information about theories with a holographic dual. In particular, the large N limit of the index for the three-dimensional ABJM theory was successfully used in [7, 8] to provide the first microscopic counting of the microstates of an AdS$_4$ black hole. The large N limit of general three-dimensional quivers with an AdS dual was studied in [9, 10]. In this paper we study the asymptotic behavior of the index, at finite N, for four-dimensional $\mathcal{N} = 1$ gauge theories.

With an eye on holography we also evaluate the index in the large N limit. We focus, in particular, on the class of $\mathcal{N}=1$ theories arising from D3-branes probing Calabi-Yau singularities, which have a well-known holographic dual in terms of compactifications on Sasaki-Einstein manifolds. Black string solutions corresponding to D3-branes at a Calabi-Yau singularity have been recently studied in details in [11–13]. They interpolate between AdS_5 and $\mathrm{AdS}_3 \times \Sigma_g$ vacua and can be interpreted as an RG flow from an UV four-dimensional $\mathcal{N}=1$ CFT and an IR two-dimensional $(0,2)$ one. The two-dimensional CFT is obtained by compactifying the four-dimensional theory on Σ_g with a topological twist parameterized by a set of background magnetic charges $\mathfrak{n}_I$. The right-moving central charge of the two-dimensional CFT has been computed in [11–14], and successfully compared with the supergravity result for a variety of models.

The topologically twisted index of a general four-dimensional $\mathcal{N}=1$ gauge theory can be interpreted as a trace over a Hilbert space of states on $\Sigma_g \times S^1$

$$Z(\mathfrak{n}, y) = \mathrm{Tr}_{\Sigma_g \times S^1} (-1)^F q^{H_L} \prod_I y_I^{J_I} \, , \tag{1.1}$$

where $q = e^{2\pi i \tau}$ and y_I are fugacities for the flavor symmetries J_I. Here, τ is the complex modulus of T^2. The Hamiltonian H_L on $\Sigma_g \times S^1$ explicitly depends on the background magnetic fluxes $\mathfrak{n}_I$. For simplicity, we restrict to the case of $\Sigma_g = S^2$, since the generalization to an arbitrary Riemann surface is straightforward [5]. The index can be evaluated using supersymmetric localization and it reduces to a matrix model. It can be written as the contour integral,

$$Z(\mathfrak{n}, y) = \frac{1}{|\mathcal{W}|} \sum_{\mathfrak{m} \in \Gamma_{\mathfrak{h}}} \oint_{\mathcal{C}} Z_{\text{int}}(\mathfrak{m}, x; \mathfrak{n}, y) \, , \tag{1.2}$$

of a meromorphic differential form in variables x living on the torus T^2 and parameterizing the Cartan subgroup of the gauge group. An important feature of the matrix model is that there is a sum over the lattice of magnetic charges $\mathfrak{m}$ of the gauge group. For each $\mathfrak{m}$ the integrand has the form of an elliptic genus as computed in [15, 16]. There exist particular choices of background magnetic fluxes $\mathfrak{n}$ for which the sum truncates to a single set of gauge fluxes $\mathfrak{m}$ [17]. However, for generic background fluxes this does not happen and we need to sum an infinite number of contributions. The strategy is then to explicitly resum the integrand [7] and consider the contour integral of

$$Z_{\text{resummed}}(x; \mathfrak{n}, y) = \frac{1}{|\mathcal{W}|} \sum_{\mathfrak{m} \in \Gamma_{\mathfrak{h}}} Z_{\text{int}}(\mathfrak{m}, x; \mathfrak{n}, y) \, , \tag{1.3}$$

which is a complicated elliptic function of x. One can write a set of algebraic equations for the position of the poles, which we call *Bethe ansatz equations* (BAEs) (they actually are the BAEs of the dimensionally reduced theory on Σ_g in the formalism of [18]), and a *Bethe potential* $\mathcal{V}$ (or Yang-Yang functional [19]) whose derivatives reproduce the BAEs. The topologically twisted index is then given by the sum of the residues of Z_{resummed} at the solutions to the BAEs. The explicit evaluation of the topologically twisted index is a hard task, even in the large N limit. However, the index greatly simplifies if we identify the modulus $\tau = i\beta/2\pi$ of the torus T^2 with a *fictitious* inverse temperature β, and take the *high-temperature* limit ($\beta \to 0$). In this limit, we can use the modular properties of the integrand under the $\mathrm{SL}(2,\mathbb{Z})$ action to simplify the result.

In the high-temperature limit, we find a number of interesting results, valid to leading order in $1/\beta$.

First, we obtain an explicit relation between the Bethe potential and the R-symmetry 't Hooft anomalies of the UV four-dimensional $\mathcal{N}=1$ theory

$$\overline{\mathcal{V}}\left(\Delta_I\right) = \frac{\pi^3}{6\beta}\left[\operatorname{Tr} R^3(\Delta_I) - \operatorname{Tr} R(\Delta_I)\right] = \frac{16\pi^3}{27\beta}\left[3c\left(\Delta_I\right) - 2a(\Delta_I)\right] , \tag{1.4}$$

where R is a choice of $\mathrm{U}(1)_R$ symmetry and the trace is over all fermions in the theory. Here, we use the chemical potentials Δ_I/π to parameterize a trial R-symmetry of the $\mathcal{N}=1$ theory. Details about this identification are given in the main text. In writing the second equality in (1.4) we used the relation between conformal and R-symmetry 't Hooft anomalies in $\mathcal{N}=1$ SCFTs [20],

$$a = \frac{9}{32}\operatorname{Tr} R^3 - \frac{3}{32}\operatorname{Tr} R\,, \qquad\qquad c = \frac{9}{32}\operatorname{Tr} R^3 - \frac{5}{32}\operatorname{Tr} R\,. \tag{1.5}$$

Secondly, the value of the index as a function of the chemical potentials Δ_I and the set of magnetic fluxes $\mathfrak{n}_I$, parameterizing the twist, can be expressed in terms of the trial left-moving central charge of the 2d $\mathcal{N}=(0,2)$ SCFT as

$$\log Z(\Delta_I, \mathfrak{n}_I) = \frac{\pi^2}{6\beta} c_l\left(\Delta_I, \mathfrak{n}_I\right) . \tag{1.6}$$

This is related to the trial right-moving central charge c_r by the gravitational anomaly k [11, 12],

$$c_r - c_l = k\,, \qquad\qquad k = -\operatorname{Tr}\gamma_3\,. \tag{1.7}$$

Here, γ_3 is the chirality operator in two dimensions.[1]

Finally, there is a simple universal formula at leading order in N for computing the index from the Bethe potential as a function of the chemical potentials Δ_I,

$$\log Z(\Delta_I, \mathfrak{n}_I) = -\frac{3}{\pi}\overline{\mathcal{V}}\left(\Delta_I\right) - \sum_I\left[\left(\mathfrak{n}_I - \frac{\Delta_I}{\pi}\right)\frac{\partial\overline{\mathcal{V}}\left(\Delta_I\right)}{\partial\Delta_I}\right] = \frac{\pi^2}{6\beta} c_r\left(\Delta_I, \mathfrak{n}_I\right) , \tag{1.8}$$

where the index I runs over the bi-fundamental and adjoint fields in the quiver. In the large N limit the Bethe potential can be written as

$$\overline{\mathcal{V}}\left(\Delta_I\right) = \frac{16\pi^3}{27\beta} a(\Delta_I)\,. \tag{1.9}$$

[1] With our choice of chirality operator the gaugino zero-modes have $\gamma_3 = 1$.

These formulae are valid for theories of D3-branes, where $\operatorname{Tr} R = \mathcal{O}(1)$ and $c = a$ at large N [21]. These topologically twisted theories have holographic duals in terms of black strings in $\mathrm{AdS}_5 \times \mathrm{SE}_5$, where SE_5 are five-dimensional Sasaki-Einstein spaces [11, 12].

There is a striking similarity with the results obtained in [7–10] for the large N limit of the topologically twisted index of three-dimensional theories, if we replace

$$
\begin{aligned}
\textit{central charge } a(\Delta_I) \quad &\Longleftrightarrow \quad \textit{free energy on } S^3 \\
\textit{central charge } c_r\left(\Delta_I, \mathfrak{n}_I\right) \quad &\Longleftrightarrow \quad \textit{black hole entropy} \\
c - \textit{extremization} \quad &\Longleftrightarrow \quad I - \textit{extremization} \,.
\end{aligned}
$$

Indeed, in three dimensions, the very same formula (1.8) holds with the Bethe potential given by the S^3 partition function F_{S^3} of the gauge theory [9, 10]. Notice that F_{S^3} is the natural replacement for a, both being monotonic along RG flows [22, 23]. Moreover, both of them can be computed, as a function of Δ_I, in terms of the volume of a family of Sasakian manifolds [23–27]. In addition, in three dimensions, the dual black string is replaced by a dual black hole and $\log Z$ computes the entropy of the black hole. As discussed in [7, 8, 13], the entropy is obtained by extremizing $\log Z$ with respect to the Δ_I (I-extremization). Similarly, as it was shown in [11, 12], the exact central charge of the 2d SCFT is obtained by extremizing the trial right-moving central charge with respect to the Δ_I. Given the relation (1.8) we see that c-extremization corresponds to I-extremization. Finally, in both three and four dimensions, the field theory extremization corresponds to the attractor mechanism [28–33] on the gravity side.

Formula (1.6) implies a Cardy-like behavior of the topologically twisted index, which is related to the modular properties of the elliptic genus [34, 35]. Analogous behaviors for other partition functions have been found in [36–46].[2]

Notice also that our results (1.8) and (1.9) are compatible with a very simple relation between the field theoretical quantities $\operatorname{Tr} R^3(\Delta_I)$ and $c_r\left(\Delta_I, \mathfrak{n}_I\right)$ that is worthwhile to state separately,

$$
c_r\left(\Delta_I, \mathfrak{n}_I\right) = -3 \operatorname{Tr} R^3(\Delta_I) - \pi \sum_I \left[\left(\mathfrak{n}_I - \frac{\Delta_I}{\pi}\right) \frac{\partial \operatorname{Tr} R^3(\Delta_I)}{\partial \Delta_I}\right] . \tag{1.10}
$$

The rest of the paper is organized as follows. In Section 2 we review the basic properties of the topologically twisted index in four dimensions. In Section 3 we analyze the high-temperature limit of the index for $\mathcal{N} = 4$ super Yang-Mills while in Section 4 we discuss the example of the conifold. Then in Section 5 we derive the formulae (1.4), (1.6), (1.8) and (1.9). The body of the paper ends with Section 6, which contains possible future problems to explore. In Appendix A we derive the asymptotics of the elliptic functions relevant for our computations. Appendix B is devoted to the study of anomaly cancellation conditions for theories on $S^2 \times T^2$.

[2]In particular, ambiguities in the definition of the partition function for 2d $\mathcal{N} = (0, 2)$ theories, the ellipitic genus, have been pointed out in [41]. It would be interesting to see if there are similar ambiguities for the topologically twisted index of $\mathcal{N} = 1$ gauge theories.

2 The topologically twisted index

The topologically twisted index of an $\mathcal{N}=1$ gauge theory with vector and chiral multiplets and a non-anomalous $\mathrm{U}(1)_R$ symmetry in four dimensions is defined as the path-integral of the theory on $S^2 \times T^2$ with a partial topological A-twist along S^2 [1]. It is a function of $q = e^{2\pi i\tau}$, where τ is the modular parameter of T^2, fugacities y for the global symmetries and flavor magnetic fluxes $\mathfrak{n}$ on S^2 parameterizing the twist. The index can be reduced to a matrix integral over zero-mode gauge variables by exploiting the localization technique. The zero-mode gauge variables $x = e^{iu}$ parameterize the Wilson lines on the two directions of the torus

$$u = 2\pi \oint_{\text{A-cycle}} A - 2\pi\tau \oint_{\text{B-cycle}} A \,, \tag{2.1}$$

and are defined modulo

$$u_i \sim u_i + 2\pi n + 2\pi m\tau \,, \qquad n\,, m \in \mathbb{Z}\,. \tag{2.2}$$

Explicitly, for a theory with gauge group G and a set of chiral multiplets transforming in representations $\mathfrak{R}_I$ of G, the topologically twisted index is given by a contour integral of a meromorphic form[3]

$$\begin{aligned} Z(\mathfrak{n}, y) = \frac{1}{|\mathcal{W}|} \sum_{\mathfrak{m}\,\in\,\Gamma_{\mathfrak{h}}} \oint_{\mathcal{C}} \prod_{\text{Cartan}} \left(\frac{dx}{2\pi i x} \eta(q)^2 \right) (-1)^{\sum_{\alpha>0}\alpha(\mathfrak{m})} \prod_{\alpha\in G} \left[\frac{\theta_1(x^\alpha; q)}{i\eta(q)} \right] \\ \times \prod_I \prod_{\rho_I \in \mathfrak{R}_I} \left[\frac{i\eta(q)}{\theta_1(x^{\rho_I} y_I; q)} \right]^{\rho_I(\mathfrak{m}) - \mathfrak{n}_I + 1} , \end{aligned} \tag{2.3}$$

where α are the roots of G and $|\mathcal{W}|$ denotes the order of the Weyl group. Given a weight ρ_I of the representation $\mathfrak{R}_I$, we use the notation $x^{\rho_I} = e^{i\rho_I(u)}$. In this formula, $\theta_1(x;q)$ is a Jacobi theta function and $\eta(q)$ is the Dedekind eta function (see Appendix A). The result is summed over a lattice of gauge magnetic fluxes $\mathfrak{m}$ on S^2 living in the co-root lattice $\Gamma_{\mathfrak{h}}$ of the gauge group G (up to gauge transformations). The integrand in (2.3) is a well-defined meromorphic function on the torus provided that the gauge and the gauge-flavor anomalies vanish (see Appendix B).

The topologically twisted index (2.3) depends on a choice of fugacities y_I for the flavor group and a choice of integer magnetic fluxes $\mathfrak{n}_I$ for the R-symmetry of the theory. It is useful to introduce complex chemical potentials $y_I = e^{i\Delta_I}$. In an $\mathcal{N}=1$ theory, the choice of the R-symmetry is not unique, and can be mixed with the U(1) flavor symmetries

$$\mathfrak{n}_I = r_I + \mathfrak{p}_I \,, \tag{2.4}$$

[3]Supersymmetric localization picks a particular contour of integration and the final result can be expressed in terms of the Jeffrey-Kirwan residue [1].

where r_I is a reference R-symmetry and $\mathfrak{p}_I$ are magnetic fluxes under the flavor symmetries of the theory. The invariance of each monomial term W in the superpotential under the symmetries of the theory imposes the following constraints

$$\prod_{I \in W} y_I = 1 \,, \qquad \sum_{I \in W} \mathfrak{n}_I = 2 \,, \tag{2.5}$$

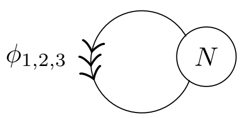
where the latter comes from supersymmetry, and, as a consequence,

$$\sum_{I \in W} \Delta_I \in 2\pi\mathbb{Z} \,. \tag{2.6}$$

Here, the product and the sum are restricted to the fields entering in the monomial W.

3 $\mathcal{N} = 4$ super Yang-Mills

We first consider the twisted compactification of four-dimensional $\mathcal{N} = 4$ super Yang-Mills (SYM) with gauge group $\mathrm{SU}(N)$ on S^2. At low energies, it results in a family of 2d theories with $\mathcal{N} = (0,2)$ supersymmetry depending on the twisting parameters $\mathfrak{n}$ [11, 12]. The theory describes the dynamics of N D3-branes wrapped on S^2 and can be pictured as the quiver gauge theory given in (3.1).



(3.1)

The superpotential

$$W = \mathrm{Tr}\left(\phi_3 \left[\phi_1, \phi_2\right]\right) \tag{3.2}$$

imposes the following constraints on the chemical potentials Δ_a and the flavor magnetic fluxes $\mathfrak{n}_a$ associated with the fields ϕ_a,

$$\sum_{a=1}^{3} \Delta_a \in 2\pi\mathbb{Z} \,, \qquad \sum_{a=1}^{3} \mathfrak{n}_a = 2 \,. \tag{3.3}$$

The topologically twisted index for the $\mathrm{SU}(N)$ SYM theory is given by

$$Z = \frac{\mathcal{A}}{N!} \sum_{\substack{\mathfrak{m} \in \mathbb{Z}^N , \\ \sum_i \mathfrak{m}_i = 0}} \int_{\mathcal{C}} \prod_{i=1}^{N-1} \frac{dx_i}{2\pi i x_i} \prod_{j \neq i}^{N} \frac{\theta_1\left(\frac{x_i}{x_j}; q\right)}{i\eta(q)} \prod_{a=1}^{3} \left[\frac{i\eta(q)}{\theta_1\left(\frac{x_i}{x_j} y_a; q\right)}\right]^{\mathfrak{m}_i - \mathfrak{m}_j - \mathfrak{n}_a + 1} \,, \tag{3.4}$$

where we defined the quantity

$$\mathcal{A} = \eta(q)^{2(N-1)} \prod_{a=1}^{3} \left[\frac{i\eta(q)}{\theta_1\left(y_a; q\right)}\right]^{(N-1)(1-\mathfrak{n}_a)} \,. \tag{3.5}$$

Here, we already imposed the SU(N) constraint $\prod_{i=1}^{N} x_i = 1$. Instead of performing a constrained sum over gauge magnetic fluxes we introduce the Lagrange multiplier w and consider an unconstrained sum. Thus, the index reads

$$Z = \frac{\mathcal{A}}{N!} \sum_{\mathfrak{m}\,\in\,\mathbb{Z}^N} \int_{\mathcal{B}} \frac{dw}{2\pi i w} w^{\sum_{i=1}^{N}\mathfrak{m}_i} \int_{\mathcal{C}} \prod_{i=1}^{N-1} \frac{dx_i}{2\pi i x_i} \prod_{j\neq i}^{N} \frac{\theta_1\left(\frac{x_i}{x_j};q\right)}{i\eta(q)} \prod_{a=1}^{3} \left[\frac{i\eta(q)}{\theta_1\left(\frac{x_i}{x_j}y_a;q\right)}\right]^{\mathfrak{m}_i-\mathfrak{m}_j-\mathfrak{n}_a+1} . \tag{3.6}$$

In order to evaluate (3.6), we employ the same strategy as in [1, 7]. The Jeffrey-Kirwan residue picks a middle-dimensional contour in $(\mathbb{C}^*)^N$. We can then take a large positive integer M and resum the contributions $\mathfrak{m} \leq M-1$. Performing the summations we get

$$Z = \frac{\mathcal{A}}{N!} \int_{\mathcal{B}} \frac{dw}{2\pi i w} \int_{\mathcal{C}} \prod_{i=1}^{N-1} \frac{dx_i}{2\pi i x_i} \prod_{i=1}^{N} \frac{\left(e^{iB_i}\right)^M}{e^{iB_i}-1} \prod_{j\neq i}^{N} \frac{\theta_1\left(\frac{x_i}{x_j};q\right)}{i\eta(q)} \prod_{a=1}^{3} \left[\frac{i\eta(q)}{\theta_1\left(\frac{x_i}{x_j}y_a;q\right)}\right]^{1-\mathfrak{n}_a} , \tag{3.7}$$

where we defined

$$e^{iB_i} = w \prod_{j=1}^{N} \prod_{a=1}^{3} \frac{\theta_1\left(\frac{x_j}{x_i}y_a;q\right)}{\theta_1\left(\frac{x_i}{x_j}y_a;q\right)} . \tag{3.8}$$

In picking the residues, we need to insert a Jacobian in the partition function and evaluate everything else at the poles, which are located at the solutions to the *Bethe ansatz equations* (BAEs),

$$e^{iB_i} = 1 , \tag{3.9}$$

such that the off-diagonal vector multiplet contribution does not vanish. We consider (3.9) as a system of N independent equations with respect to N independent variables $\{x_1, \ldots, x_{N-1}, w\}$. In the final expression, the dependence on the cut-off M disappears and we find

$$Z = \mathcal{A} \sum_{I\in\mathrm{BAEs}} \frac{1}{\det\mathbb{B}} \prod_{j\neq i}^{N} \frac{\theta_1\left(\frac{x_i}{x_j};q\right)}{i\eta(q)} \prod_{a=1}^{3} \left[\frac{i\eta(q)}{\theta_1\left(\frac{x_i}{x_j}y_a;q\right)}\right]^{1-\mathfrak{n}_a} , \tag{3.10}$$

where the summation is over all solutions I to the BAEs (3.9). The matrix $\mathbb{B}$ appearing in the Jacobian has the following form

$$\mathbb{B} = \frac{\partial\left(e^{iB_1}, \ldots, e^{iB_N}\right)}{\partial\left(\log x_1, \ldots, \log x_{N-1}, \log w\right)} . \tag{3.11}$$

3.1 Bethe potential at high temperature

We also introduce the "Bethe potential", a function that has critical points at the solutions to the BAEs (3.9). Here, we will not give the general expression for the Bethe potential as it is quite involved. Instead, we will try to make our problem easier by looking at the *high-temperature* limit, *i.e.* $q \to 1$ ($\tau \to i0$).

Let us start by considering the BAEs (3.9) at high temperature. Taking the logarithm of the BAEs (3.9), we obtain

$$0 = -2\pi i n_i + \log w - \sum_{j=1}^{N} \sum_{a=1}^{3} \left\{ \log \left[\theta \left(\frac{x_i}{x_j} y_a ; q \right) \right] - \log \left[\theta \left(\frac{x_j}{x_i} y_a ; q \right) \right] \right\} , \tag{3.12}$$

where n_i is an integer that parameterizes the angular ambiguity. It is convenient to use the variables u_i, Δ_a, v, defined modulo 2π:

$$x_i = e^{i u_i} , \qquad y_a = e^{i \Delta_a} , \qquad w = e^{i v} . \tag{3.13}$$

Then, using the asymptotic formulæ (A.6) and (A.8) we obtain the high-temperature limit of the BAEs (3.12), up to exponentially suppressed corrections,

$$0 = -2\pi i n_i + i v + \frac{1}{\beta} \sum_{j=1}^{N} \sum_{a=1}^{3} \left[F' \left(u_i - u_j + \Delta_a \right) - F' \left(u_j - u_i + \Delta_a \right) \right] , \tag{3.14}$$

where $i/(2\pi\tau) = 1/\beta$ is the formal "temperature" variable. Here, we have introduced the polynomial functions

$$F(u) = \frac{u^3}{6} - \frac{1}{2} \pi u^2 \mathrm{sign}[\mathbb{R}\mathrm{e}(u)] + \frac{\pi^2}{3} u , \qquad F'(u) = \frac{u^2}{2} - \pi u \, \mathrm{sign}[\mathbb{R}\mathrm{e}(u)] + \frac{\pi^2}{3} . \tag{3.15}$$

The high-temperature limit of the Bethe potential can be found directly by integrating the BAEs (3.14) with respect to u_i and summing over i. It reads

$$\begin{aligned} \mathcal{V}(\{u_i\}) = & \sum_{i=1}^{N} \left(2\pi n_i - v \right) u_i + \frac{i(N-1)}{\beta} \sum_{a=1}^{3} F \left(\Delta_a \right) \\ & + \frac{i}{2\beta} \sum_{i \neq j}^{N} \sum_{a=1}^{3} \left[F \left(u_i - u_j + \Delta_a \right) + F \left(u_j - u_i + \Delta_a \right) \right] . \end{aligned} \tag{3.16}$$

It is easy to check that the BAEs (3.14) can be obtained as critical points of the above Bethe potential. We introduced a Δ_a-dependent integration constant in order to have precisely one contribution $F \left(u_i - u_j + \Delta_a \right)$ for each component of the adjoint multiplet.

It is natural to restrict the Δ_a to the fundamental domain. In the high-temperature limit, we can assume that Δ_a are real and $0 < \Delta_a < 2\pi$. Moreover, since (3.3) must hold, $\sum_{a=1}^{3} \Delta_a$ can only be $0, 2\pi, 4\pi$ or 6π. We have checked that $\sum_{a=1}^{3} \Delta_a = 0, 6\pi$ lead to a singular index, and those for 2π and 4π are related by a discrete symmetry of the index *i.e.* $y_a \to 1/y_a$ ($\Delta_a \to 2\pi - \Delta_a$). Thus, without loss of generality, we will assume $\sum_{a=1}^{3} \Delta_a = 2\pi$ in the following.

The solution for $\sum_a \Delta_a = 2\pi$. We seek for solutions to the BAEs (3.14) assuming that

$$0 < \mathbb{R}\mathrm{e}\,(u_j - u_i) + \Delta_a < 2\pi\,, \qquad \forall \quad i, j, a\,. \tag{3.17}$$

Thus, the high-temperature limit of the BAEs (3.14) takes the simple form

$$\frac{2}{\beta}\sum_{a=1}^{3}(\Delta_a - \pi)\sum_{k=1}^{N}(u_j - u_k) = i\,(2\pi n_j - v)\;, \quad \text{for} \quad j = 1, 2, \ldots, N\,. \tag{3.18}$$

Imposing the constraints $\sum_{a=1}^{3}\Delta_a = 2\pi$ for the chemical potentials as well as SU(N) constraint $\sum_{i=1}^{N} u_i = 0$ we obtain the following set of equations

$$\begin{aligned} \frac{iN}{\beta}u_j &= n_j - \frac{v}{2\pi}\,, \quad \text{for} \quad j = 1, \ldots, N-1\,, \\ -\frac{iN}{\beta}\sum_{j=1}^{N-1}u_j &= n_N - \frac{v}{2\pi}\,. \end{aligned} \tag{3.19}$$

Summing up all equations we obtain the solution for v, which is given by

$$v = \frac{2\pi}{N}\sum_{i=1}^{N} n_i\,. \tag{3.20}$$

The solution for eigenvalues u_i reads

$$u_i = -\frac{i\beta}{N}\left(n_i - \frac{1}{N}\sum_{i=1}^{N} n_i\right)\,. \tag{3.21}$$

Notice that, the tracelessness condition is automatically satisfied in this case.

To proceed further, we need to provide an estimate on the value of the constants n_i. Whenever any two integers are equal $n_i = n_j$, we find that the off-diagonal vector multiplet contribution to the index, which is an elliptic generalization of the Vandermonde determinant, vanishes. Moreover, the high-temperature expansion (A.8) breaks down as subleading terms start blowing up. Hence, we should make another ansatz for the phases n_i such that

$$n_i - n_j \neq 0 \mod N\,. \tag{3.22}$$

To understand how much freedom we have, let us first note that eigenvalues u_i are variables defined on the torus T^2 and thus they should be periodic in β. Due to (3.21), this means that integers n_i are defined modulo N and hence, without loss of generality, we can consider only integers lying in the domain $[1, N]$ with the condition (3.22) modified to $n_i \neq n_j\,, \forall\, i, j$. This leaves us with the only choice $n_i = i$ and its permutations.

Substituting (3.21) and (3.20) into the Bethe potential (3.16), we obtain

$$\mathcal{V}\left(\Delta_a\right)\big|_{\text{BAEs}} = \frac{i\left(N^2-1\right)}{\beta}\sum_{a=1}^{3} F\left(\Delta_a\right) = \frac{i\left(N^2-1\right)}{2\beta}\Delta_1\Delta_2\Delta_3\,, \tag{3.23}$$

up to terms $\mathcal{O}(\beta)$.

There is an interesting relation between the "on-shell" Bethe potential (3.23) and the central charge of the UV four-dimensional theory. Note that, given the constraint $\sum_{a=1}^{3}\Delta_a = 2\pi$, the quantities Δ_a can be used to parameterize the most general R-symmetry of the theory

$$R(\Delta_a) = \sum_{a=1}^{3}\Delta_a\frac{R_a}{2\pi}\,, \tag{3.24}$$

where R_a gives charge 2 to ϕ_a and zero to ϕ_b with $b \neq a$. Observe also that the cubic R-symmetry 't Hooft anomaly is given by

$$\begin{aligned}\operatorname{Tr} R^3\left(\Delta_a\right) &= \left(N^2-1\right)\left[1+\sum_{a=1}^{3}\left(\frac{\Delta_a}{\pi}-1\right)^3\right]\\ &= \frac{3\left(N^2-1\right)}{\pi^3}\Delta_1\Delta_2\Delta_3\,,\end{aligned} \tag{3.25}$$

where the trace is taken over the fermions of the theory. Therefore, the "on-shell" value of the Bethe potential (3.23) can be rewritten as

$$\mathcal{V}\left(\Delta_a\right)\big|_{\text{BAEs}} = \frac{i\pi^3}{6\beta}\operatorname{Tr} R^3\left(\Delta_a\right) = \frac{16 i\pi^3}{27\beta}a\left(\Delta_a\right)\,, \tag{3.26}$$

where in the second equality we used the relation (1.5). Note that the linear R-symmetry 't Hooft anomaly is zero for $\mathcal{N}=4$ SYM.

3.2 The topologically twisted index at high temperature

We are interested in the high-temperature limit of the logarithm of the partition function (3.10). We shall use the asymptotic expansions (A.6) and (A.8) in order to calculate the vector and hypermultiplet contributions to the twisted index in the $\beta \to 0$ limit.

The contribution of the off-diagonal vector multiplets can be computed as

$$\log\prod_{i\neq j}^{N}\left[\frac{\theta_1\left(\frac{x_i}{x_j};q\right)}{i\eta(q)}\right] = -\frac{1}{\beta}\sum_{i\neq j}^{N}F'\left(u_i-u_j\right) - \frac{iN(N-1)\pi}{2}\,, \tag{3.27}$$

in the asymptotic limit $q \to 1$ ($\beta \to 0$). The contribution of the matter fields is instead

$$\begin{aligned}\log\prod_{i\neq j}^{N}\prod_{a=1}^{3}\left[\frac{i\eta(q)}{\theta_1\left(\frac{x_i}{x_j}y_a;q\right)}\right]^{1-\mathfrak{n}_a} &= -\frac{1}{\beta}\sum_{i\neq j}^{N}\sum_{a=1}^{3}\left[\left(\mathfrak{n}_a-1\right)F'\left(u_i-u_j+\Delta_a\right)\right]\\ &+\frac{iN(N-1)\pi}{2}\sum_{a=1}^{3}\left(1-\mathfrak{n}_a\right)\,, \qquad \text{as}\ \ \beta\to 0\,.\end{aligned} \tag{3.28}$$

The prefactor $\mathcal{A}$ in the partition function (3.5) at high temperature contributes

$$\log\left\{\eta(q)^{2(N-1)}\prod_{a=1}^{3}\left[\frac{i\eta(q)}{\theta_1\left(y_a;q\right)}\right]^{(N-1)(1-\mathfrak{n}_a)}\right\} = -\frac{N-1}{\beta}\left[\frac{\pi^2}{3}+\sum_{a=1}^{3}\left(\mathfrak{n}_a-1\right)F'(\Delta_a)\right] \\ -(N-1)\left[\log\left(\frac{\beta}{2\pi}\right)-\frac{i\pi}{2}\sum_{a=1}^{3}\left(1-\mathfrak{n}_a\right)\right]. \tag{3.29}$$

The last term to work out is $-\log\det\mathbb{B}$. The matrix $\mathbb{B}$, imposing $e^{iB_i}=1$, reads

$$\mathbb{B}=\frac{\partial\left(B_1,\ldots,B_N\right)}{\partial\left(u_1,\ldots,u_{N-1},v\right)}\,,\qquad \text{as}\ \ \beta\to 0\,, \tag{3.30}$$

and has the following entries

$$\begin{aligned} \frac{\partial B_k}{\partial u_j} &= \frac{2\pi i}{\beta}N\delta_{kj}\,, \quad \text{for} \quad k,j=1,2,\ldots,N-1\,,\\ \frac{\partial B_N}{\partial u_k} &= -\frac{2\pi i}{\beta}N\,, \quad \frac{\partial B_k}{\partial v}=1\,, \quad \text{for} \quad k=1,2,\ldots,N-1\,,\\ \frac{\partial B_N}{\partial v} &= 1\,. \end{aligned} \tag{3.31}$$

Here, we have already imposed the constraint $\sum_{a=1}^{3}\Delta_a=2\pi$. Therefore, we obtain

$$-\log\det\mathbb{B}=(N-1)\left[\log\left(\frac{\beta}{2\pi}\right)-\frac{i\pi}{2}\right]-N\log N\,. \tag{3.32}$$

Putting everything together we can write the high-temperature limit of the twisted index, at finite N,

$$\begin{aligned} \log Z = &-\frac{1}{\beta}\sum_{i\neq j}^{N}\left[F'\left(u_i-u_j\right)+\sum_{a=1}^{3}\left(\mathfrak{n}_a-1\right)F'\left(u_i-u_j+\Delta_a\right)\right]\\ &-\frac{N-1}{\beta}\left[\frac{\pi^2}{3}+\sum_{a=1}^{3}\left(\mathfrak{n}_a-1\right)F'\left(\Delta_a\right)\right]-N\log N\,, \end{aligned} \tag{3.33}$$

up to exponentially suppressed corrections. We may then evaluate the index by substituting the pole configurations (3.21) back into the functional (3.33) to get,

$$\begin{aligned} \log Z &= -\frac{N^2-1}{\beta}\left[\frac{\pi^2}{3}+\sum_{a=1}^{3}\left(\mathfrak{n}_a-1\right)F'\left(\Delta_a\right)\right]-N\log N\\ &= -\frac{N^2-1}{2\beta}\sum_{\substack{a<b\\ (\neq c)}}\Delta_a\Delta_b\mathfrak{n}_c-N\log N\,, \end{aligned} \tag{3.34}$$

which, to leading order in $1/\beta$, can be rewritten in a more intriguing form:

$$\log Z = i\sum_{a=1}^{3}\mathfrak{n}_a\frac{\partial\mathcal{V}(\Delta_a)\big|_{\text{BAEs}}}{\partial\Delta_a}\,. \tag{3.35}$$

We can relate the index to the trial left-moving central charge of the two-dimensional $\mathcal{N}=(0,2)$ theory on T^2. The latter reads [11, 12]

$$c_l = c_r - k\,, \tag{3.36}$$

where k is the gravitational anomaly

$$k = -\operatorname{Tr}\gamma_3 = -\left(N^2-1\right)\left[1+\sum_{a=1}^{3}(\mathfrak{n}_a-1)\right] = 0\,, \tag{3.37}$$

and c_r is the trial right-moving central charge

$$\begin{aligned} c_r\left(\Delta_a,\mathfrak{n}_a\right) = -3\operatorname{Tr}\gamma_3 R^2\left(\Delta_a\right) &= -3\left(N^2-1\right)\left[1+\sum_{a=1}^{3}(\mathfrak{n}_a-1)\left(\frac{\Delta_a}{\pi}-1\right)^2\right] \\ &= -\frac{3\left(N^2-1\right)}{\pi^2}\sum_{\substack{a<b\\(\neq c)}}\Delta_a\Delta_b\mathfrak{n}_c\,. \end{aligned} \tag{3.38}$$

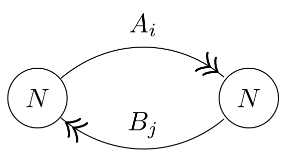
Here, the trace is taken over the fermions and γ_3 is the chirality operator in 2d. In the twisted compactification, each of the chiral fields ϕ_a give rise to 2d fermions. The difference between the number of fermions of opposite chiralities is $\mathfrak{n}_a-1$, thus explaining the above formulae. We used Δ_a/π to parameterize the trial R-symmetry. We find that the index is given by

$$\log Z = \frac{\pi^2}{6\beta}c_r\left(\Delta_a,\mathfrak{n}_a\right) = -\frac{16\pi^3}{27\beta}\sum_{a=1}^{3}\mathfrak{n}_a\frac{\partial a(\Delta_a)}{\partial\Delta_a}\,. \tag{3.39}$$

As shown in [11, 12], the exact central charge of the 2d CFT is obtained by extremizing $c_r\left(\Delta_I,\mathfrak{n}_I\right)$ with respect to the Δ_I. Given the above relation (3.39), we see that this is equivalent to extremizing the $\log Z$ at high temperature.

4 The Klebanov-Witten theory

In this section we study the Klebanov-Witten theory [47] describing the low energy dynamics of N D3-branes at the conifold singularity. This is the Calabi-Yau cone over the homogeneous Sasaki-Einstein five-manifold $T^{1,1}$ which can be described as the coset $\mathrm{SU}(2)\times\mathrm{SU}(2)/\mathrm{U}(1)$. This theory has $\mathcal{N}=1$ supersymmetry and has $\mathrm{SU}(N)\times\mathrm{SU}(N)$ gauge group with bi-fundamental chiral multiplets A_i and B_j, $i,j=1,2$, transforming in the $\left(\mathbf{N},\overline{\mathbf{N}}\right)$ and $\left(\overline{\mathbf{N}},\mathbf{N}\right)$ representations of the two gauge groups. This can be pictured as



(4.1)

It has a quartic superpotential,

$$W = \operatorname{Tr}\left(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1\right) . \tag{4.2}$$

We assign chemical potentials $\Delta_{1,2} \in (0, 2\pi)$ to A_i and $\Delta_{3,4} \in (0, 2\pi)$ to B_i. Invariance of the superpotential under the global symmetries requires

$$\sum_{I=1}^{4} \Delta_I \in 2\pi\mathbb{Z} , \qquad \sum_{I=1}^{4} \mathfrak{n}_I = 2 . \tag{4.3}$$

For the Klebanov-Witten theory, the topologically twisted index can be written as

$$\begin{aligned} Z = & \frac{1}{(N!)^2} \sum_{\mathfrak{m}, \widetilde{\mathfrak{m}} \in \mathbb{Z}^N} \int_{\mathcal{B}} \frac{dw}{2\pi i w} \frac{d\widetilde{w}}{2\pi i \widetilde{w}} w^{\sum_{i=1}^N \mathfrak{m}_i} \, \widetilde{w}^{\sum_{i=1}^N \widetilde{\mathfrak{m}}_i} \times \\ & \times \int_{\mathcal{C}} \prod_{i=1}^{N-1} \frac{dx_i}{2\pi i x_i} \frac{d\tilde{x}_i}{2\pi i \tilde{x}_i} \eta(q)^{4(N-1)} \prod_{i \neq j}^{N} \left[\frac{\theta_1\left(\frac{x_i}{x_j}; q\right)}{i\eta(q)} \frac{\theta_1\left(\frac{\tilde{x}_i}{\tilde{x}_j}; q\right)}{i\eta(q)} \right] \times \\ & \times \prod_{i,j=1}^{N} \prod_{a=1,2} \left[\frac{i\eta(q)}{\theta_1\left(\frac{x_i}{\tilde{x}_j} y_a; q\right)} \right]^{\mathfrak{m}_i - \widetilde{\mathfrak{m}}_j - \mathfrak{n}_a + 1} \prod_{b=3,4} \left[\frac{i\eta(q)}{\theta_1\left(\frac{\tilde{x}_j}{x_i} y_b; q\right)} \right]^{\widetilde{\mathfrak{m}}_j - \mathfrak{m}_i - \mathfrak{n}_b + 1} . \end{aligned} \tag{4.4}$$

Here, we assumed that eigenvalues x_i and $\tilde{x}_i$ satisfy the SU(N) constraint $\prod_{i=1}^N x_i = \prod_{i=1}^N \tilde{x}_i = 1$. Hence, the integration is performed over $(N-1)$ variables instead of N. In order to impose the SU(N) constraints for the magnetic fluxes, *i.e.*

$$\sum_{i=1}^{N} \mathfrak{m}_i = \sum_{i=1}^{N} \widetilde{\mathfrak{m}}_i = 0 , \tag{4.5}$$

we have introduced two Lagrange multipliers $w = e^{iv}$ and $\widetilde{w} = e^{i\tilde{v}}$. Now, we can resum over gauge magnetic fluxes $\mathfrak{m}_i \leq M - 1$ and $\widetilde{\mathfrak{m}}_j \geq 1 - M$ for some large positive integer cut-off M. We obtain

$$\begin{aligned} Z = & \frac{1}{(N!)^2} \int_{\mathcal{B}} \frac{dw}{2\pi i w} \frac{d\widetilde{w}}{2\pi i \widetilde{w}} \int_{\mathcal{C}} \prod_{i=1}^{N-1} \frac{dx_i}{2\pi i x_i} \frac{d\tilde{x}_i}{2\pi i \tilde{x}_i} \prod_{i \neq j}^{N} \left[\frac{\theta_1\left(\frac{x_i}{x_j}; q\right)}{i\eta(q)} \frac{\theta_1\left(\frac{\tilde{x}_i}{\tilde{x}_j}; q\right)}{i\eta(q)} \right] \times \\ & \times \mathcal{P} \prod_{i=1}^{N} \frac{(e^{iB_i})^M}{e^{iB_i} - 1} \prod_{j=1}^{N} \frac{(e^{i\widetilde{B}_j})^M}{e^{i\widetilde{B}_j} - 1} , \end{aligned} \tag{4.6}$$

where we defined the quantities

$$\mathcal{P} = \eta(q)^{4(N-1)} \prod_{i,j=1}^{N} \prod_{a=1,2} \left[\frac{i\eta(q)}{\theta_1\left(\frac{x_i}{\tilde{x}_j} y_a; q\right)} \right]^{1-\mathfrak{n}_a} \prod_{b=3,4} \left[\frac{i\eta(q)}{\theta_1\left(\frac{\tilde{x}_j}{x_i} y_b; q\right)} \right]^{1-\mathfrak{n}_b} , \tag{4.7}$$

and

$$e^{iB_i} = w \prod_{j=1}^{N} \frac{\prod_{b=3,4} \theta_1\left(\frac{\tilde{x}_j}{x_i} y_b; q\right)}{\prod_{a=1,2} \theta_1\left(\frac{x_i}{\tilde{x}_j} y_a; q\right)}, \qquad e^{i\widetilde{B}_j} = \widetilde{w}^{-1} \prod_{i=1}^{N} \frac{\prod_{b=3,4} \theta_1\left(\frac{\tilde{x}_j}{x_i} y_b; q\right)}{\prod_{a=1,2} \theta_1\left(\frac{x_i}{\tilde{x}_j} y_a; q\right)}. \tag{4.8}$$

Then, similarly to the case of $\mathcal{N} = 4$ SYM, the following BAEs equations

$$e^{iB_i} = 1\,, \qquad e^{i\widetilde{B}_j} = 1\,. \tag{4.9}$$

determine the poles of the integrand. In order to calculate the index we simply insert a Jacobian of the transformation from $\{\log x_i, \log \tilde{x}_i, \log w, \log \widetilde{w}\}$ to $\{e^{iB_i}, e^{i\tilde{B}_i}\}$ variables and evaluate everything else at the solutions to BAEs. In the final expression, the dependence on the cut-off M disappears. We can then write the partition function as,

$$Z = \sum_{I \in \text{BAEs}} \frac{1}{\det \mathbb{B}} \prod_{i \neq j}^{N} \left[\frac{\theta_1\left(\frac{x_i}{x_j}; q\right)}{i\eta(q)} \frac{\theta_1\left(\frac{\tilde{x}_i}{\tilde{x}_j}; q\right)}{i\eta(q)} \right] \mathcal{P}\,, \tag{4.10}$$

where $\mathbb{B}$ is a $2N \times 2N$ matrix

$$\mathbb{B} = \frac{\partial\left(e^{iB_1}, \ldots, e^{iB_N}, e^{i\widetilde{B}_1}, \ldots, e^{i\widetilde{B}_N}\right)}{\partial\left(\log x_1, \ldots, \log x_{N-1}, \log w, \log \tilde{x}_1, \ldots, \log \tilde{x}_{N-1}, \log \widetilde{w}\right)}. \tag{4.11}$$

4.1 Bethe potential at high temperature

Let us now look at the Bethe potential at high temperature, *i.e.* $\beta \to 0$ limit. Taking the logarithm of the BAEs (4.9) we obtain

$$\begin{aligned} 0 &= -2\pi i n_i + \log w - \sum_{j=1}^{N} \left\{ \sum_{a=1,2} \log\left[\theta_1\left(\frac{x_i}{\tilde{x}_j} y_a; q\right)\right] - \sum_{b=3,4} \log\left[\theta_1\left(\frac{\tilde{x}_j}{x_i} y_b; q\right)\right] \right\}, \\ 0 &= -2\pi i \tilde{n}_j - \log \widetilde{w} - \sum_{i=1}^{N} \left\{ \sum_{a=1,2} \log\left[\theta_1\left(\frac{x_i}{\tilde{x}_j} y_a; q\right)\right] - \sum_{b=3,4} \log\left[\theta_1\left(\frac{\tilde{x}_j}{x_i} y_b; q\right)\right] \right\}, \end{aligned} \tag{4.12}$$

where $n_i\,, \tilde{n}_j$ are integers that parameterize the angular ambiguities. In order to compute the high-temperature limit of the above BAEs, we go to the variables $u_i\,, \tilde{u}_j\,, \Delta_I\,, v\,, \tilde{v}$, defined modulo 2π, and employ the asymptotic expansions (A.6) and (A.8). We find

$$\begin{aligned} 0 &= -2\pi i n_i + iv + \frac{1}{\beta} \sum_{j=1}^{N} \left[\sum_{a=1,2} F'\left(u_i - \tilde{u}_j + \Delta_a\right) - \sum_{b=3,4} F'\left(\tilde{u}_j - u_i + \Delta_b\right) \right], \\ 0 &= -2\pi i \tilde{n}_j - i\tilde{v} + \frac{1}{\beta} \sum_{i=1}^{N} \left[\sum_{a=1,2} F'\left(u_i - \tilde{u}_j + \Delta_a\right) - \sum_{b=3,4} F'\left(\tilde{u}_j - u_i + \Delta_b\right) \right], \end{aligned} \tag{4.13}$$

where the polynomial function $F'(u)$ is defined in (3.15). The BAEs (4.13) can be obtained as critical points of the Bethe potential

$$\begin{aligned}\mathcal{V}\left(\{u_i,\tilde{u}_i\}\right) = &2\pi \sum_{i=1}^{N}\left(n_i u_i - \tilde{n}_i\tilde{u}_i\right) - \sum_{i=1}^{N}\left(v\, u_i + \tilde{v}\,\tilde{u}_i\right)\\ &+ \frac{i}{\beta}\sum_{i,j=1}^{N}\left[\sum_{a=1,2} F(u_i - \tilde{u}_j + \Delta_a) + \sum_{b=3,4} F(\tilde{u}_j - u_i + \Delta_b)\right] .\end{aligned} \tag{4.14}$$

We next turn to find solutions to the BAEs (4.13). The constraints (4.3) imply that $\sum_{I=1}^{4}\Delta_I$ can only be $0, 2\pi, 4\pi, 6\pi$ or 8π. For the conifold theory, it turns out that the solutions with $\sum_{I=1}^{4}\Delta_I = 0, 8\pi$ lead to a singular index, those for 2π and 6π are related by a discrete symmetry of the index, *i.e.* $y_I \to 1/y_I$ $(\Delta_I \to 2\pi - \Delta_I)$, and there are no consistent solutions for $\sum_{I=1}^{4}\Delta_I = 4\pi$. Thus, without loss of generality, we assume again $\sum_{I=1}^{4}\Delta_I = 2\pi$ in the following.

The solution for $\sum_I \Delta_I = 2\pi$. We assume that

$$0 < \mathbb{R}\mathrm{e}\,(\tilde{u}_j - u_i) + \Delta_{3,4} < 2\pi\,, \qquad -2\pi < \mathbb{R}\mathrm{e}\,(\tilde{u}_j - u_i) - \Delta_{1,2} < 0\,, \qquad \forall \quad i,j\,. \tag{4.15}$$

Hence, the BAEs (4.13) become

$$\begin{aligned} 0 &= -2\pi i n_j + i\,v - \frac{1}{\beta}\sum_{k=1}^{N}\left[\Delta_1\Delta_2 - \Delta_3\Delta_4 - 2\pi\left(\tilde{u}_k - u_j\right)\right] ,\\ 0 &= -2\pi i \tilde{n}_k - i\,\tilde{v} - \frac{1}{\beta}\sum_{j=1}^{N}\left[\Delta_1\Delta_2 - \Delta_3\Delta_4 - 2\pi\left(\tilde{u}_k - u_j\right)\right] .\end{aligned} \tag{4.16}$$

Here, we have already imposed the constraint $\sum_{I=1}^{4}\Delta_I = 2\pi$. Imposing the SU$(N)$ constraints for u_i , $\tilde{u}_i$ we can rewrite the BAEs in the following form

$$\frac{iN}{\beta}\,u_j = n_j - \frac{v}{2\pi} + \frac{iN}{2\pi\beta}\left(\Delta_3\,\Delta_4 - \Delta_1\,\Delta_2\right) , \quad \text{for} \quad j = 1,\dots,N-1\,, \tag{4.17}$$

$$-\frac{iN}{\beta}\sum_{j=1}^{N-1} u_j = n_N - \frac{v}{2\pi} + \frac{iN}{2\pi\beta}\left(\Delta_3\,\Delta_4 - \Delta_1\,\Delta_2\right) , \tag{4.18}$$

$$\frac{iN}{\beta}\,\tilde{u}_j = -\tilde{n}_j - \frac{\tilde{v}}{2\pi} - \frac{iN}{2\pi\beta}\left(\Delta_3\,\Delta_4 - \Delta_1\,\Delta_2\right) , \quad \text{for} \quad j = 1,\dots,N-1\,, \tag{4.19}$$

$$-\frac{iN}{\beta}\sum_{j=1}^{N-1} \tilde{u}_j = -\tilde{n}_N - \frac{\tilde{v}}{2\pi} - \frac{iN}{2\pi\beta}\left(\Delta_3\,\Delta_4 - \Delta_1\,\Delta_2\right) . \tag{4.20}$$

Equations (4.17) and (4.19) can be considered as equations defining u_i and $\tilde{u}_i$. In order to find v and $\tilde{v}$ we need to sum $(N-1)$ equations (4.17) with (4.18) and equations

(4.19) with (4.20). This leads to

$$\begin{aligned} v &= \frac{iN}{\beta}\left(\Delta_3\,\Delta_4-\Delta_1\,\Delta_2\right)+\frac{2\pi}{N}\sum_{j=1}^{N} n_j\,, \qquad u_j=-\frac{i\beta}{N}\left(n_j-\frac{1}{N}\sum_{i=1}^{N} n_i\right)\,, \\ \tilde{v} &= -\frac{iN}{\beta}\left(\Delta_3\,\Delta_4-\Delta_1\,\Delta_2\right)-\frac{2\pi}{N}\sum_{j=1}^{N} \tilde{n}_j\,, \qquad \tilde{u}_j=\frac{i\beta}{N}\left(\tilde{n}_j-\frac{1}{N}\sum_{i=1}^{N} \tilde{n}_i\right)\,. \end{aligned} \tag{4.21}$$

According to our prescription, all solutions which lead to zeros of the off-diagonal vector multiplet should be avoided. Therefore, the allowed parameter space for integers n_i and $\tilde{n}_i$ is determined by

$$n_j-n_i\neq 0 \quad \mathrm{mod} \quad N\,, \qquad\qquad \tilde{n}_j-\tilde{n}_i\neq 0 \quad \mathrm{mod} \quad N\,. \tag{4.22}$$

Given the solution (4.21) to the BAEs, the integers n_i and $\tilde{n}_i$ are defined modulo N due to the β-periodicity of eigenvalues on T^2. Thus we are left with $\{n_i,\tilde{n}_i\}\in[1,N]$. The only possible choice is then given by $n_i=\tilde{n}_i=i$ and its permutations.

Finally, plugging the solution (4.21) to the BAEs back into (4.14), we obtain the "on-shell" value of the Bethe potential

$$\mathcal{V}(\Delta_I)\big|_{\text{BAEs}}=\frac{iN^2}{2\beta}\sum_{a<b<c}\Delta_a\Delta_b\Delta_c\,, \tag{4.23}$$

up to terms $\mathcal{O}(\beta)$. The relation between the "on-shell" Bethe potential and the 4d conformal anomaly coefficients also holds for the conifold theory. The R-symmetry 't Hooft anomalies can be expressed as

$$\begin{aligned} \operatorname{Tr} R\left(\Delta_I\right) &= 2\left(N^2-1\right)+N^2\sum_{I=1}^{4}\left(\frac{\Delta_I}{\pi}-1\right)=-2\,, \\ \operatorname{Tr} R^3\left(\Delta_I\right) &= 2\left(N^2-1\right)+N^2\sum_{I=1}^{4}\left(\frac{\Delta_I}{\pi}-1\right)^3 \\ &= \frac{3N^2}{\pi^3}\sum_{a<b<c}\Delta_a\Delta_b\Delta_c-2\,, \end{aligned} \tag{4.24}$$

where we used Δ_I/π to parameterize the trial R-symmetry of the theory. Hence, Eq. (4.23) can be rewritten as

$$\mathcal{V}(\Delta_I)\big|_{\text{BAEs}}=\frac{i\pi^3}{6\beta}\left[\operatorname{Tr} R^3\left(\Delta_I\right)-\operatorname{Tr} R\left(\Delta_I\right)\right]=\frac{16i\pi^3}{27\beta}\left[3c\left(\Delta_I\right)-2a(\Delta_I)\right]\,. \tag{4.25}$$

Here, we employed Eq. (1.5) to write the second equality.

4.2 The topologically twisted index at high temperature

The twisted index, at high temperature, can be computed by evaluating the contribution of the saddle point configurations to (4.10). The procedure for computing the index is very similar to that presented in section 3.2. The off-diagonal vector multiplet contributes

$$\log \prod_{i\neq j}^{N} \left[\frac{\theta_1\left(\frac{x_i}{x_j};q\right)}{i\eta(q)} \frac{\theta_1\left(\frac{\tilde{x}_i}{\tilde{x}_j};q\right)}{i\eta(q)} \right] = -\frac{1}{\beta} \sum_{i\neq j}^{N} \left[F'\left(u_i-u_j\right) + F'\left(\tilde{u}_i-\tilde{u}_j\right) \right] - iN(N-1)\pi\,. \tag{4.26}$$

The quantity $\mathcal{P}$, Eq. (4.7), contributes

$$\begin{aligned} \log \mathcal{P} = & -\frac{1}{\beta}\Bigg\{ \frac{2\pi^2}{3}(N-1) + \sum_{i,j=1}^{N} \sum_{\substack{I=1,2:+\\ I=3,4:-}} (\mathfrak{n}_I - 1) F'\left[\pm\left(u_i - \tilde{u}_j \pm \Delta_I\right)\right] \Bigg\} \\ & + \frac{iN^2\pi}{2} \sum_{I=1}^{4} (1-\mathfrak{n}_I) - 2(N-1)\log\left(\frac{\beta}{2\pi}\right)\,. \end{aligned} \tag{4.27}$$

The Jacobian (4.11) has the following entries

$$\begin{aligned} & \frac{\partial B_k}{\partial u_j} = -\frac{\partial \widetilde{B}_k}{\partial \tilde{u}_j} = \frac{2\pi i}{\beta} N \delta_{kj}\,, \quad \text{for} \quad k,j=1,2,\ldots,N-1\,, \\ & \frac{\partial B_N}{\partial u_k} = -\frac{\partial \widetilde{B}_N}{\partial \tilde{u}_k} = -\frac{2\pi i}{\beta} N\,, \quad \frac{\partial B_k}{\partial v} = -\frac{\partial \widetilde{B}_k}{\partial \tilde{v}} = 1\,, \quad \text{for} \quad k=1,2,\ldots,N-1\,, \\ & \frac{\partial B_N}{\partial v} = -\frac{\partial \widetilde{B}_N}{\partial \tilde{v}} = 1\,, \quad \frac{\partial B_k}{\partial \tilde{u}_j} = \frac{\partial \widetilde{B}_k}{\partial u_j} = \frac{\partial B_k}{\partial \tilde{v}} = \frac{\partial \widetilde{B}_k}{\partial v} = 0\,, \quad \text{for} \quad k,j=1,\ldots,N\,. \end{aligned} \tag{4.28}$$

Now, it is straightforward to find the determinant of the matrix $\mathbb{B}$:

$$-\log\det\mathbb{B} = 2(N-1)\left[\log\left(\frac{\beta}{2\pi}\right) - \frac{i\pi}{2}\right] - 2N\log N + \pi i N\,. \tag{4.29}$$

The high-temperature limit of the index, at finite N, may then be written as

$$\begin{aligned} \log Z = & -\frac{1}{\beta}\Bigg\{ \sum_{i\neq j}^{N} \left[F'\left(u_i-u_j\right) + F'\left(\tilde{u}_i-\tilde{u}_j\right) \right] + \frac{2\pi^2}{3}(N-1) \\ & + \sum_{i,j=1}^{N} \sum_{\substack{I=1,2:+\\ I=3,4:-}} (\mathfrak{n}_I - 1) F'\left[\pm\left(u_i - \tilde{u}_j \pm \Delta_I\right)\right] \Bigg\} \\ & - 2N\log N + \pi i(N+1)\,. \end{aligned} \tag{4.30}$$

Plugging the solution (4.21) to the BAEs back into the index (4.30) we find

$$\log Z = -\frac{N^2}{2\beta} \sum_{\substack{a<b\\ (\neq c)}} \Delta_a \Delta_b \mathfrak{n}_c + \frac{2\pi^2}{3\beta} - 2N\log N + \pi i(N+1)\,. \tag{4.31}$$

As in the case of $\mathcal{N}=4$ SYM we can also write, to leading order in $1/\beta$,

$$\log Z = \frac{\pi^2}{6\beta} c_l\left(\Delta_I, \mathfrak{n}_I\right) = -\frac{16\pi^3}{27\beta} \sum_{I=1}^{4} \mathfrak{n}_I \frac{\partial a(\Delta_I)}{\partial \Delta_I}\,, \tag{4.32}$$

where the second equality is written assuming that N is large. Here, c_l is the left-moving central charge of the 2d $\mathcal{N}=(0,2)$ SCFT obtained by the twisted compactification on S^2. This is related to the trial right-moving central charge c_r by the gravitational anomaly, *i.e.* $c_l = c_r - k$. The central charge c_r takes contribution from the 2d massless fermions, the gauginos and the zero-modes of the chiral fields (the difference between the number of modes of opposite chirality being $\mathfrak{n}_I - 1$) [11–13],

$$c_r\left(\Delta_I, \mathfrak{n}_I\right) = -3 \operatorname{Tr} \gamma_3 R^2\left(\Delta_I\right) = -3\left[2\left(N^2-1\right) + N^2 \sum_{I=1}^{4} \left(\mathfrak{n}_I - 1\right)\left(\frac{\Delta_I}{\pi} - 1\right)^2\right], \tag{4.33}$$

while the gravitational anomaly k reads

$$k = -\operatorname{Tr} \gamma_3 = -2\left(N^2-1\right) - N^2 \sum_{I=1}^{4} \left(\mathfrak{n}_I - 1\right) = 2\,. \tag{4.34}$$

The extremization of $c_r\left(\Delta_I, \mathfrak{n}_I\right)$ with respect to the Δ_I reproduces the exact central charge of the 2d CFT [11, 12]. Notice that all the non-anomalous symmetries, including the baryonic one, enter in the formula (4.33), which depends on three independent fluxes and three independent fugacities. As pointed out in [13], the inclusion of baryonic charges is crucial when performing c-extremization.

5 High-temperature limit of the index for a generic theory

We can easily generalize the previous discussion to the case of general four-dimensional $\mathcal{N}=1$ SCFTs. Our goal is to compute the partition function of $\mathcal{N}=1$ gauge theories on $S^2 \times T^2$ with a partial topological A-twist along S^2. We identify, as before, the modulus of the torus with the *fictitious* inverse temperature β, and we are interested in the *high-temperature* limit ($\beta \to 0$) of the index. As we take β to zero, we can use the asymptotic expansions (A.6) and (A.8) for the elliptic functions appearing in the supersymmetric path integral (2.3). We focus on quiver gauge theories with bi-fundamental and adjoint chiral multiplets and a number $|G|$ of $\mathrm{SU}(N)^{(a)}$ gauge groups. Eigenvalues $u_i^{(a)}$ and gauge magnetic fluxes $\mathfrak{m}_i^{(a)}$ have to satisfy the tracelessness condition, *i.e.*

$$\sum_{i=1}^{N} u_i^{(a)} = 0\,, \qquad \sum_{i=1}^{N} \mathfrak{m}_i^{(a)} = 0\,. \tag{5.1}$$

The magnetic fluxes and the chemical potentials for the global symmetries of the theory fulfill the constraints (2.5) and (2.6). We also assume that $0 < \Delta_I < 2\pi$.

As in the previous examples, the solution to the BAEs is given by

$$u_i^{(a)} = \mathcal{O}(\beta) \,, \qquad \forall \quad i, a \,, \tag{5.2}$$

and exists (up to discrete symmetries) only for $\sum_{I \in W} \Delta_I = 2\pi$, for each monomial term W in the superpotential, as we checked in many examples. Due to this constraint, Δ_I/π behaves at all effects like a trial R-symmetry of the theory.

5.1 Bethe potential at high temperature

In this section we give the general rules for constructing the *high-temperature* "on-shell" Bethe potential for $\mathcal{N}=1$ quiver gauge theories to leading order in $1/\beta$:

1. A bi-fundamental field with chemical potential $\Delta_{(a,b)}$ transforming in the $(\mathbf{N}, \overline{\mathbf{N}})$ representation of $\mathrm{SU}(N)_a \times \mathrm{SU}(N)_b$, contributes

$$\frac{i N^2}{\beta} F\left(\Delta_{(a,b)}\right) \,, \tag{5.3}$$

   where the function F is defined in (3.15).

2. An adjoint field with chemical potential $\Delta_{(a,a)}$ contributes

$$\frac{i\left(N^2-1\right)}{\beta} F\left(\Delta_{(a,a)}\right) \,. \tag{5.4}$$

5.2 The topologically twisted index at high temperature

Using the dominant solution (5.2) to the BAEs we can proceed to compute the topologically twisted index. Here are the rules for constructing the logarithm of the index at high temperature to leading order in $1/\beta$:

1. For each group a, the contribution of the off-diagonal vector multiplet is

$$-\frac{\left(N^2-1\right)}{\beta} \frac{\pi^2}{3} \,. \tag{5.5}$$

2. A bi-fundamental field with magnetic flux $\mathfrak{n}_{(a,b)}$ and chemical potential $\Delta_{(a,b)}$ transforming in the $(\mathbf{N}, \overline{\mathbf{N}})$ representation of $\mathrm{SU}(N)_a \times \mathrm{SU}(N)_b$, contributes

$$-\frac{N^2}{\beta}\left(\mathfrak{n}_{(a,b)}-1\right) F'\left(\Delta_{(a,b)}\right) \,. \tag{5.6}$$

3. An adjoint field with magnetic flux $\mathfrak{n}_{(a,a)}$ and chemical potential $\Delta_{(a,a)}$, contributes

$$-\frac{N^2-1}{\beta}\left(\mathfrak{n}_{(a,a)}-1\right) F'\left(\Delta_{(a,a)}\right) \,. \tag{5.7}$$

5.3 An index theorem for the twisted matrix model

The high-temperature behavior of the index, to leading order in $1/\beta$ and N, can be captured by a simple universal formula involving the Bethe potential and its derivatives. Let us recall some of the essential ingredients that we need in the following.

The R-symmetry 't Hooft anomaly of UV four-dimensional $\mathcal{N}=1$ SCFTs is given by

$$\operatorname{Tr} R^\alpha(\Delta_I) = |G| \dim \mathrm{SU}(N) + \sum_I \dim \mathfrak{R}_I \left(\frac{\Delta_I}{\pi} - 1\right)^\alpha , \tag{5.8}$$

where the trace is taken over all the bi-fundamental fermions and gauginos and $\dim \mathfrak{R}_I$ is the dimension of the respective matter representation with R-charge Δ_I/π. On the other hand, the trial right-moving central charge of the IR two-dimensional $\mathcal{N}=(0,2)$ SCFT on T^2 can be computed from the spectrum of massless fermions [11–13]. These are gauginos with chirality $\gamma_3 = 1$ for all the gauge groups and fermionic zero modes for each chiral field, with a difference between the number of fermions of opposite chiralities equal to $\mathfrak{n}_I - 1$. The central charge is related by the $\mathcal{N}=2$ superconformal algebra to the R-symmetry anomaly [11, 12], and is given by

$$\begin{aligned} c_r\left(\Delta_I, \mathfrak{n}_I\right) &= -3 \operatorname{Tr} \gamma_3 R^2\left(\Delta_I\right) \\ &= -3\left[|G| \dim \mathrm{SU}(N) + \sum_I \dim \mathfrak{R}_I \left(\mathfrak{n}_I - 1\right) \left(\frac{\Delta_I}{\pi} - 1\right)^2\right] . \end{aligned} \tag{5.9}$$

By an explicit calculation we see that Eq. (5.9) can be rewritten as

$$c_r\left(\Delta_I, \mathfrak{n}_I\right) = -3 \operatorname{Tr} R^3\left(\Delta_I\right) - \pi \sum_I \left[\left(\mathfrak{n}_I - \frac{\Delta_I}{\pi}\right) \frac{\partial \operatorname{Tr} R^3\left(\Delta_I\right)}{\partial \Delta_I}\right] , \tag{5.10}$$

where we used the relation (5.8).[4] Moreover, the trial left-moving central charge of the 2d $\mathcal{N}=(0,2)$ theory reads

$$c_l = c_r - k \, , \tag{5.11}$$

where k is the gravitational anomaly and is given by

$$k = -\operatorname{Tr} \gamma_3 = -|G| \dim \mathrm{SU}(N) - \sum_I \dim \mathfrak{R}_I \left(\mathfrak{n}_I - 1\right) . \tag{5.12}$$

For theories of D3-branes with an AdS dual, to leading order in N, the linear R-symmetry 't Hooft anomaly of the 4d theory vanishes, *i.e.* $\operatorname{Tr} R = \mathcal{O}(1)$ and $a = c$ [21]. Using the parameterization of a trial R-symmetry in terms of Δ_I/π, this is equivalent to

$$\pi |G| + \sum_I \left(\Delta_I - \pi\right) = 0 \, , \tag{5.13}$$

[4] Notice that, in evaluating the right hand side of (5.10), we can consider all the Δ_I as independent variables and impose the constraints $\sum_{I \in W} \Delta_I = 2\pi$ only after differentiation. This is due to the form of the differential operator in (5.10) and the constraints $\sum_{I \in W} \mathfrak{n}_I = 2$.

where the sum is taken over all matter fields (bi-fundamental and adjoint) in the quiver. Similarly, we have

$$|G| + \sum_I (\mathfrak{n}_I - 1) = 0 \,. \tag{5.14}$$

This is simply $k = -\operatorname{Tr}\gamma_3 = \mathcal{O}(1)$, to leading order in N.

The *index theorem* can be expressed as:

Theorem 1. *The topologically twisted index of any $\mathcal{N} = 1$ $\mathrm{SU}(N)$ quiver gauge theory placed on $S^2 \times T^2$ to leading order in $1/\beta$ is given by*

$$\log Z\left(\Delta_I, \mathfrak{n}_I\right) = \frac{\pi^2}{6\beta} c_l\left(\Delta_I, \mathfrak{n}_I\right) \,, \tag{5.15}$$

which is Cardy's universal formula for the asymptotic density of states in a CFT_2 [48]. We can write the extremal value of the Bethe potential $\overline{\mathcal{V}}\left(\Delta_I\right)$ as

$$\begin{aligned} \overline{\mathcal{V}}\left(\Delta_I\right) \equiv -i\mathcal{V}\left(\Delta_I\right)\big|_{BAEs} &= \frac{\pi^3}{6\beta}\left[\operatorname{Tr} R^3\left(\Delta_I\right) - \operatorname{Tr} R\left(\Delta_I\right)\right] \\ &= \frac{16\pi^3}{27\beta}\left[3c\left(\Delta_I\right) - 2a(\Delta_I)\right] \,. \end{aligned} \tag{5.16}$$

For theories of D3-branes at large N, the index can be recast as

$$\begin{aligned} \log Z\left(\Delta_I, \mathfrak{n}_I\right) &= -\frac{3}{\pi}\overline{\mathcal{V}}\left(\Delta_I\right) - \sum_I \left[\left(\mathfrak{n}_I - \frac{\Delta_I}{\pi}\right)\frac{\partial\overline{\mathcal{V}}\left(\Delta_I\right)}{\partial\Delta_I}\right] \\ &= \frac{\pi^2}{6\beta} c_r\left(\Delta_I, \mathfrak{n}_I\right) \,, \end{aligned} \tag{5.17}$$

where $\overline{\mathcal{V}}\left(\Delta_I\right)$ reads

$$\overline{\mathcal{V}}\left(\Delta_I\right) \equiv -i\mathcal{V}\left(\Delta_I\right)\big|_{BAEs} = \frac{16\pi^3}{27\beta} a(\Delta_I) \,. \tag{5.18}$$

Proof. Observe first that again we can consider all the Δ_I in (5.17) as independent variables and impose the constraints $\sum_{I\in W}\Delta_I = 2\pi$ only after differentiation. This is due to the form of the differential operator in (5.17) and $\sum_{I\in W}\mathfrak{n}_I = 2$. To prove the first equality in (5.17), we promote the explicit factors of π, appearing in (5.3) and (5.4), to a formal variable $\boldsymbol{\pi}$. Notice that the "on-shell" Bethe potential $\overline{\mathcal{V}}$, at large N, is a homogeneous function of Δ_I and $\boldsymbol{\pi}$, *i.e.*

$$\overline{\mathcal{V}}(\lambda\Delta_I, \lambda\boldsymbol{\pi}) = \lambda^3\,\overline{\mathcal{V}}(\Delta_I, \boldsymbol{\pi}) \,. \tag{5.19}$$

Hence,

$$\frac{\partial\overline{\mathcal{V}}(\Delta_I, \boldsymbol{\pi})}{\partial\boldsymbol{\pi}} = \frac{1}{\boldsymbol{\pi}}\left[3\,\overline{\mathcal{V}}(\Delta_I) - \sum_I \Delta_I \frac{\partial\overline{\mathcal{V}}(\Delta_I)}{\partial\Delta_I}\right] \,. \tag{5.20}$$

Now, we consider a generic quiver gauge theory with matters in bi-fundamental and adjoint representations of the gauge group. They contribute to the Bethe potential $\overline{\mathcal{V}}(\Delta_I, \boldsymbol{\pi})$ as written in (5.3) and (5.4), respectively. Let us calculate the derivative of $\overline{\mathcal{V}}(\Delta_I, \boldsymbol{\pi})$ with respect to Δ_I:

$$-\sum_I \mathfrak{n}_I \frac{\partial \overline{\mathcal{V}}(\Delta_I, \boldsymbol{\pi})}{\partial \Delta_I} = -\frac{N^2}{\beta} \sum_I \mathfrak{n}_I \, F'\left(\Delta_I\right) . \tag{5.21}$$

Next, we take the derivative of the Bethe potential with respect to $\boldsymbol{\pi}$:

$$-\sum_I \frac{\partial \overline{\mathcal{V}}(\Delta_I, \boldsymbol{\pi})}{\partial \boldsymbol{\pi}} = \frac{N^2}{\beta} \sum_I \; F'\left(\Delta_I\right) - \frac{N^2}{\beta} \sum_I \left(\frac{\boldsymbol{\pi}^2}{3} - \frac{\boldsymbol{\pi}}{3} \Delta_I \right) . \tag{5.22}$$

Using (5.20) and combining (5.21) with the first term of (5.22) as in the right hand side of Eq. (5.17), we obtain the contribution of matter fields (5.6) and (5.7) to the index. The contribution of the second term in (5.22) to Eq. (5.17) can be written as

$$-\frac{N^2}{\beta} \frac{\boldsymbol{\pi}}{3} \sum_I \left(\boldsymbol{\pi} - \Delta_I \right) = -\frac{N^2}{\beta} \frac{\boldsymbol{\pi}^2}{3} |G| \, , \tag{5.23}$$

where we used the constraint (5.13). This is precisely the contribution of the off-diagonal vector multiplets (5.5) to the index at large N.

Parameterizing the trial R-symmetry of an $\mathcal{N} = 1$ theory in terms of Δ_I / π, we can prove (5.16):

$$\begin{aligned} \overline{\mathcal{V}}\left(\Delta_I\right) &= \frac{1}{\beta} \sum_I \; \dim \mathfrak{R}_I \; F\left(\Delta_I\right) = \frac{1}{6\beta} \sum_I \; \dim \mathfrak{R}_I \left[\left(\Delta_I - \pi\right)^3 - \pi^2 \left(\Delta_I - \pi\right) \right] \\ &= \frac{\pi^3}{6\beta} \left[\sum_I \; \dim \mathfrak{R}_I \left(\frac{\Delta_I}{\pi} - 1 \right)^3 - \sum_I \; \dim \mathfrak{R}_I \left(\frac{\Delta_I}{\pi} - 1 \right) \right] \\ &= \frac{\pi^3}{6\beta} \left[\operatorname{Tr} R^3 \left(\Delta_I\right) - \operatorname{Tr} R \left(\Delta_I\right) \right] \, , \end{aligned} \tag{5.24}$$

which at large N, due to (5.13), is equal to (5.18).

Finally, we need to show that the high-temperature limit of the index is given by the Cardy formula (5.15). Bi-fundamental and adjoint fields contribute to the index according to (5.6) and (5.7), respectively. We thus have

$$\begin{aligned} \log Z\left(\Delta_I, \mathfrak{n}_I\right) &= -\frac{1}{\beta} \left[\frac{\pi^2}{3} |G| \; \dim \mathrm{SU}(N) + \sum_I \; \dim \mathfrak{R}_I \; \left(\mathfrak{n}_I - 1\right) F'\left(\Delta_I\right) \right] \\ &= -\frac{\pi^2}{6\beta} \left\{ 2|G| \; \dim \mathrm{SU}(N) + \sum_I \; \dim \mathfrak{R}_I \; \left(\mathfrak{n}_I - 1\right) \left[3 \left(\frac{\Delta_I}{\pi} - 1 \right)^2 - 1 \right] \right\} \\ &= \frac{\pi^2}{6\beta} \left[c_r \left(\Delta_I, \mathfrak{n}_I\right) + \operatorname{Tr} \gamma_3 \right] = \frac{\pi^2}{6\beta} c_l \left(\Delta_I, \mathfrak{n}_I\right) \, , \end{aligned} \tag{5.25}$$

where we used (5.9) and (5.11) in the third and the fourth equality, respectively. For quiver gauge theories fulfilling the constraint (5.14) the above formula reduces to the second equality in (5.17) at large N. This completes the proof. $\square$

It is worth stressing that, when using formula (5.17), the linear relations among the Δ_I can be imposed after differentiation. It is always possible, ignoring some linear relations, to parameterize $\overline{\mathcal{V}}(\Delta_I)$ such that it becomes a homogeneous function of degree 3 in the chemical potentials Δ_I [49]. With this parameterization the index theorem becomes

$$\log Z\left(\Delta_I, \mathfrak{n}_I\right) = -\sum_I \mathfrak{n}_I \frac{\partial \overline{\mathcal{V}}\left(\Delta_I\right)}{\partial \Delta_I}\,. \tag{5.26}$$

As we have seen, this is indeed the case for $\mathcal{N}=4$ SYM and the Klebanov-Witten theory. We note that our result is very similar to that obtained for the large N limit of the topologically twisted index of three-dimensional $\mathcal{N} \geq 2$ Yang-Mills-Chern-Simons-matter theories placed on $S^2 \times S^1$ [9, 10].

6 Future directions

There are various directions to explore. Let us mention some of them.

1. We can refine the index by turning on angular momentum along the two-dimensional compact manifold S^2 [1]. It would be quite interesting to understand the results for the refined index in the context of rotating black string solutions in five-dimensional gauged supergravity which are still to be found.

2. The critical points of the Bethe potential $\mathcal{V}(\{u_i^{(a)}\})$ coincide with the Bethe equations for the vacua of a quantum integrable system [18, 50–54].[5] It would be very interesting to understand if the quantum integrability picture can shed new light on the microscopic origin of black holes/strings entropy.

3. Regular asymptotically $\mathrm{AdS}_5 \times S^5$ rotating black holes, characterized by three electric charges and two angular momenta, have been found in five-dimensional $\mathrm{U}(1)^3$ gauged supergravity [57–60]. Our general results for 4d/5d static black holes/strings may suggest new approaches for understanding the statistical meaning of the Bekenstein-Hawking entropy for this class of black holes in terms of states in the dual $\mathcal{N}=4$ SYM theory.

We hope to come back to these questions soon.

[5]See [55, 56] for a discussion about the Bethe equations in the context of factorization and holomorphic blocks.

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A Elliptic functions and their asymptotics

The Dedekind eta function is defined by

$$\eta(q) = \eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \,, \qquad \mathbb{I}\mathrm{m}\, \tau > 0 \,, \tag{A.1}$$

where $q = e^{2\pi i \tau}$. It has the following modular properties

$$\eta(\tau + 1) = e^{\frac{i\pi}{12}} \eta(\tau) \,, \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\, \eta(\tau) \,. \tag{A.2}$$

The Jacobi theta function reads

$$\begin{aligned} \theta_1(x; q) = \theta_1(u; \tau) &= -i q^{\frac{1}{8}} x^{\frac{1}{2}} \prod_{k=1}^{\infty} \left(1 - q^k\right) \left(1 - x q^k\right) \left(1 - x^{-1} q^{k-1}\right) \\ &= -i \sum_{n \in \mathbb{Z}} (-1)^n e^{iu\left(n + \frac{1}{2}\right)} e^{\pi i \tau \left(n + \frac{1}{2}\right)^2} \,, \end{aligned} \tag{A.3}$$

where $x = e^{iu}$ and q is as before. The function $\theta_1(u; \tau)$ has simple zeros in u at $u = 2\pi\mathbb{Z} + 2\pi\tau\mathbb{Z}$ and no poles. Its modular properties are,

$$\theta_1\left(u; \tau + 1\right) = e^{\frac{i\pi}{4}} \theta_1\left(u; \tau\right) \,, \qquad \theta_1\left(\frac{u}{\tau}; -\frac{1}{\tau}\right) = -i\sqrt{-i\tau}\, e^{\frac{iu^2}{4\pi\tau}} \theta_1\left(u; \tau\right) \,. \tag{A.4}$$

We also note the following useful formula,

$$\theta_1\left(q^m x; q\right) = (-1)^{-m} x^{-m} q^{-\frac{m^2}{2}} \theta_1(x; q) \,, \qquad m \in \mathbb{Z} \,. \tag{A.5}$$

The asymptotic behavior of the $\eta(q)$ and $\theta_1(x; q)$ as $q \to 1$ can be derived by using their modular properties. To this purpose, we first need to perform an S-transformation, *i.e.* $\tau \to -1/\tau$, and then expand the resulting functions in series of q, which is now a small parameter in the $\tau \to i0$ limit.

Let us start with the Dedekind η-function. The action of modular transformation is written in (A.2). We will identify the "inverse temperature" β with the modular parameter τ of the torus: $\tau = i\beta/2\pi$. Then, expanding the S-transformed η-function we get

$$\begin{aligned} \log\left[\eta(\tau)\right] &= -\frac{1}{2}\log\left(-i\tau\right) + \log\left[\eta\left(-\frac{1}{\tau}\right)\right] \\ &= -\frac{1}{2}\log\left(\frac{\beta}{2\pi}\right) - \frac{\pi^2}{6\beta} + \mathcal{O}\left(e^{-1/\beta}\right) . \end{aligned} \tag{A.6}$$

Similarly, we can consider the asymptotic expansion of the Jacobi θ-function:

$$\begin{aligned} \log\left[\theta_1(u;\tau)\right] &= \frac{i\pi}{2} - \frac{1}{2}\log\left(-i\tau\right) - \frac{iu^2}{4\pi\tau} + \log\left[\theta_1\left(\frac{u}{\tau};-\frac{1}{\tau}\right)\right] \\ &= -\frac{\pi^2}{2\beta} - \frac{u^2}{2\beta} - \frac{1}{2}\log\left(\frac{\beta}{2\pi}\right) + \log\left[2\sinh\left(\frac{\pi u}{\beta}\right)\right] + \mathcal{O}\left(e^{-1/\beta}\right) , \end{aligned}$$

Writing $2\sinh\left(\frac{\pi u}{\beta}\right) = e^{\pi u/\beta}\left(1 - e^{-2\pi u/\beta}\right)$, we have the following expansion

$$\log\left[2\sinh\left(\frac{\pi u}{\beta}\right)\right] = \frac{\pi}{\beta}u\,\mathrm{sign}\left[\mathbb{R}\mathrm{e}(u)\right] - \sum_{k=1}^{\infty}\frac{1}{k}e^{-\frac{2k\pi}{\beta}u\,\mathrm{sign}\left[\mathbb{R}\mathrm{e}(u)\right]} . \tag{A.7}$$

Putting all pieces together, we find

$$\log\left[\theta_1(u;\tau)\right] = -\frac{\pi^2}{2\beta} - \frac{u^2}{2\beta} - \frac{1}{2}\log\left(\frac{\beta}{2\pi}\right) + \frac{\pi}{\beta}u\,\mathrm{sign}\left[\mathbb{R}\mathrm{e}(u)\right] + \mathcal{O}\left(e^{-1/\beta}\right) . \tag{A.8}$$

B Anomaly cancellation

Here we obtain the conditions for which the integrand in (2.3) is a well-defined meromorphic function on the torus. To this aim the one-loop contributions must be invariant under the transformation $x^\rho \to q^{\rho(\gamma)}x^\rho$ where γ live in the co-root lattice $\Gamma_{\mathfrak{h}}$ of the gauge group.

The off-diagonal vector multiplets contribute to the index as

$$Z_{1-\text{loop}}^{\text{gauge, off}} = (-1)^{\sum_{\alpha>0}\alpha(\mathfrak{m})}\prod_{\alpha\in G}\left[\frac{\theta_1(x^\alpha;q)}{i\eta(q)}\right] . \tag{B.1}$$

Applying $x^\rho \to q^{\rho(\gamma)}x^\rho$ and using Eq. (A.5) we find

$$Z_{1-\text{loop}}^{\text{gauge, off}} \to Z_{1-\text{loop}}^{\text{gauge, off}}\prod_{\alpha\in G}(-1)^{-\alpha(\gamma)}\,e^{-i\pi\tau\alpha(\gamma)^2}\,e^{-i\alpha(u)\alpha(\gamma)} . \tag{B.2}$$

The one-loop contribution of a chiral multiplet is given by

$$Z_{1-\text{loop}}^{\text{chiral}} = \prod_{\rho_I\in\mathfrak{R}_I}\left[\frac{i\eta(q)}{\theta_1(x^{\rho_I}y^{\nu_I};q)}\right]^B , \qquad B = \rho_I(\mathfrak{m}) - \mathfrak{n}_I + 1 , \tag{B.3}$$

where ν_I is the weight of the multiplet under the flavor symmetry group. It transforms as

$$Z^{\text{chiral}}_{1-\text{loop}} \to Z^{\text{chiral}}_{1-\text{loop}} \prod_{\rho_I \in \mathfrak{R}_I} (-1)^{\rho_I(\gamma)B}\, e^{i\pi\tau\rho_I(\gamma)^2 B}\, e^{i\rho_I(u)\rho_I(\gamma)B}\, e^{i\rho_I(\gamma)\nu_I(\Delta)B}\,. \tag{B.4}$$

Putting everything together, the total prefactor in the integrand vanishes if we demand the following anomaly cancellation conditions:

$$\sum_{\alpha\in G}\alpha(\gamma)^2 + \sum_I \sum_{\rho_I\in\mathfrak{R}_I} (\mathfrak{n}_I - 1)\, \rho_I(\gamma)^2 = 0\,, \qquad \text{U(1)}_R\text{-gauge-gauge anomaly}\,, \tag{B.5}$$

$$\sum_{\alpha\in G}\alpha(\gamma)\alpha(u) + \sum_I \sum_{\rho_I\in\mathfrak{R}_I} (\mathfrak{n}_I - 1)\, \rho_I(\gamma)\rho(u) = 0\,, \qquad \text{U(1)}_R\text{-gauge-gauge anomaly}\,, \tag{B.6}$$

$$\sum_I \sum_{\rho_I\in\mathfrak{R}_I} \rho_I(\gamma)^2\, \rho_I(\mathfrak{m}) = 0\,, \qquad \text{gauge}^3 \text{ anomaly}\,, \tag{B.7}$$

$$\sum_I \sum_{\rho_I\in\mathfrak{R}_I} \rho_I(\gamma)\, \rho(u)\, \rho_I(\mathfrak{m}) = 0\,, \qquad \text{gauge}^3 \text{ anomaly}\,, \tag{B.8}$$

$$\sum_I \sum_{\rho_I\in\mathfrak{R}_I} \rho_I(\gamma)\, \rho_I(\mathfrak{m})\, \nu_I(\Delta) = 0\,, \qquad \text{gauge-gauge-flavor anomaly}\,, \tag{B.9}$$

$$\sum_I \sum_{\rho_I\in\mathfrak{R}_I} (\mathfrak{n}_I - 1)\, \rho_I(\gamma)\, \nu_I(\Delta) = 0\,, \qquad \text{U(1)}_R\text{-gauge-flavor anomaly}\,. \tag{B.10}$$

The signs cancel out automatically for all D3-brane quivers since the number of arrows entering a node equals the number of arrows leaving it.

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