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Seminar 1 (Hamiltonian formalism, Legendre transformation)

Theory

* One way to derive properties of mechanical system is to use Lagrange function $L(q_j; \dot{q}_j; t)$ and equation of motion is then Euler-Lagrange equations

$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0;}$ But we can describe mechanical systems in terms of $(p_j; q_j; t)$ rather than $(q_j; \dot{q}_j; t)$. To go from one set of coordinates we use Legendre transformation

Consider $f(x,y)$ function such that

$$df = u\,dx + v\,dy, \text{ where } u = \frac{\partial f}{\partial x}; \; v = \frac{\partial f}{\partial y};$$

Let's now consider function g of u and y, defined by $g = f - ux$; thus $dg = df - u\,dx - x\,du = v\,dy - x\,du$

So now x and v are now functions of the variables u and y given by:

$$\boxed{x = -\frac{\partial g}{\partial u}; \quad v = \frac{\partial g}{\partial y};}$$

Now let's find equations of motion for this formalism

For this let's write down differential of L

$$dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt,$$ canonical momentum is defined as $p_i = \frac{\partial L}{\partial \dot{q}_i}$. Substituting this into Lagrange equation we get $\dot{p}_i = \frac{\partial L}{\partial q_i}$. So now making Legendre transformation $H(q, p, t) = \dot{q}_i p_i - L(q, \dot{q}, t)$, differential is

$$dH = \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt$$, at the same time

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt$$ thus we get new set of equations.

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}; \quad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}; \quad -\dot{p}_i = \frac{\partial H}{\partial q_i};}$$

This are our new equations of motion. Note that this are first order equations, but there are $2n$ equations for system with n.

② The Hamiltonian of a relativistic particle of mass

Problem 1 m has the form $H=\sqrt{m^2c^4+p^2c^2}$ where $\bar{p}$ is 3-dimensional momentum and c is the speed of light. Find a Lagrangian corresponding to this Hamiltonian.

Following logic of theory, we conclude the following algorithm of finding Lagrangian starting with hamiltonian or vise versa

Algorithm Hamiltonian $\rightarrow$ Lagrangian

I using hamilton equation $\dot{q}_i = \frac{\partial H}{\partial p_i}$ we can find

p_i as function of $(\dot{q}_i, q_i, t)$

II Make Legendre transformation

$L = p\dot{q} - H$. On this stage we get "mixed" function of $p, \dot{q}, q$ and t variables

III Final step is going from $(p, \dot{q}, q)$ set of variables to $(\dot{q}, q, t)$ variables using result of step I

* Lagrangian - Hamiltonian

The algorithm is very much the same

I using definition of canonical momentum $p_i = \frac{\partial L}{\partial \dot{q}_i}$ we express $\dot{q}_i$ through p_i

II Again we make Legendre transformation

$H = p\dot{q} - L$ and get function of $(p, \dot{q}, q, t)$

III Final step is going from $(p, \dot{q}, q)$ to (p, q, t) using results of step I.

Now let's go through this steps:

I First let's use hamiltonian equation

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{p_i c^2}{\sqrt{m^2c^4 + \bar{p}^2c^2}} = \frac{p_i c^2}{H}$$ inverting this expression

gives $p_i = \frac{H\dot{q}_i}{c^2}$, now we can write H is terms of $\dot{q}$ and q

③ $H = \sqrt{m^2c^4 + \frac{H^2\dot{q}^2}{c^4}\cdot c^2} \Rightarrow H^2\left(1 - \frac{\dot{q}^2}{c^2}\right) = m^2c^4;$

$$\boxed{H = \frac{mc^2}{\sqrt{1 - \frac{\dot{q}^2}{c^2}}}}$$

Ⓘ Now let's perform Legendre transformation

$$L = \bar{p}\cdot\dot{\bar{q}} - H = \bar{p}\cdot\dot{\bar{q}} - \frac{mc^2}{\sqrt{1-\frac{\dot{q}^2}{c^2}}} = \frac{H\dot{\bar{q}}^2}{c^2} - H = -mc^2\sqrt{1-\frac{\dot{\bar{q}}^2}{c^2}};$$

$\boxed{L = -\frac{mc^2}{\gamma(\dot{\bar{q}})}}$ where $\gamma(\dot{\bar{q}}) = \frac{1}{\sqrt{1-\frac{\dot{\bar{q}}^2}{c^2}}}$ is Lorentz contraction factor

Note We will deal with this Lagrangian further in special relativity part of the course.

Relativistic action is $S = \int dt L = -\int dt \frac{mc^2}{\gamma} = -mc^2\int\sqrt{1-\frac{v^2}{c^2}}\,dt \Rightarrow$

$\boxed{S = -mc\int ds}$ where $\int ds$ is the length of the world line.

Problem 2 A dynamical system has the Lagrangian

$$L = \frac{5}{2}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2 + \dot{q}_1\dot{q}_2\cdot\cos(q_1 - q_2) + 3\cos q_1 + \cos q_2;$$

Before solving this problem let's derive general formula of Hamiltonian corresponding to Lagrangian

$$L(\bar{q}, \dot{\bar{q}}, t) = L_0(q, t) + \dot{\bar{q}}^T\bar{a} + \frac{1}{2}\dot{\bar{q}}^T\cdot T\cdot\dot{\bar{q}};$$

where $\bar{q}$ and $\dot{\bar{q}}$ are vectors with dimension equall to number of degrees of freedom of our system. Canonical momentum is the, by definitio is given by:

$\bar{p} = \frac{\partial L}{\partial\dot{q}} = T\dot{\bar{q}} + \bar{a}$, now we can extract $\dot{\bar{q}}$ from here:

$\dot{\bar{q}} = T^{-1}(\bar{p} - \bar{a})$; and $\dot{\bar{q}}^T = (\bar{p} - \bar{a})^T T^{-1}$;

Now we can do Legendre transformation:

$$H = \dot{\bar{q}}^T\bar{p} - L = \dot{\bar{q}}^T(\bar{p} - \bar{a}) - \frac{1}{2}\dot{\bar{q}}^T T\dot{\bar{q}} - L_0$$

④

$$H=(\bar{p}-\bar{a})^T T^{-1}(\bar{p}-\bar{a}) - \frac{1}{2}(\bar{p}-\bar{a})^T T^{-1}(\bar{p}-\bar{a}) - L_0(q,t)$$, finally

$$\boxed{H=\frac{1}{2}(\bar{p}-\bar{a})^T T^{-1}(\bar{p}-\bar{a}) - L_0(q,t);}$$

Now we can directly use this formula to derive Hamiltonians.

For example in our case $\bar{q}=\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$, and

$$T=\begin{bmatrix} 5 & \cos(q_1-q_2) \\ \cos(q_1-q_2) & 1 \end{bmatrix}; \quad \bar{a}=0 \quad \text{and} \quad L_0(\bar{q})=3\cos q_1+\cos q_2;$$

First of all we should find inverse matrix. It is found in the following way: $T^{-1}=\frac{T_c^T}{|T|}$ where T_c is the cofactor matrix whose elements $(T_c)_{jk}$ are $(-1)^{j+k}$ times the determinant of matrix obtained by striking j^{th} row and i^{th} column of T. In our example:

$$(T^{-1})_{11}=\frac{1}{|T|}; \; (T^{-1})_{12}=-\frac{\cos(q_1-q_2)}{|T|}; \; (T^{-1})_{21}=-\frac{\cos(q_1-q_2)}{|T|}; \; (T^{-1})_{22}=\frac{5}{|T|};$$

and $|T|=5-\cos^2(q_1-q_2)$; So we finally get:

$$T^{-1}=\frac{1}{5-\cos^2(q_1-q_2)}\begin{bmatrix} 1 & -\cos(q_1-q_2) \\ -\cos(q_1-q_2) & 5 \end{bmatrix};$$ Thus hamiltonian is given by

$$H=\frac{1}{2}\frac{1}{5-\cos^2(q_1-q_2)}\cdot[p_1; p_2]\begin{bmatrix} 1 & -\cos(q_1-q_2) \\ -\cos(q_1-q_2) & 5 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}-3\cos q_1-\cos q_2;$$

finally

$$\boxed{H=\frac{1}{10-2\cos^2(q_1-q_2)}(p_1^2+5p_2^2-2\cos(q_1-q_2)p_1p_2)-3\cos q_1-\cos q_2;}$$

ⓑ The same thing for Lagrangian:

$$L=\frac{1}{2}\left[(\dot{q}_1-\dot{q}_2)^2+a\dot{q}_1^2t^2\right]-a\cos q_2;$$

⑤ This is Lagrangian of canonical form too

$$L = \frac{1}{2}\dot{q}_1^2(1+at^2) + \frac{1}{2}\dot{q}_2^2 - \dot{q}_1\dot{q}_2 - a\cos q_2;$$

This is Lagrangian exacetely of form we want with

$\bar{a}=0$; $T = \begin{bmatrix} 1+at^2 & -1 \\ -1 & 1 \end{bmatrix}$; $L_0(q,t) = -a\cos q_2$;

First of all we should find inverse matrix T^{-1}

$|T| = at^2$; $(T^{-1})_{11} = \frac{1}{at^2}$; $(T^{-1})_{22} = \frac{1}{at^2}+1$; $(T^{-1})_{12} = \frac{1}{at^2}$; $(T^{-1})_{22} = \frac{1}{at^2}+1$;

$T^{-1} = \frac{1}{at^2}\begin{bmatrix} 1 & 1 \\ 1 & 1+at^2 \end{bmatrix}$ So we can now directly apply

general formula:

$$H = \frac{1}{2}\bar{p}\,T^{-1}\,\bar{p} - L_0(q,t) = \frac{1}{2at^2}[p_1\ p_2]\begin{bmatrix} 1 & 1 \\ 1 & 1+at^2 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + a\cos q_2;$$

or $$H = \frac{1}{2at^2}(p_1^2 + p_2^2 + p_2^2\cdot at^2 + 2p_1p_2) + a\cos q_2 \Rightarrow$$

$$\Rightarrow \boxed{H = \frac{1}{2at^2}(p_1+p_2)^2 + \frac{p_2^2}{2} + a\cdot\cos q_2;}$$

Problem 3 A dynamical system has the Hamiltonian

$$H = p_1p_2 + q_1q_2$$

Find a Lagrangian corresponding to this Hamiltonian

* First way is to follow general algorithm of finding Lagrangian when Hamiltonian is given

Ⅰ $\dot{q}_1 = \frac{\partial H}{\partial p_1} = p_2$; $\dot{q}_2 = \frac{\partial H}{\partial p_2} = p_1$; - here we have just used Hamilton equations to relate canonical momentum and $\dot{q}$

Ⅱ Now let's make Legendre transformation

$$L = p_1\dot{q}_1 + p_2\dot{q}_2 - H = 2\dot{q}_1\dot{q}_2 - \dot{q}_1\dot{q}_2 - q_1q_2 = \dot{q}_1\dot{q}_2 - q_1q_2$$

$$\boxed{L = \dot{q}_1\dot{q}_2 - q_1q_2;}$$

* Second equivalent way is to derive general formula in the way we derived previous formula

(6) Let's write down general form of hamiltonian

$$H = H_0(q,t) + \bar{p}^T \bar{a} + \frac{1}{2}\bar{p}^T T \bar{p}$$

We define $\dot{\bar{q}}$ from Hamilton equations $\dot{\bar{q}} = \frac{\partial H}{\partial \bar{p}} = \bar{a} + T\bar{p}$

thus inverting this expression we get $\bar{p} = T^{-1}(\dot{\bar{q}} - \bar{a})$

Now if we make Legendre transformation

$$L = \bar{p}^T \dot{\bar{q}} - H(\bar{q},t) = \bar{p}^T(\dot{\bar{q}} - \bar{a}) - \frac{1}{2}\bar{p}^T T \bar{p} - H_0(q,t)$$

$$\boxed{L(\dot{\bar{q}}, \bar{q}, t) = \frac{1}{2}(\dot{\bar{q}} - \bar{a})^T T^{-1}(\dot{\bar{q}} - \bar{a}) - H_0(q,t);}$$

In our case $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $T^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = T$; $\bar{a} = \bar{0}$;

$H_0(\bar{q}, t) = q_1 q_2$ and substituting all this values into our derived formula.

$$L = \frac{1}{2}\dot{\bar{q}}^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \dot{\bar{q}} - q_1 q_2 = \dot{q}_1 \dot{q}_2 - q_1 q_2$$ so we have got

$\boxed{L(\dot{\bar{q}}, \bar{q}, t) = \dot{q}_1 \dot{q}_2 - q_1 q_2;}$ which coincides with the Lagrangian observed previously.

Problem 4 Hamiltonian is given by

$$H = \frac{p^2}{2a} - bqpe^{-\alpha t} + \frac{ab}{2}q^2 e^{-\alpha t}(\alpha + be^{-\alpha t}) + \frac{kq^2}{2};$$ where

a, b, α and k are constants.

(a) Find a Lagrangian corresponding to this Hamiltonian

$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{a} - bqe^{-\alpha t}$ - Hamilton equation gives relation between p and $\dot{q}$; $p = a(\dot{q} + bqe^{-\alpha t})$;

Now Legendre transformation gives:

$$L = p\dot{q} - H = a(\dot{q}^2 + bq\dot{q}e^{-\alpha t}) - \frac{a}{2}(\dot{q} + bqe^{-\alpha t})^2 + bqa(\dot{q} + bqe^{-\alpha t}) -$$

$$- \frac{kq^2}{2} - \frac{1}{2}abq^2\alpha e^{-\alpha t} - \frac{1}{2}ab^2q^2e^{-2\alpha t}; \Rightarrow$$

$$\Rightarrow L = \frac{a}{2}\dot{q}^2 + \cancel{abq\dot{q}}e^{-\alpha t} - \cancel{abq\dot{q}}e^{-\alpha t} - \frac{1}{2}\cancel{ab^2q^2}e^{-2\alpha t} + abq\dot{q}e^{-\alpha t} +$$

$$+ \cancel{ab^2q^2}e^{-2\alpha t} - \frac{kq^2}{2} - \frac{1}{2}abq^2\alpha e^{-\alpha t} - \frac{1}{2}\cancel{ab^2q^2}e^{-2\alpha t}$$

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We finally get:

$$\boxed{L = \frac{a}{2}\dot{q}^2 - \frac{kq^2}{2} + \frac{1}{2}ab\left(2q\dot{q}e^{-\alpha t} - \alpha q^2 e^{-\alpha t}\right);}$$

(b) Find equivalent Lagrangian that doesn't depend on time explicitely.

First of all note that our Lagrangian can be rewritten in the following way

$$L = \frac{a}{2}\dot{q}^2 - \frac{kq^2}{2} + \frac{1}{2}ab\frac{d}{dt}\left(q^2 e^{-\alpha t}\right)$$

Now let's consider 2 general lagrangians that differ by additional total derivative: L and $L' = L + \frac{d}{dt}(F(q,t))$

If L' satisfies Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}} - \frac{\partial L'}{\partial q} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} + \frac{d}{dt}\left(\frac{\partial F}{\partial q}\right) - \frac{d}{dt}\frac{\partial F}{\partial q} =$$

$$= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

so we can see that $\frac{d}{dt}F(q,t)$ doesn't contribute to equations of motion and thus 2 Lagrangians that differ by total derivative $\frac{d}{dt}F(q,t)$ describe the same mechanical system.

Note: in derivation we have used $\frac{d}{dt}F(q,t) = \frac{\partial F}{\partial t} + \dot{q}\frac{\partial F}{\partial q}$;

Using this fact we can see that our Lagrangian is equivalent to

$$\boxed{L = \frac{a}{2}\dot{q}^2 - \frac{kq^2}{2}}$$

which is Lagrangian of harmonic oscillator

(c) What is the Hamiltonian corresponding to second Lagrangian and what is the relationship between the two Hamiltonians?

Derived Lagrangian is of simple form $L = T - V$ so we can immideatly write down Hamiltonian $H = T + V$;

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where $T = \frac{1}{2} a\dot{q}^2 = \frac{p'^2}{2a}$; $V = \frac{kq^2}{2}$ and $p' = a\dot{q}$

thus $\boxed{H'(p',q) = \frac{p'^2}{2a} + \frac{kq^2}{2}}$

But let's fast derive same result. Canonical momentum is given by $p' = \frac{\partial L'}{\partial \dot{q}} = a\dot{q}$ and thus:

$$H' = p\dot{q} - L = \frac{p'^2}{a} - \frac{p'^2}{2a} + \frac{kq^2}{2} \Rightarrow \boxed{H'(p',q) = \frac{p'^2}{2a} + \frac{kq^2}{2}};$$

Now lets understand how this new hamiltonian is related to the initial one. Let's again consider systems corresponding to two Lagrangians L and $L' = L + \frac{d}{dt}F(q,t)$; First we derive hamiltonian H corresponding to L. Canonical momentum is given by $p = \frac{\partial L}{\partial \dot{q}}$ and $H = p\dot{q} - L$. For the second system

$$p' = \frac{\partial L'}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{q}} + \frac{\partial F}{\partial q} \Rightarrow \underline{p' = p + \frac{\partial F}{\partial q}}; \text{ and } H(p') = p'\dot{q} - L' =$$

$$= p\dot{q} + \dot{q}\frac{\partial F}{\partial q} - L - \dot{q}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial t}, \text{ thus } \boxed{H'(p') = H(p) - \frac{\partial F}{\partial t}}.$$

This is general formula. Let's check this for our case:

$$H'(p') = \frac{p'^2}{2a} + \frac{kq^2}{2} ; \; p' = p - abq\, e^{-\alpha t} \Rightarrow$$

$$\Rightarrow H'(p') = \frac{p^2}{2a} - bpq\, e^{-\alpha t} + \frac{1}{2} ab^2 q^2 e^{-2\alpha t} + \frac{kq^2}{2} \quad \text{then}$$

$$H(p) - H'(p') = \cancel{\frac{p^2}{2a}} - \cancel{bqp}e^{-\alpha t} + \frac{ab}{2} q^2 e^{-\alpha t}(\alpha + b\, e^{-\alpha t}) + \frac{kq^2}{2} - \cancel{\frac{p^2}{2a}} +$$

$$+ \cancel{bqp}\, e^{-\alpha t} - \frac{kq^2}{2} - \frac{1}{2} ab^2 q^2 e^{-2\alpha t} = \frac{1}{2} ab\alpha q^2 e^{-\alpha t} \text{ and, indeed,}$$

in our case F function is given by

$F(q,t) = -\frac{1}{2} ab\, q^2 e^{-\alpha t}$ and $\frac{\partial F}{\partial t} = \frac{\alpha}{2} ab\, q^2 e^{-\alpha t}$ thus we have checked that equation $H'(p') = H(p) - \frac{\partial F}{\partial t}$ is indeed true!!!

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Additional example

In problem 2B we have found Hamiltonian corresponding to Lagrangian $L = \frac{1}{2}\left[(\dot{q}_1 - \dot{q}_2)^2 + a\dot{q}_1^2 t^2\right] - a\cos q_2$ using general formula that we derived. But let's do it streght forward for this example, just for fun :)

Canonical momenta are given by:

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1 - \dot{q}_2 + a^2 t^2 \dot{q}_1 \quad ; \quad p_2 = \frac{\partial L}{\partial \dot{q}_2} = \dot{q}_2 - \dot{q}_1$$

inverting this equations we get:

$$\dot{q}_1 = \frac{p_1 + p_2}{a^2 t^2} \quad ; \quad \dot{q}_2 = \frac{p_1}{a^2 t^2} + p_2\left(1 + \frac{1}{a^2 t^2}\right)$$

Now we do Legendre transformation and get:

$$H = p_1\dot{q}_1 + p_2\dot{q}_2 - L = \frac{p_1^2}{a^2 t^2} + 2\frac{p_1 p_2}{a^2 t^2} + p_2^2\left(1 + \frac{1}{a^2 t^2}\right) - \frac{1}{2}p_2^2 - \frac{1}{2}\frac{(p_1 + p_2)^2}{a^2 t^2} + a\cos q_2,$$

and finally $\boxed{H = \frac{1}{2a^2 t^2}(p_1 + p_2)^2 + \frac{1}{2}p_2^2 + a\cos q_2}$, which coincides with the expression observed before.

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Seminar 2 (Hamiltonians, cyclic variablies)

Theory Just to remind: we have 2 ways of mechanical systems description - Lagrange when we work in terms of $(\dot{q}, q, t)$ variables and Hamilton when we describe system in terms of (p, q, t).
In Lagrange description for system with n degrees of freedom we get n Euler-Lagrange equations $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$
for Hamiltonian description we get 2n first order equations $\dot{q}_i = \frac{\partial H}{\partial p_i}$; $\dot{p}_i = -\frac{\partial H}{\partial q_i}$; Lagrangian and Hamiltonian are related with Legendre transformation
$H = p\dot{q} - L$ and canonical momentum is defined as $p_i = \frac{\partial L}{\partial \dot{q}_i}$

Problem 1 A dynamical system has the Hamiltonian

$$H = q_1 p_2 - q_2 p_1 + a(p_1^2 + p_2^2);$$

Find a Lagrangian corresponding to Hamiltonian

There are 2 ways of deriving corresponding

* First one is just following algorithm we have written in previous class:

I First using Hamilton equations we write down

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = -q_2 + 2ap_1 \; ; \; \dot{q}_2 = \frac{\partial H}{\partial p_2} = q_1 + 2ap_2;$$

inverting this we get:

$$\boxed{p_1 = \frac{\dot{q}_1 + q_2}{2a} \; ; \; p_2 = \frac{\dot{q}_2 - q_1}{2a} \; ;}$$

II + III Now we make Legendre transformation:

$$L = p\dot{q} - H = p_1\dot{q}_1 + p_2\dot{q}_2 + q_2 p_1 - q_1 p_2 - a(p_1^2 + p_2^2) =$$

$$= \frac{1}{2a}(\dot{q}_1 + q_2)^2 + \frac{1}{2a}(\dot{q}_2 - q_1)^2 - \frac{1}{4a}(\dot{q}_1 + q_2)^2 - \frac{1}{4a}(\dot{q}_2 - q_1)^2 =$$

$$= \frac{1}{4a}(\dot{q}_1 + q_2)^2 + \frac{1}{4a}(\dot{q}_2 - q_1)^2;$$

(2) So we finally get:

$$\boxed{L = \frac{1}{4a}(\dot{q}_1 + q_2)^2 + \frac{1}{4a}(\dot{q}_2 - q_1)^2;}$$

* Second way to solve this formula is just to apply general formula for Hamiltonian - Lagrangian transformation we have derived on the previous lesson.

For general form of Hamiltonian

$H = H_0(q,t) + \bar{p}^T \bar{a} + \frac{1}{2}\bar{p}^T T \bar{p}$; corresponding Lagrangian is

$L = \frac{1}{2}(\dot{\bar{q}} - \bar{a})^T T^{-1} (\dot{\bar{q}} - \bar{a}) - H_0(q,t)$; In our case

$$T = 2a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad T^{-1} = \frac{1}{2a}\mathbb{1}; \quad \bar{a} = \begin{bmatrix} -q_2 \\ q_1 \end{bmatrix}; \quad H_0(q,t) = 0;$$

Thus we immideately get $L = \frac{1}{4a}(\dot{\bar{q}} - \bar{a})^T (\dot{\bar{q}} - \bar{a}) \Rightarrow$

$\Rightarrow \boxed{L = \frac{1}{4a}(\dot{q}_1 + q_2)^2 + \frac{1}{4a}(\dot{q}_2 - q_1)^2;}$ which coincides with expression observed by first methode.

Problem 2 A dynamical system has the Hamiltonian

$$H = \frac{1}{2}\left(p_1^2(p_2^2 + q_2^2) + q_1^2\right)$$

Derive canonical equations and find their general solution.

(1) $\dot{q}_1 = \frac{\partial H}{\partial p_1} = p_1(p_2^2 + q_2^2)$; (2) $\dot{q}_2 = \frac{\partial H}{\partial p_2} = p_2 p_1^2$;

(3) $\dot{p}_1 = -\frac{\partial H}{\partial q_1} = -q_1$; (4) $\dot{p}_2 = -\frac{\partial H}{\partial q_2} = -q_2 p_1^2$;

$\Rightarrow$ These are canonical equations.

First of all let's note that

$$\frac{d}{dt}(p_2^2 + q_2^2) = 2(p_2 \dot{p}_2 + q_2 \dot{q}_2) = 2(-p_2 q_2 p_1^2 + p_2 q_2 p_1^2) = 0;$$

So $p_2^2 + q_2^2 = const = \Omega^2$ is constant of motion

as $p_2^2 + q_2^2$ is constant we can find q_1 second derivative

$\ddot{q}_1 = \dot{p}_1(p_2^2 + q_2^2) = -q_1\Omega^2$ which has general solution

$\boxed{q_1 = A\cos(\Omega t + \delta);}$ as from eq (1) $p_1 = \frac{\dot{q}_1}{\Omega^2}$ we

get $\boxed{p_1 = -\frac{A}{\Omega}\sin(\Omega t + \delta);}$ now we go to the second pair

③ of equations (2) and (4) to find q_2 and p_2 as $p_2^2+q_2^2=\Omega^2$ - p_2 and q_2 always lie on a circle of radius Ω and it is reasonable to parametrise them in the following way

$p_2=\Omega\cos\varphi(t)$ and now using this parametrisation
$q_2=\Omega\sin\varphi(t)$ out of equations (2) and (4) we get

$\dot q_2=\Omega\dot\varphi(t)\cos\varphi(t)$; $p_2p_1^2=\Omega\cos\varphi(t)\frac{A^2}{\Omega^2}\sin^2(\Omega t+\delta)$ thus we get $\dot\varphi(t)=\frac{A^2}{\Omega^2}\sin^2(\Omega t+\delta)=\frac{A^2}{2\Omega^2}(1-\cos(2\Omega t+2\delta))$

integrating this we get

$$\varphi(t)=\frac{A^2}{2\Omega^2}\left(t-\frac{1}{2\Omega}\sin(2\Omega t+2\delta)\right)+\varphi_0;$$

So, summing up, general solution for this mechanical system is

$$\boxed{\begin{gathered}q_1=A\cos(\Omega t+\delta);\ q_2=\Omega\sin\varphi(t);\ p_1=\frac{A}{\Omega}\sin(\Omega t+\delta);\ p_2=\Omega\cos\varphi(t)\\ \varphi(t)=\frac{A^2}{2\Omega^2}\left(t-\frac{1}{2\Omega}\sin(2\Omega t+2\delta)\right)+\varphi_0;\end{gathered}}$$

and $A,\varphi_0,\Omega,\delta$ are integration constants here.

Problem 3 A dynamical system has the Lagrangian

$$L=\frac{mR^2}{2}\left(\dot\theta^2+\frac{\dot\varphi^2}{\sin^2\theta}\right)-mgR\cos\theta$$

Find a Hamiltonian corresponding to this Lagrangian. Find a cyclic variable and reduce the problem to a family of problems with one degree of freedom. Draw the phase portraits for the 1-dimensional systems.

First of all let's find canonical momentum:

$p_\theta=\frac{\partial L}{\partial\dot\theta}=mR^2\dot\theta$; $p_\varphi=\frac{\partial L}{\partial\dot\varphi}=\frac{mR^2\dot\varphi}{\sin^2\theta}$, inverting this equations we get: $\dot\theta=\frac{p_\theta}{mR^2}$; $\dot\varphi=\frac{\sin^2\theta\, p_\varphi}{mR^2}$;

Now we make Legendre transformation

$$H=p_\theta\cdot\dot\theta+p_\varphi\dot\varphi-L=\frac{p_\theta^2}{mR^2}+\frac{\sin^2\theta}{mR^2}p_\varphi^2-\frac{1}{2}\frac{p_\theta^2}{mR^2}-\frac{\sin^2\theta}{2mR^2}p_\varphi^2+mgR\cos\theta;$$

(4) Then we finally get

$$\boxed{H=\frac{p_\theta^2}{2mR^2}+\frac{\sin^2\theta}{2mR^2}p_\varphi^2+mgR\cos\theta};$$

As φ doesn't enter Hamiltonian explicitely it is obviously cyclic variable. Then due to canonical equation of motion $\dot{p}_\varphi=-\frac{\partial H}{\partial \varphi}=0$ thus $p_\varphi=const=\alpha$;

$$H=\frac{p_\theta^2}{2mR^2}+\underbrace{\frac{\alpha^2}{2mR^2}(\sin^2\theta+2C\cdot\cos\theta)}_{V_{eff}}$$ here $C=\frac{m^2gR^3}{\alpha^2}\geqslant 0$ is positive constant.

Second term now plays role of effective potential for single variable θ;

Phase portrait pics trajectories of system in phase space. Equations of motion for the system are given by

$$\begin{cases}\dot{p}_\theta=-\frac{\partial H}{\partial\theta}=-\frac{\alpha^2}{mR^2}(\cos\theta-C)\cdot\sin\theta\\ \dot{\theta}=\frac{\partial H}{\partial p_\theta}=\frac{p_\theta}{mR^2}\end{cases}$$

To draw sketch of systems phase portrait we usually find system's fixed points (i.e. points for which $\dot{p}_\theta=0$ and $\dot{\theta}=0$ simultaneously) and linearise equations of motion around this points.

For our system fixed points are given by $p_\theta^0=0$ - for all fixed points and $\theta^0=\pi k$, $k\in\mathbb{Z}$ or $\theta^0=\arccos C$ which exists only in case $C\leqslant 1$

Let's consider 2 cases separately.

(I) $C\leqslant 1$

Linearisation is done using Taylor expansion. Assume we have general system of the form $\dot{\bar{x}}=\bar{f}(\bar{x})$ linearising we get

(5) $\bar f(\bar x) = \bar f(\bar x^0) + \sum_i \frac{\partial \bar f}{\partial x^i}\Big|_{\bar x = \bar x^0} \cdot (x - x^0)^i + \ldots;$ $\bar f(\bar x^0) = \bar 0$ by the definition of fixed point. So we get linear system of the form $\dot{\bar x} = A\bar x$ where A is 2×2 matrix. If we substitute ansatz $\bar x = \bar v e^{\lambda t}$ then we get $A\bar v = \lambda \bar v$ so we get eigenproblem. General solution of equation is thus $\bar x = \bar v_1 e^{\lambda_1 t} + \bar v_2 e^{\lambda_2 t}$ where $\bar v_i$ are eigenvectors and λ_i are eigenvalues of A. In our case

$$\bar f = \begin{bmatrix} -\frac{\ell^2}{mR^2}(\cos\theta - C)\sin\theta \\ \frac{1}{mR^2} p_\theta \end{bmatrix}; \quad \frac{\partial \bar f}{\partial \theta} = \begin{bmatrix} -\frac{\ell^2}{mR^2}(\cos 2\theta - C\cos\theta) \\ 0 \end{bmatrix}; \quad \frac{\partial \bar f}{\partial p_\theta} = \begin{bmatrix} 0 \\ \frac{1}{mR^2} \end{bmatrix};$$

Then

$$\begin{cases} \dot p_\theta = -\frac{\ell^2}{mR^2}(1 \mp C)\,\theta \\ \dot\theta = \frac{1}{mR^2} p_\theta \end{cases}$$

– here upper sign corresponds to $k = 2n$ (even) and lower sign – to $k = 2n+1$ (odd)

as $C < 1$ any way $1 \mp C > 0$, and we get

$\ddot p_\theta = -\omega^2 p_\theta$ where $\omega_\mp^2 = \frac{\ell^2}{(mR^2)^2}(1 \mp C)$

$\ddot\theta = -\omega^2 \theta$

Solution is just oscillation $p_\theta \propto \cos(\omega_\mp t + \delta)$

$\theta \propto \sin(\omega_\mp t + \delta)$

and thus phase portrait looks like ellips around $\theta = \pi k$ points.

Now let's consider second fixed point $\cos\theta^0 = C$. At this point $\cos 2\theta^0 = 2\cos^2\theta^0 - 1 = 2C^2 - 1$; and we get linearised equations of motion in the following form

$$\begin{cases} \dot p_\theta = (1 - C^2)\frac{\ell^2}{mR^2}\delta_\theta \\ \dot\delta_\theta = \frac{1}{mR^2} p_\theta \end{cases}$$

, where $\delta_\theta = \theta - \theta^0$ is deviation from fixed point.

In this case

$$A = \frac{1}{mR^2}\begin{bmatrix} 0 & \ell^2(1 - C^2) \\ 1 & 0 \end{bmatrix}$$

⑥ eigenvalues are $\lambda = \pm\sqrt{1-c^2}\,\frac{d}{mR^2}$ eigenvectors are given by:

* for $\lambda_+ = \sqrt{1-c^2}\,\frac{d}{mR^2}$

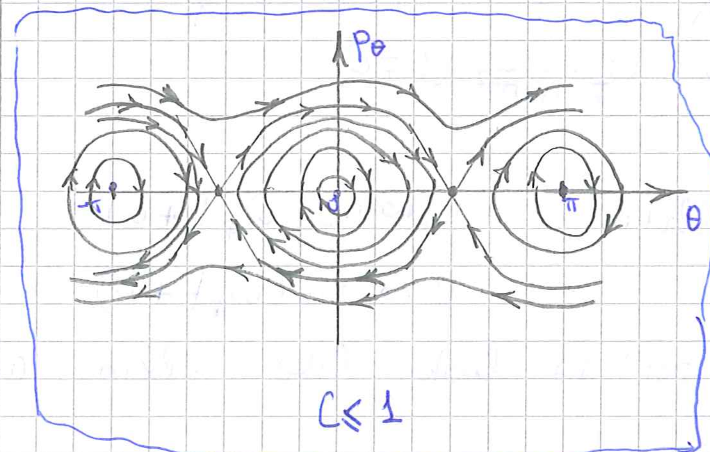
$$A - \lambda I = \frac{d}{mR^2}\begin{bmatrix} -\sqrt{1-c^2} & d(1-c^2) \\ \frac{1}{d} & -\sqrt{1-c^2} \end{bmatrix}$$ corresponding eigenvector is $\bar{v}_+ = \begin{bmatrix} d\sqrt{1-c^2} \\ 1 \end{bmatrix}$;

Now for negative eigenvalue $\lambda_- = -\frac{d}{mR^2}\sqrt{1-c^2}$;

$$A - \lambda I = \begin{bmatrix} \sqrt{1-c^2} & d(1-c^2) \\ \frac{1}{d} & +\sqrt{1-c^2} \end{bmatrix}\frac{d}{mR^2}$$ and thus $\bar{v}_- = \begin{bmatrix} d\sqrt{1-c^2} \\ -1 \end{bmatrix}$

General solution is then $\begin{bmatrix} p_\theta \\ \delta_\theta \end{bmatrix} = C_+ \bar{v}_+ e^{\lambda_+ t} + C_- \bar{v}_- e^{\lambda_- t}$

First term has positive exponent and thus is growing ⇒ ⇒ solution is growing along $\bar{v}_+$ direction, and in the same way we conclude that solution decays along v_- direction. Summing up qualitative phase diagramm in case $C \leq 1$ looks like.



$C \leq 1$

Ⓘ $C > 1$

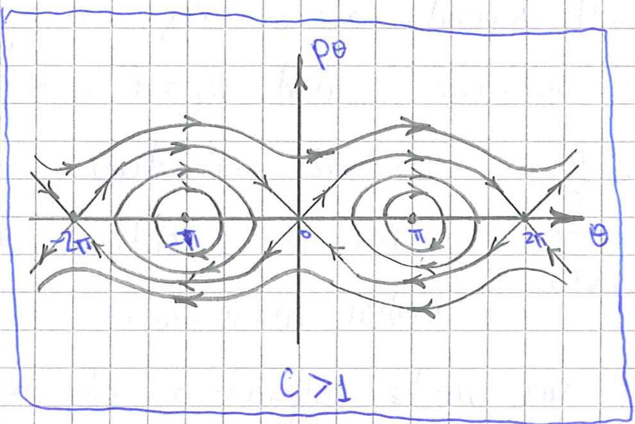
In this case only $(p_\theta; \theta) = (0, \pi k)$ points exist. Linearised equations of motion are the same as before

$$\begin{cases} \dot{p}_\theta = -\frac{d^2}{mR^2}(1 \mp C)\theta \\ \dot{\theta} = \frac{1}{mR^2} p_\theta \end{cases}$$

here it differs wether we take even or odd k in $\theta = \pi k$.

if k is odd we still have $(1+C) > 0$ and around this points we get ellipse. But for even k $1-C < 0$ and we get saddle point as this case is totally analogical to the previous case of $\theta^0 = \arccos C$;

Then phase portrat became:



Problem 4 A point particle of mass m moves in a 2-dimensional space in the central potential.

$$V(r) = -\frac{q}{r};$$

Find a Hamiltonian for the system using polar coordinates. Find a cyclic variable and reduce the problem to a family of one-dimensional problems. Draw the phase portraits.

This is simple mechanical system with Lagrangian in form $L = T - V$ where T and V are kinetic and potential energy correspondingly. In cartesian coordinates $T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$. To go to polar coordinates we use

$$x = r\cos\varphi \qquad \Rightarrow \qquad \dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi$$
$$y = r\sin\varphi \qquad\qquad \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi ;$$

$$T = \frac{m}{2}\{\dot{r}^2\cos^2\varphi - r\dot{r}\dot{\varphi}\sin 2\varphi + r^2\sin^2\varphi \cdot \dot{\varphi}^2 + \dot{r}^2\sin^2\varphi + r\dot{r}\dot{\varphi}\sin 2\varphi +$$
$$+ \dot{\varphi}^2 r^2\cos^2\varphi\} \Rightarrow T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2)$$ Thus Lagrangian in polar coordinates takes form:

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{q}{r};$$ Canonical momenta are given by

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}; \quad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}$$ then

$$H = p_r\cdot\dot{r} + p_\varphi\cdot\dot{\varphi} - L = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} - \frac{q}{r};$$ thus, finally

$$\boxed{H = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} - \frac{q}{r};}$$

8

As Hamiltonian doesn't depend explicitely on φ, obviously φ is cyclic variable and $p_\varphi = l = const$; thus

$$H = \frac{p_r^2}{2m} + \underbrace{\frac{l^2}{2mr^2} - \frac{q}{r}}_{V_{eff}(r)}$$

So we introduce effective potential for radial movement

$V_{eff}(r) = \frac{l^2}{2mr^2} - \frac{q}{r}$; If we write equations of motion for radial movement we get

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}; \quad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{l^2}{mr^3} - \frac{q}{r^2};$$

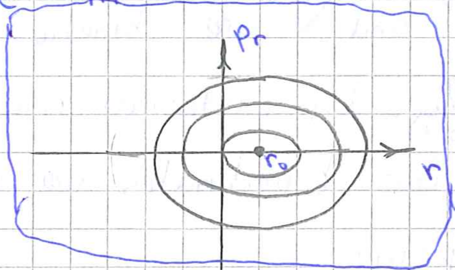
fixed point now is $p_r^0 = 0$ and $r^0 = \frac{l^2}{mq}$

linearisation gives $\frac{\partial}{\partial r}\left(\frac{l^2}{mr^3} - \frac{q}{r^2}\right)\Big|_{r=r_0} = -\frac{3l^2}{mr^4} + \frac{2q}{r^3}\Big|_{r=r^0} = -\frac{3q}{r_0^3} + \frac{2q}{r_0^3}$

thus linearised equation of motion is:

$$\begin{cases} \dot{p}_r = -\frac{q}{r_0^3}(r - r_0) \\ \dot{r} = \frac{p_r}{m} \end{cases}$$

as $q > 0$ we get oscillation and phase diagramm is ellipse.



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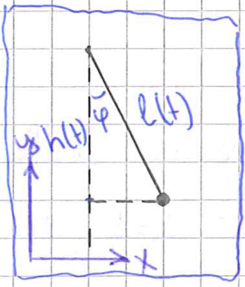
Seminar 3 (canonical transformation I)

Problem 1 The length l of a mathematical pendulum varies with time $l = l(t)$. Derive the Lagrangian and Hamiltonian of the system



Lagrangian is given by
$L = T - V$, where T is kinetic energy and V is potential one.

$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$ Let's write it down in terms of l and φ:

$x = l\sin\varphi$; $\Rightarrow \dot{x}^2 + \dot{y}^2 = \dot{l}^2 + l^2\dot{\varphi}^2$, thus

$y = -l\cos\varphi$; $T = \frac{m}{2}(\dot{l}^2 + l^2\dot{\varphi}^2)$; and finally

$V = -mg\,h(t) = -mg\,l(t)\cos\varphi$; Thus Lagrangian is given by $\boxed{L = \frac{m}{2}(\dot{l}^2 + l^2\dot{\varphi}^2) + mgl(t)\cdot\cos\varphi;}$

Now we can derive Hamiltonian:

(I) Canonical momenta are given by

$p_l = \frac{\partial L}{\partial \dot{l}} = m\dot{l}$; $p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = ml^2\dot{\varphi}$ inverting this expressions

we can obtain $\dot{l} = \frac{p_l}{m}$; $\dot{\varphi} = \frac{p_\varphi}{ml^2}$;

(II)(III) Now we make Legendre transformation

$$H = p_\varphi\cdot\dot{\varphi} + p_l\cdot\dot{l} - L = \frac{p_l^2}{m} + \frac{p_\varphi^2}{ml^2} - \frac{1}{2}\frac{p_l^2}{m} - \frac{1}{2}\frac{p_\varphi^2}{ml^2} - mgl(t)\cos\varphi;$$

and finally $\boxed{H = \frac{p_\varphi^2}{2ml^2} + \frac{p_l^2}{2m} - mgl\cdot\cos\varphi;}$

Theory Canonical transformations

Assume we have Hamiltonian $H(p,q,t)$ depending on p and q coordinates. Let's consider transformation of general form:

$Q_i = Q_i(q,p,t)$ We want this coordinates to be canonical

$P_i = P_i(q,p,t)$ and thus there should exist some function

② $K(P,Q,t)$ (the new hamiltonian) such that:

$\dot{Q}_i = \frac{\partial K}{\partial P_i}$; $\dot{P}_i = -\frac{\partial K}{\partial Q_i}$; or in other formulation

$$\delta \int_{t_1}^{t_2} dt\,(p_i \dot{q}_i - H(p,q,t)) = 0$$ and

$$\delta \int_{t_1}^{t_2} dt\,(P_i \dot{Q}_i - K(P,Q,t)) = 0$$ should be satisfied

simultaneously. This can happen if:

$\lambda(p_i \dot{q}_i - H(p,q)) = P_i \dot{Q}_i - K + \frac{dF}{dt}$; λ here is scale transformation, we will take it identical: $\lambda = 1$, so that

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF}{dt};$$

F is called generating function and it defines our canonical transformation. To understand how let's consider different types of generating functions

Ⅰ First type

$F = F_1(q,Q,t)$, then $p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF_1}{dt} = P_i \dot{Q}_i - K + \frac{\partial F_1}{\partial t} +$

$+ \dot{q}_i \frac{\partial F_1}{\partial q_i} + \dot{Q}_i \frac{\partial F_1}{\partial Q_i}$; so that $\dot{q}_i \left(p_i - \frac{\partial F_1}{\partial q_i}\right) + \dot{Q}_i \left(-P_i - \frac{\partial F_1}{\partial Q_i}\right) +$

$+ \left(K - H - \frac{\partial F_1}{\partial t}\right) = 0$ To satisfy this we assume

$$\boxed{p_i = \frac{\partial F_1}{\partial q_i}; \quad P_i = -\frac{\partial F_1}{\partial Q_i}; \quad K = H + \frac{\partial F_1}{\partial t};}$$

This means that using generating function $F_1(q,Q,t)$ and equation $p_i = \frac{\partial F_1}{\partial q_i}$ we express Q_i as function of (p_i, q_i, t), then using second equation we define $P_i(p,q,t)$ and, finally, we find new hamiltonian K as function of $(Q; P; t)$

Ⅱ Second type

$F = F_2(q,P,t) - Q_i P_i$

Substituting this into $p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \frac{dF}{dt} = \cancel{P_i \dot{Q}_i} - K - \cancel{P_i \dot{Q}_i} -$

$- Q_i \dot{P}_i + \frac{\partial F_2}{\partial t} + \dot{q}_i \frac{\partial F_2}{\partial q_i} + \dot{P}_i \frac{\partial F_2}{\partial P_i}$; thus:

(3)

$$\boxed{Q_i = \frac{\partial F_2}{\partial P_i} \; ; \; p_i = \frac{\partial F_2}{\partial q_i} \; ; \; K = H + \frac{\partial F_2}{\partial t} \; ;}$$

III Third type

$F = F_3(p, Q, t) + q_i p_i$, In this case we get:

$$\cancel{p_i \dot{q}_i} - H = P_i \dot{Q}_i - K + \frac{dF}{dt} = P_i \dot{Q}_i - K + \cancel{\dot{q}_i p_i} + q_i \dot{p}_i + \frac{\partial F_3}{\partial t} + \dot{p}_i \frac{\partial F_3}{\partial p_i} + \dot{Q}_i \frac{\partial F_3}{\partial Q_i}$$

$$\dot{p}_i \left(q_i + \frac{\partial F_3}{\partial p_i}\right) + \dot{Q}_i \left(P_i + \frac{\partial F_3}{\partial Q_i}\right) + \left(-K + H + \frac{\partial F_3}{\partial t}\right)$$ thus

$$\boxed{P_i = -\frac{\partial F_3}{\partial Q_i} \; ; \; q_i = -\frac{\partial F_3}{\partial p_i} \; ; \; K = H + \frac{\partial F_3}{\partial t} \; ;}$$

IV Fourth type

$F = F_4(p, P, t) + q_i p_i - Q_i P_i$; In this case:

$$\cancel{p_i \dot{q}_i} - H = \cancel{P_i \dot{Q}_i} - K + \cancel{\dot{q}_i p_i} + q_i \dot{p}_i - \cancel{\dot{Q}_i P_i} - Q_i \dot{P}_i + \frac{\partial F_4}{\partial t} + \dot{p}_i \frac{\partial F_4}{\partial p_i} + \dot{P}_i \frac{\partial F_4}{\partial P_i}$$

Then we get: $\boxed{Q_i = \frac{\partial F_4}{\partial P_i} \; ; \; q_i = -\frac{\partial F_4}{\partial p_i} \; ; \; K = H + \frac{\partial F_4}{\partial t} \; ;}$

All generating functions are related with each other through Legendre transformation.

Problem 2 Find canonical transformations defined by the following generating function

(a) $F = \ln(qt)\, e^P$

as F is function of q and P this is obviously generating function of second type, thus:

$Q = \frac{\partial F_2}{\partial P} = \ln(qt) e^P \; ; \; p = \frac{\partial F_2}{\partial q} = \frac{1}{q} e^P$ first, let's invert second equations $\boxed{P = \log(pq) \; ; \; Q = pq \log(qt) \; ;}$

(b) $F = q \ln P$

as in previous case F is function of q and P and this is again generating function of second type.

$Q = \frac{\partial F_2}{\partial P} = \frac{q}{P} \; ; \; p = \frac{\partial F_2}{\partial q} = \ln P$ Inverting this functions we get $P = e^p$, and $Q = \frac{q}{P} = q e^{-p}$; $\boxed{P = e^p ; \; Q = q e^{-p} ;}$

④ **Problem 3**

A dynamical system has the Hamiltonian

$$H = \frac{1}{2}q^2 + at^2q^2 - 2tpq + \frac{1}{a}p^2;$$

where a is constant. Apply a canonical transformation defined by the generating function

$$F = \frac{1}{2}atq^2 - qP;$$

Find the new Hamiltonian in terms of P and Q;

We see that F is function of P and q and thus this is generating function of second type

First let's determin transformation formulas:

$p = \frac{\partial F}{\partial q} = atq - P$; $Q = \frac{\partial F}{\partial P} = -q$; so that $\boxed{Q = -q;\ P = -p + atq;}$

inverting this equation we observe $\boxed{q = -Q;\ p = -P - atQ;}$

Then new Hamiltonian is given by

$$K = H + \frac{\partial F_2}{\partial t} = \frac{1}{2}q^2 + at^2q^2 - 2tpq + \frac{1}{a}p^2 + \frac{1}{2}aq^2 = q^2\left(\frac{1}{2} + at^2 + \frac{1}{2}a\right) +$$

$$+ \frac{1}{a}p^2 - 2tpq = Q^2\left(\frac{1}{2} + \cancel{at^2} + \frac{a}{2}\right) + \frac{1}{a}P^2 + \cancel{2tPQ} + \cancel{at^2Q^2} - \cancel{2tPQ} - \cancel{2at^2Q^2} \Rightarrow$$

$\Rightarrow \boxed{K = \frac{a+1}{2}Q^2 + \frac{1}{a}P^2;}$ so we have obtained how complicated Hamiltonian can be reduced to simple Harmonic Oscillator Hamiltonian.

Cross check Canonical equations give $\dot{Q} = \frac{\partial K}{\partial P} = \frac{2P}{a} = 2tq - \frac{2p}{a}$; at the same time $\dot{Q} = \frac{d}{dt}(-q) = -\dot{q} = -\frac{\partial H}{\partial p} = 2ta - \frac{2p}{a}$ – here everything is self-consitent.

$$\dot{P} = -\frac{\partial K}{\partial Q} = -(1+a)Q = (1+a)q \quad \text{at the same time}$$

$$\dot{P} = \frac{d}{dt}(atq - p) = aq + at\dot{q} - \dot{p} = aq + at\frac{\partial H}{\partial p} + \frac{\partial H}{\partial q} = aq + at\left(-2tq + \frac{2p}{a}\right) +$$

$+ (q + 2at^2q - 2tp) = (1+a)q$, so here everything is self-consistent too

Seminar 4 (Canonical transformations and Poisson brackets)

Problem 1 A 1 dimensional free point-particle is described by the Hamiltonian $H = \frac{p^2}{2m}$. Find a time-independent canonical transformation such that the new momentum P coincides with H. Give a general solution of the equations of motion using this canonical transformation.

So we want to find such canonical transformation, that $P = \frac{p^2}{2m}$. We can use different types of canonical transformations. Let's remind how to operate with different types.

(I) $F_1(q, Q, t)$; $p_i = \frac{\partial F_1}{\partial q_i}$; $P_i = -\frac{\partial F_1}{\partial Q_i}$;

(II) $F_2(q, P, t)$; $p_i = \frac{\partial F_2}{\partial q_i}$; $Q_i = \frac{\partial F_2}{\partial P_i}$;

(III) $F_3(p, Q, t)$; $q_i = -\frac{\partial F_3}{\partial p_i}$; $P_i = -\frac{\partial F_3}{\partial Q_i}$;

(IV) $F_4(p, P, t)$; $q_i = -\frac{\partial F_4}{\partial p_i}$; $Q_i = \frac{\partial F_4}{\partial P_i}$;

Derivatives of generating function should be equal to p or P so that we can use equation $P = \frac{p^2}{2m}$ to define generating function. So we are not OK with F_4 function. And we are not OK with F_1 becaus we will get 2 equations $p = p(q, Q, t)$ and $P = P(q, Q, t)$ and it is not so convinient to work with. So we will use type-II and type-III

Type-II generating function $F_2(q, P, t)$

$$p = \frac{\partial F_2}{\partial q} = \sqrt{2mP} \Rightarrow F_2(q, P) = \sqrt{2mP}\, q + f(P);$$

where $f(P)$ is some arbitrary function of P, then we can

(2) find Q: $\boxed{Q = \frac{\partial F}{\partial P} = \frac{q\sqrt{m}}{\sqrt{2P}} + \frac{\partial f(P)}{\partial P} \; ; \; P = \frac{p^2}{2m}}$ so we have found formulas defining canonical transformations.

Note that time-independent canonical transformations don't change Hamiltonian, which is obvious from transformation formula $K = H + \frac{\partial F_2}{\partial t} = H$, so in our case $\boxed{K = P}$

Now using canonical equations we can write.

$$\dot{P} = -\frac{\partial K}{\partial Q} = 0 \quad ; \quad P = \alpha = \text{const} \qquad \dot{Q} = \frac{\partial K}{\partial P} = 1 \Rightarrow Q = Q + t$$

Now if we invert transformations:

$$p = \sqrt{2m\alpha} = \text{const} \; ; \; q = \sqrt{\frac{2P}{m}}\,(Q - f') \; ; \Rightarrow q = \sqrt{\frac{2\alpha}{m}}\,(t + Q_0 - f'(\alpha$$

here we noted $f'(\alpha) = \left.\frac{\partial f(P)}{\partial P}\right|_{P=\alpha}$;

$$\boxed{\begin{aligned} &p = \sqrt{2m\alpha}\,; \\ &q = \sqrt{\frac{2\alpha}{m}}\left(Q_0 + t - f'(\alpha)\right); \end{aligned}}$$

Type-III generating function.

If we take generating function of form: $F_3(p, Q, t)$;

$q = -\frac{\partial F_3}{\partial p}$; $P = -\frac{\partial F_3}{\partial Q}$; it's derivatives will look like this

$-\frac{\partial F_3}{\partial Q} = \frac{p^2}{2m} \Rightarrow F_3 = -\frac{p^2 Q}{2m} + f_3(p)$ where $f_3(p)$ is arbitrary function of the momentum. Now we can define q as $q = -\frac{\partial F_3}{\partial p} = \frac{pQ}{m} + f_3'(p)$ where $f_3'(p) = \frac{\partial f_3(p)}{\partial p}$; thus $q = \sqrt{\frac{2P}{m}}\,Q + f_3'(P)\cdot\sqrt{\frac{2P}{m}}$ (here $f_3'(P) = \frac{\partial f_3(P)}{\partial P}$, and Hamiltonian is given by $K = P$

So we see that hamiltonian and canonical transformation formulas remain the same: $\boxed{K = P \; ; \; p = \sqrt{2mP} \; ; \; q = \sqrt{\frac{2P}{m}}\left(Q + f_3'(P)\right);}$

and thus solution remains the same too.

③

Theory   Poisson Brackets

By definition: $\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p^i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p^i}$

Poisson brackets satisfy following properties:

(I) antisymmetry $\{f, g\} = -\{g, f\}$;

(II) $\{f, gh\} = \{f, g\}h + g\{f, h\}$

Proof $\{f, gh\} = \frac{\partial f}{\partial q}\frac{\partial (gh)}{\partial p} - \frac{\partial (gh)}{\partial q}\frac{\partial f}{\partial p} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}h - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}h +$

$+ \frac{\partial f}{\partial q} g \frac{\partial h}{\partial p} - \frac{\partial f}{\partial p} g \frac{\partial h}{\partial q} = \{f, g\}h + g\{f, h\}$, q.e.d

(III) Jacobi identity

$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$ – this identity need much algebra to be proved, so we omit it.

* Usefull and important place of application:

Let's consider some function $f(p, q, t)$ and find it's total time derivative:

$$\frac{d}{dt} f(p, q, t) = \dot{p}\frac{\partial f}{\partial p} + \dot{q}\frac{\partial f}{\partial q} + \frac{\partial f}{\partial t} = -\frac{\partial H}{\partial q}\frac{\partial f}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial f}{\partial q} + \frac{\partial f}{\partial t} \Rightarrow$$

$$\Rightarrow \boxed{\frac{d}{dt} f(p, q, t) = \{f, H\} + \frac{\partial f}{\partial t};}$$

This equation you may recognise from quantum mechanics, there this is just Heisenberg equation describing time evolution of operators. To obtain it we should replace Poisson brackets with commutator.

Fundamental Poisson brackets are defined as

$\{q_i, p_j\} = \delta_{ij}$; $\{p_i; p_j\} = 0$; $\{q_i; q_j\} = 0$;

Another important relations include:

$\{q, g\} = \frac{\partial g}{\partial p}$; $\{f, p\} = \frac{\partial f}{\partial q}$;

Important property of Poisson brackets is their invariance under canonical transformations.

(4) Assume we make canonical transformation $(p,q) \to (P,Q)$;
Using infinitesemal canonical transformation we can show that $\{Q,P\}_{q,p} = 1$; Now using this we can show that Poisson brackets for 2 arbitrary functions is invariant of canonical transformations too

$$\{f,g\}_{q,p} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} = \left(\frac{\partial f}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial f}{\partial P}\frac{\partial P}{\partial q}\right)\left(\frac{\partial g}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial g}{\partial P}\frac{\partial P}{\partial p}\right) -$$

$$-\left(\frac{\partial f}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial f}{\partial P}\frac{\partial P}{\partial p}\right)\left(\frac{\partial g}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial g}{\partial P}\frac{\partial P}{\partial q}\right) = \frac{\partial f}{\partial Q}\frac{\partial g}{\partial P}\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} +$$

$$+ \frac{\partial f}{\partial P}\frac{\partial g}{\partial Q}\frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial f}{\partial Q}\frac{\partial g}{\partial P}\frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} - \frac{\partial f}{\partial P}\frac{\partial g}{\partial Q}\frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} =$$

$$= \frac{\partial f}{\partial Q}\frac{\partial g}{\partial P}\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right) - \frac{\partial f}{\partial P}\frac{\partial g}{\partial Q}\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right) =$$

$$= \left(\frac{\partial f}{\partial Q}\frac{\partial g}{\partial P} - \frac{\partial f}{\partial P}\frac{\partial g}{\partial Q}\right)\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right) = \{f,g\}_{Q,P} \cdot \{Q;P\}_{q,p}$$

as we have seen before $\{Q,P\}_{q,p} = 1$ thus we have just proved that:

$$\boxed{\{f,g\}_{q,p} = \{f,g\}_{Q,P}}$$

Problem 2 A change of variables in the phase space plane is defined by the equations:

$$P = p\cos t + q\sin t \;;\quad Q = q\cos t - p\sin t;$$

Show that this is canonical transformation. Find the generating function $F(q,P,t)$;

As we have said before transformation is canonical if and only if it preserves fundamental poisson brackets:

$$\{Q,P\}_{q,p} = \{q\cos t - p\sin t; p\cos t + q\sin t\} = \cos^2 t\,\{q,p\} - \sin^2 t\,\{p,q\} -$$

$$-\sin t\cdot\cos t\,\underbrace{\{p,p\}}_{0} + \cos t\cdot\sin t\,\underbrace{\{q,q\}}_{0} = (\cos^2 t + \sin^2 t)\,\{q,p\} = \{q,p\} = 1;$$

So we get $\boxed{\{Q;P\} = 1 \quad q.e.d.}$ i.e. transformation is indeed canonical.

If we have generating function $F(P,q,t)$; then it's partial derivatives are defined as:

$$Q = \frac{\partial F}{\partial P} = q\cos t - p\sin t; \quad p = \frac{\partial F}{\partial q} = \frac{P}{\cos t} - q\tan t$$

Let express r.h.s. of this equations through P and q variables using equations of canonical transformations:

$p = \frac{P}{\cos t} - q\cdot\tan t$; $Q = \frac{q}{\cos t} - P\tan t$; Using this relations we can write:

$$\frac{\partial F}{\partial q} = \frac{P}{\cos t} - q\cdot\tan t \Rightarrow F(P,q,t) = \frac{Pq}{\cos t} - \frac{1}{2}q^2\cdot\tan t + f_1(P,t)$$

substituting this solution into second equation

$$\frac{\partial F}{\partial P} = \frac{q}{\cos t} - P\tan t \Rightarrow \frac{\partial F}{\partial P} = \frac{q}{\cos t} + \frac{\partial f_1}{\partial P} = \frac{q}{\cos t} - P\tan t, \text{ thus}$$

$\frac{\partial f_1}{\partial P} = -P\tan t$ and finally $f_1(P,t) = -\frac{1}{2}P^2\tan t + g(t)$ where $g(t)$ is arbitrary function of time. Thus finally we get:

$$\boxed{F(P,q) = \frac{Pq}{\cos t} - \frac{1}{2}(P^2+q^2)\tan t + g(t);}$$

Problem 3 A dynamical system has the Hamiltonian $H = \frac{pq^3}{2t}$. Find a new Hamiltonian after the canonical transformations

$$P = pq^3\left(1 + t\cdot\exp\left(\frac{1}{q^2}\right)\right); \quad Q = -\frac{1}{2}\left(\frac{1}{q^2} + \ln\left(\frac{pq^3}{t}\right)\right);$$

To find new Hamiltonian we should find canonical transformation generating function. Let's find generating function $F(P,q)$

$$p = \frac{\partial F}{\partial q} = \frac{P}{q^3\left(1 + t\exp\frac{1}{q^2}\right)}; \quad Q = \frac{\partial F}{\partial P} = -\frac{1}{2}\left(\frac{1}{q^2} + \ln\left(\frac{pq^3}{t}\right)\right);$$

$\frac{pq^3}{t} = \frac{P}{t\left(1 + t\exp\left(\frac{1}{q^2}\right)\right)}$; so that we can rewrite second equation in the following form:

$$\frac{\partial F}{\partial P} = -\frac{1}{2}\left(\frac{1}{q^2} + \ln P - \ln t - \ln\left(1 + t\exp\left(\frac{1}{q^2}\right)\right)\right) \text{ integrating this}$$

we get $F(P,q) = -\frac{1}{2}\left[\frac{P}{q^2} - P\ln\{t(1 + t\exp\frac{1}{q^2})\} + P\ln P - P\right] + f(q,t)$

Now subsituting this function into second equation we get:

(6) $\frac{\partial F}{\partial q} = \frac{P}{q^3} - \frac{1}{2}\cdot\frac{2}{q^3}\frac{P\, t\exp\left(\frac{1}{q^2}\right)}{1+t\exp\left(\frac{1}{q^2}\right)} + \frac{\partial f(q,t)}{\partial q} = \frac{P}{q^3}\left(1+t\exp\frac{1}{q^2}\right)^{-1} + \frac{\partial f(q,t)}{\partial q} =$

$= \frac{P}{q^3}\left(1+t\exp\left(\frac{1}{q^2}\right)\right)^{-1}$ thus we obtain following condition for

$f(q,t)$: $\frac{\partial f}{\partial q} = 0 \Rightarrow f(q,t) = g(t)$ - some function of time.

So, finally, we get the following generating function:

$$\boxed{F(P,q) = -\frac{1}{2}\left[\frac{P}{q^2} - P\ln\{t(1+t\exp\tfrac{1}{q^2})\} + P\ln P - P\right] + g(t);}$$

$H = \frac{pq^3}{2t}$; and $K = H + \frac{\partial F}{\partial t}$, and $H = \frac{pq^3}{2t} = \frac{P}{2t(1+t\exp(\frac{1}{q^2}))}$

$\frac{\partial F}{\partial t} = g'(t) + \frac{P}{2t} + \frac{P\exp\left(\frac{1}{q^2}\right)}{2(1+t\exp\frac{1}{q^2})} = g'(t) + \frac{P(1+2t\exp(\frac{1}{q^2}))}{2t(1+t\exp\frac{1}{q^2})}$

Then finally $K = \frac{P(1+t\exp\frac{1}{q^2})}{t(1+\exp\frac{1}{q^2})} + g'(t) = \frac{P}{t} + g'(t)$. So, final

answer is $\boxed{K(P,Q,t) = \frac{1}{t}P + g'(t)}$

Problem 4 Find the Poisson Bracket of the following functions on phase-space

$\Phi = p^2 + q^2$; $\Psi = \arctan\left(\frac{p}{q}\right)$;

$\{\Phi, \Psi\} = \{p^2+q^2; \arctan\frac{p}{q}\}$ now we can use property II

of Poisson Brackets: $\{f; gh\} = \{f;g\}h + g\{f;h\}$ then

$\{f; g^2\} = \{f;g\}g + g\{f;g\} = 2g\{f;g\}$. Then:

$\{\Phi;\Psi\} = 2p\{p; \arctan\frac{p}{q}\} + 2q\{q; \arctan\frac{p}{q}\}$, now, using

$\{q; g\} = \frac{\partial g}{\partial p}$; $\{f; p\} = \frac{\partial f}{\partial q}$; we transform it into:

$\{\Phi;\Psi\} = -2p\frac{\partial}{\partial q}\arctan\frac{p}{q} + 2q\frac{\partial}{\partial p}\arctan\frac{p}{q} = \frac{2p^2}{q^2}\frac{1}{1+\frac{p^2}{q^2}} + \frac{2}{1+\frac{p^2}{q^2}} =$

$= 2 \Rightarrow \boxed{\{\Phi;\Psi\} = 2;}$

1

Seminar 5 (Poisson brackets N2)

Theory

Poisson brackets

Def: $\{f;g\} = \frac{\partial f}{\partial q^i}\cdot\frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i}\cdot\frac{\partial g}{\partial q^i}$;

Properties of Poisson brackets:

I antisymmetry $\{f;g\} = -\{g;f\}$;

II $\{f;gh\} = \{f,g\}h + g\{f,h\}$; $\Rightarrow \{f,g^2\} = 2g\{f,g\}$;

III Jacobi identity $\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0$

$\{q_i;p_j\} = \delta_{ij}$; $\{p_i,p_j\} = 0$; $\{q_i,q_j\} = 0$;

Poisson brackets are preserved in canonical transformation i.e. $\{Q_i, P_j\} = \delta_{ij}$ is transformation $Q_i = Q_i(q,p,t)$ and $P_i = P_i(q,p,t)$ is canonical

important relations are $\{q,g\} = \frac{\partial g}{\partial p}$; $\{f,p\} = \frac{\partial f}{\partial q}$;

Total time derivative of some quantaty $f(p,q,t)$ equals to $\frac{df}{dt} = \{f,H\} + \frac{\partial f}{\partial t}$, so if this quantaty is conserved we want: $\boxed{\frac{\partial f}{\partial t} = -\{f,H\};}$

Another important property is the following:
If u and v are 2 conserved quantaties we can generate one more conserved quantaty w using Poisson brackets: $w = \{u,v\}$ Let's prove this. As u and v are conserved the following identities are valid: $\frac{\partial u}{\partial t} = -\{u;H\}$; $\frac{\partial v}{\partial t} = -\{v,H\}$;
So that $\frac{\partial w}{\partial t} = \{\frac{\partial u}{\partial t}, v\} + \{u;\frac{\partial v}{\partial t}\} = -\{\{u,H\},v\} - \{u,\{v,H\}\} =$

$= -\{\{u,H\},v\} - \{\{H,v\},u\} = \{\{v,u\},H\} = -\{w,H\}$. We have used Jacobi identity here, and eventually shawn that $\frac{\partial w}{\partial t} = -\{w,H\}$; and thus $\frac{dw}{dt} = \{w,H\} + \frac{\partial w}{\partial t} = 0$, so w is indeed conserved quantaty, q.e.d.

②

Problem 1 A canonical transformation is given by the generating function $F(q,P) = P^2 + \ln(P+q)$

ⓐ Find the new variables P and Q as functions of p and q.

Now using values of partial derivatives:

$$p = \frac{\partial F}{\partial q}; \quad Q = \frac{\partial F}{\partial P}; \quad \text{thus} \quad p = \frac{1}{P+q}; \quad Q = 2P + \frac{1}{P+q};$$

thus $P = \frac{1}{p}(1 - pq)$; and substituting this into second equation we get: $Q = \frac{2}{p} - 2q + p$. This are desired formulas of canonical transformation.

$$\boxed{P = \frac{1}{p}(1-pq); \quad Q = \frac{2}{p} - 2q + p;}$$

ⓑ Verify that the new variables have the correct Poisson Brackets.

Now let's calculate Poisson Brackets of this coordinates

$\{Q, P\} = \{\frac{2}{p} - 2q + p; \frac{1}{p} - q\}$, due to antisymmetry property

$\{\frac{2}{p} - 2q; \frac{1}{p} - q\} = 0$ so we are left with:

$\{Q, P\} = \{p; \frac{1}{p} - q\} = -\{\frac{1}{p} - q; p\} = -\frac{\partial}{\partial q}\left(\frac{1}{p} - q\right) = 1$, so we have shown that fundamental Poisson brackets are preserved after our transformation and thus this transformation is indeed canonical, q.e.d.

Problem 2 A dynamical system has the Hamiltonian $H = p_1 p_2 + q_1 q_2$. Show that the functions $\Phi_1 = p_1^2 + q_2^2$; $\Phi_2 = p_2^2 + q_1^2$; are conserved quantaties.

To shaw that this quantaties are conserved we will use information from "Theory" part. As Φ_1 and Φ_2 are time-independent, if $\{\Phi, H\} = 0$ Φ appears to be conserved quantaty

(3) Let's check Poisson Bracket's for given functions Φ_1, Φ_2 and H:

$$\{\Phi_1; H\} = \{p_1^2 + q_2^2; p_1p_2 + q_1q_2\} = 2p_1\{p_1; p_1p_2 + q_1q_2\} +$$
$$+ 2q_2\{q_2; p_1p_2 + q_1q_2\} = -2p_1 \frac{\partial}{\partial q_1}(p_1p_2 + q_1q_2) + 2q_2 \frac{\partial}{\partial p_2}(p_1p_2 + q_1q_2) =$$
$$= -2p_1q_2 + 2q_2p_1 = 0$$, so $\{\Phi_1; H\} = 0$ and thus

$\frac{d}{dt}\Phi_1 = \{\Phi_1, H\} + \frac{\partial \Phi_1}{\partial t} = 0$ Φ_1 is indeed conserved quantaty, q.e.d

Now we go for Φ_2. Here we can immideately conclude that $\{\Phi_2, H\} = 0$ Because $\Phi_2(p_2, q_1) = \Phi_1(p_1 \to p_2; q_2 \to q_1)$ and Hamiltonian is symmetric with respect to permutation $p_1 \leftrightarrow p_2$ and $q_1 \leftrightarrow q_2$;
But let's show this explicitely.

$$\{\Phi_2, H\} = \{p_2^2 + q_1^2, p_1p_2 + q_1q_2\} = 2p_2\{p_2, p_1p_2 + q_1q_2\} + 2q_1\{q_1, p_1p_2 + q_1q_2\} =$$
$$= -2p_2 \frac{\partial}{\partial q_2}(p_1p_2 + q_1q_2) + 2q_1 \frac{\partial}{\partial}(p_1p_2 + q_1q_2) = -2p_2q_1 + 2q_1p_2 = 0$$

So we conclude that $\frac{d\Phi_2}{dt} = \{\Phi_2, H\} + \frac{\partial \Phi_2}{\partial t} = 0$ and Φ_2 is concerved q.e.d.

Problem 3 Show that the functions
$$\Phi_1 = \frac{p_1}{q_2}; \quad \Phi_2 = (p_2 - q_2)e^{-t};$$
are conserved quantaties. for the Hamiltonian $H = p_1q_1 - p_2q_2 + q_2^2$
Find one more integral of the motion using Poisson brackets.
Find the general solution of the equations of motion using the conserved quantities.

Let's find Poisson Brackets of Φ_1 and Φ_2 with H:

$$\{\Phi_1, H\} = \left\{\frac{p_1}{q_2}, p_1q_1 - p_2q_2 + q_2^2\right\} = p_1\left\{\frac{p_1}{q_2}, q_1\right\} + q_1\left\{\frac{p_1}{q_2}, p_1\right\} -$$
$$- p_2\left\{\frac{p_1}{q_2}, q_2\right\} - q_2\left\{\frac{p_1}{q_2}, p_2\right\} + 2q_2\left\{\frac{p_1}{q_2}, q_2\right\} = -p_1\frac{\partial}{\partial p_1}\left(\frac{p_1}{q_2}\right) + q_1\underbrace{\cancel{\frac{\partial}{\partial q_1}}\left(\frac{p_1}{q_2}\right)}_{0}$$
$$+ p_2\underbrace{\cancel{\frac{\partial}{\partial p_2}}\left(\frac{p_1}{q_2}\right)}_{0} - q_2\frac{\partial}{\partial q_2}\frac{p_1}{q_2} - 2q_2\underbrace{\cancel{\frac{\partial}{\partial p_2}}\frac{p_1}{q_2}}_{0} = -\frac{p_1}{q_2} + \frac{p_1}{q_2} = 0$$, so that

(4) $\frac{d}{dt}\Phi_1 = \{\Phi_1, H\} + \frac{\partial \Phi_1}{\partial t} = 0$ so Φ_1 is indeed conserved quantity

$$\{\Phi_2, H\} = e^{-t}\{p_2 - q_2 ; p_1q_1 - p_2q_2 + q_2^2\} = -\left(\frac{\partial}{\partial q_2} + \frac{\partial}{\partial p_2}\right)(p_1q_1 - p_2q_2 + q_2^2)e^{-t} =$$

$= -(2q_2 - p_2 - q_2)e^{-t} = (p_2 - q_2)e^{-t}$; So total time derivative is:

$\frac{d}{dt}\Phi_2 = \{\Phi_2, H\} + \frac{\partial \Phi_2}{\partial t} = (p_2 - q_2)e^{-t} - (p_2 - q_2)e^{-t} \neq 0$, so

Φ_2 is conservative quatity., q.e.d.

Now as we have seen if Φ_1 and Φ_2 are conserved then their Poisson brackets $\Phi_3 = \{\Phi_1, \Phi_2\}$ is conserved quantaty too. Let's find it:

$$\Phi_3 = \{\Phi_1, \Phi_2\} = \left\{\frac{p_1}{q_2}, (p_2 - q_2)e^{-t}\right\} = e^{-t}\left\{\frac{p_1}{q_2}, p_2\right\} - e^{-t}\left\{\frac{p_1}{q_2}, q_2\right\} =$$

$$= e^{-t}\frac{\partial}{\partial q_2}\frac{p_1}{q_2} + e^{-t}\frac{\partial}{\partial p_2}\frac{p_1}{q_2} = -e^{-t}\frac{p_1}{q_2^2}; \quad \boxed{\Phi_3 = -e^{-t}\frac{p_1}{q_2^2};}$$

Let's check if this quntity is indeed conserved.

$$\{\Phi_3, H\} = -e^{-t}\left\{\frac{p_1}{q_2^2}; p_1q_1 - p_2q_2 + q_2^2\right\} = -e^{-t}q_1\left\{\frac{p_1}{q_2^2}, p_1\right\} - e^{-t}\left\{\frac{p_1}{q_2^2}, q_1\right\}p_1 +$$

$$+ e^{-t}p_2\left\{\frac{p_1}{q_2^2}, q_2\right\} + e^{-t}q_2\left\{\frac{p_1}{q_2^2}, p_2\right\} - 2e^{-t}q_2\left\{\frac{p_1}{q_2^2}, q_2\right\} =$$

$$= -e^{-t}\cancel{q_1\frac{\partial}{\partial q_1}}\frac{p_1}{q_2^2} + e^{-t}p_1\frac{\partial}{\partial p_1}\frac{p_1}{q_2^2} - e^{-t}\cancel{p_2\frac{\partial}{\partial p_2}}\frac{p_1}{q_2^2} + e^{-t}q_2\frac{\partial}{\partial q_2}\frac{p_1}{q_2^2} =$$

$= e^{-t}\frac{p_1}{q_2^2} - 2e^{-t}\frac{p_1}{q_2^2} = -e^{-t}\frac{p_1}{q_2^2}$; So we get:

$\frac{d\Phi_3}{dt} = \{\Phi_3, H\} + \frac{\partial \Phi_3}{\partial t} = -e^{-t}\frac{p_1}{q_2^2} + e^{-t}\frac{p_1}{q_2^2} = 0$, so Φ_3 is indeed conserved quantity.

Now we can find equations of motion and solve them but we don't even need to do it to find general solution. We have 4 conserved quantities Φ_1, Φ_2, Φ_3 and, as Hamiltonian is time-independent it is conserved quantity too. Now we should express p_1, q_1, p_2, q_2 through this conserved quantities.

$\Phi_1 = \frac{p_1}{q_2}$; $\Phi_2 = (p_2 - q_2)e^{-t}$; $\Phi_3 = -e^{-t}\frac{p_1}{q_2^2}$; $H = p_1q_1 - p_2q_2 + q_2^2$;

(5) From first equation we get $p_1 = \Phi_1 q_2$; Substituting this into third equation $\Phi_3 = -e^{-t}\frac{\Phi_1}{q_2} \Rightarrow q_2 = -\frac{\Phi_1}{\Phi_3}e^{-t}$; $p_1 = -\frac{\Phi_1^2}{\Phi_3}e^{-t}$;

$p_2 = e^t\Phi_2 + q_2 = e^t\Phi_2 + e^{-t}\frac{\Phi_1}{\Phi_3}$; Subsituting obtained expressions for p_1, p_2, q_2 we get:

$$H = -\frac{\Phi_1^2}{\Phi_3}e^{-t}q_1 + \frac{\Phi_1\Phi_2}{\Phi_3} - \cancel{\frac{\Phi_1^2}{\Phi_3}e^{-2t}} + \cancel{\frac{\Phi_1^2}{\Phi_3^2}e^{-2t}}$$, thus

$$q_1 = e^t\frac{\Phi_3}{\Phi_1^2}\left(\frac{\Phi_1\Phi_2}{\Phi_3} - H\right) = e^t\left(\frac{\Phi_2}{\Phi_1} - \frac{\Phi_3}{\Phi_1^2}H\right);$$

So, the answer is:

$$\boxed{q_2 = -\frac{\Phi_1}{\Phi_3}e^{-t}; \quad p_1 = -\frac{\Phi_1^2}{\Phi_3}e^{-t}; \quad p_2 = e^t\Phi_2 - e^{-t}\frac{\Phi_1}{\Phi_3}; \quad q_1 = e^t\left(\frac{\Phi_2}{\Phi_1} - \frac{\Phi_3}{\Phi_1^2}H\right);}$$

Problem 4 Using Poisson brackets, show that the components of the Laplace-Runge-Lenz vector $\bar{A} = \bar{p}\times\bar{L} - \frac{mk\bar{r}}{r}$; are conserved quantities in the Kepler problem with Hamiltonian $H = \frac{p^2}{2m} - \frac{k}{r}$;

First of all let's rewrite LRL vector components

$A^i = \varepsilon^{ijk}p^jL^k - \frac{mk}{r}r^i$ now as $L = r\times p \Rightarrow L^i = \varepsilon^{ijk}r^jp^k$;

So $\varepsilon^{ijk}\varepsilon^{klm}p^jr^lp^m$, using identity for Levi-Civita symbol

$\varepsilon^{ijk}\varepsilon^{klm} = \varepsilon^{kij}\varepsilon^{klm} = \delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}$, we can rewrite:

$\varepsilon^{ijk}p^jL^k = (\delta^{il}\delta^{jm} - \delta^{im}\delta^{jl})p^jr^lp^m = p^mp^mr^i - p^jr^jp^i = \bar{p}^2r^i - (\bar{p}\bar{r})p^i$;

So $\boxed{A^i = \bar{p}^2r^i - (\bar{p}\bar{r})p^i - \frac{mk}{r}r^i;}$

Now we will find Poisson brackets of A^i with H:

$\{A^i, H\} = \{A^i, \frac{\bar{p}^2}{2m} - \frac{k}{r}\}$; Let's consider this brackets term by term:

(1) $\frac{1}{2m}\{\bar{p}^2r^i, \bar{p}^2\} = \frac{1}{2m}r^i\underbrace{\{\bar{p}^2, \bar{p}^2\}}_{0 \text{ due to antisymmetry}} + \frac{1}{2m}\bar{p}^2\{r^i, p^jp^j\} = \frac{1}{m}\bar{p}^2p^j\{r^i, p^j\} =$

⑥ $= \frac{1}{m}\bar{p}^2 p^j \delta^{ij} = \frac{1}{m}\bar{p}^2 p^i$;

② $-mk\left\{\frac{r^i}{r}, \frac{\bar{p}^2}{2m}\right\} = -\frac{k}{2}\left\{\frac{r^i}{r}, p^j p^j\right\} = -kp^j\left\{\frac{r^i}{r}, p^j\right\} = -kp^j \frac{\partial}{\partial r^j}\left(\frac{r^i}{r}\right) =$

$= -\frac{kp^i}{r} + \frac{k(\bar{p}\bar{r})r^i}{r^3}$;

③ $-\left\{(\bar{r}\bar{p})p^i ; \frac{\bar{p}^2}{2m}\right\} = -\frac{1}{2m} p^i p^j \{r^j ; \bar{p}^2\} = -\frac{1}{m} p^i p^j p^k \{r^j, p^k\} =$

$= -\frac{1}{m}\bar{p}^2 p^i$;

④ $-\left\{\bar{p}^2 r^i ; \frac{k}{r}\right\} = -2r^i p^j k \left\{p^j, \frac{1}{r}\right\} = 2r^i p^j k \frac{\partial}{\partial r^j}\frac{1}{r} = -2k\,\frac{r^i(\bar{p}\bar{r})}{r^3}$;

⑤ $k\left\{(\bar{r}\bar{p})p^i, \frac{1}{r}\right\} = k\left\{r^j p^j p^i, \frac{1}{r}\right\} = kr^j p^j \left\{p^i, \frac{1}{r}\right\} + kr^j p^i \left\{p^j, \frac{1}{r}\right\} =$

$= -kr^j p^j \frac{\partial}{\partial r^i}\frac{1}{r} - kr^j p^i \frac{\partial}{\partial r^j}\frac{1}{r} = \frac{k(\bar{p}\bar{r})r^i}{r^3} + \frac{k}{r}p^i$;

Summing all this terms we get:

$\{A^i, H\} = \frac{1}{m}\bar{p}^2 p^i - \frac{kp^i}{r} + \frac{k}{r^3}(\bar{p}\bar{r})r^i - \frac{1}{m}\bar{p}^2 p^i - \frac{2k}{r^3}(\bar{p}\bar{r})r^i + \frac{k}{r^3}(\bar{r}\bar{p})r^i +$
$+\frac{kp^i}{r} = 0$, so as $\frac{\partial A^i}{\partial t} = 0$ ($\bar{A}$ components are time independent) and $\frac{dA^i}{dt} = 0$; so Laplace-Runge-Lenz vector components are conserved quantities.

1

Seminar 6 (Hamilton-Jacobi method)

Theory

Let's consider type-II canonical transformation with generating function $F_2(q, P, t)$. The new hamiltonian in this case will be given by

$K = H + \frac{\partial F_2}{\partial t}$ and we can choose it in such a way that

$K=0$. In this case equations of motion for new canonical coordinates are especially simple as:

$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0$; $\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0$; so Q_i and P_i appear to be constants of motion. How all this help us in solving mechanical system? Equation $K=0$ is identical to equation:

$H(q, p, t) + \frac{\partial F_2}{\partial t} = 0$ Further we will write S instead of F_2. Using formulas for partial derivatives of type-II generating function: $p_i = \frac{\partial F_2}{\partial q_i}$; $Q_i = \frac{\partial F_2}{\partial P_i}$; we can express all terms in equation above through q_i and $\frac{\partial F_2}{\partial q_i}$, so we get:

$$\boxed{H\left(q_1, \dots, q_n; \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_n}; t\right) + \frac{\partial S}{\partial t} = 0}$$ – this is

called Hamilton-Jacobi equation and S is called Hamilton's principal function.

We can solve this PDE for principal function and after use relations for it's partial derivatives; assuming $P_i = \alpha_i = \text{const}$ and $Q_i = \beta_i = \text{const}$ due to equations of motion for new identically vanishing Hamiltonian. then we get: $p_i = \frac{\partial S(q, \alpha, t)}{\partial q_i}$ and $\beta_i = \frac{\partial S(q, \alpha, t)}{\partial \alpha_i}$; which will give us general solution for the system. If initial Hamiltonian doesn't depend on time explicitely it is useful to assume $S(q, \alpha, t) = W(q, \alpha) - at$;

(2)

Problem 1 The Hamiltonian of a system has the form $H=\frac{1}{2}\left(\frac{1}{q^2}+p^2q^4\right)$ Find the equation of motion for q. Find a canonical transformation which reduces H to the form of a harmonic oscillator.

From Hamiltonian we easily derive canonical equations

$$\dot{q}=\frac{\partial H}{\partial p}=pq^4;\quad \dot{p}=-\frac{\partial H}{\partial q}=\frac{1}{q^3}-2p^2q^3;$$

$$\boxed{\dot{q}=pq^4;\quad \dot{p}=\frac{1}{q^3}-2p^2q^3;}$$

Now let's try to observe harmonic oscillator hamiltonian from what we have. Natural assumption is that transformation we need looks like:

$Q^2=\frac{1}{q^2}$; $P^2=p^2q^4$; let's choose the following combination:

$Q=\frac{1}{q}$; $P=-pq^2$; Let's assume that we have type-II generating function $F_2(q,P,t)$. Formulas for it's partial derivatives are the following one $p=\frac{\partial F_2}{\partial q}$; $Q=\frac{\partial F_2}{\partial P}$

now as $Q=\frac{1}{q}$ and $p=-\frac{P}{q^2}$ we get following differential equations:

$\frac{\partial F_2}{\partial P}=\frac{1}{q} \Rightarrow F_2=\frac{P}{q}+C_1(q)$ substituting this to another equation we get $\frac{\partial F_2}{\partial q}=-\frac{P}{q^2}+C_1'(q)=-\frac{P}{q^2}$ so

$C_1'(q)=0 \Rightarrow C_1(q)=C_1=const$. And Hamiltonian we get is given by $K=H+\frac{\partial F_2}{\partial t}=H=\frac{1}{2}(Q^2+P^2)$;

Let's check that transformation we have obtained is indeed canonical. First way is just direct check of new coordinates equations of motion:

$\dot{Q}=\frac{\partial K}{\partial P}=P$ at the same time $Q=\frac{1}{q}$ and $\dot{Q}=-\frac{\dot{q}}{q^2}=-pq^2=P$ ✓

$\dot{P}=-\frac{\partial K}{\partial Q}=-Q$ at the same time $P=-pq^2$ and $\dot{P}=-\dot{p}q^2-2pq\dot{q}=$

$=-\frac{1}{q}+2p^2q^5-2p^2q^5=-\frac{1}{q}=-Q$ ✓

3 Another and simplier way to ckeck if transformation is canonical is to find its Poisson Brackets:

$$\{Q,P\} = -\{\tfrac{1}{q}, pq^2\} = -p\underbrace{\{\tfrac{1}{q}, q^2\}}_{0} - q^2\{\tfrac{1}{q}, p\} = -q^2\frac{\partial}{\partial q}\frac{1}{q} = 1 \Rightarrow$$

$\{Q,P\}=1 \Rightarrow$ transformation is indeed canonical.

Problem 2 Find Hamilton's principal function for a 1d free particle of mass m. Derive the general solution of the equations of motion p(t) and q(t).

Hamiltonian for 1d free particle looks like

$H = \frac{p^2}{2m}$. We can imideatelly write down Hamilton-Jacobi equation: $\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{\partial S}{\partial t} = 0$. Hamiltonian doesn't depend expliciteley on time so we can us ansatz:

$S(q,t) = W(q) - \alpha t$; Then HJ equation turns into

$\frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{-2m\frac{\partial S}{\partial t}} = \sqrt{2m\alpha}$, so that $\boxed{W = \sqrt{2m\alpha}\, q}$

Then Hamilton principal function is given by:

$\boxed{S = \sqrt{2m\alpha}\, q - \alpha t}$; We take new canonical momentum to be $P = \alpha$; Then we get $p = \frac{\partial S}{\partial q} = \sqrt{2m\alpha}$; and $Q = \beta = \frac{\partial S}{\partial \alpha} = \sqrt{\frac{m}{2\alpha}}\, q - t$;

$$\boxed{\begin{aligned} q &= \sqrt{\tfrac{2\alpha}{m}}(\beta + t); \\ p &= \sqrt{2m\alpha}; \end{aligned}}$$

$\rightarrow$ this is general solutions for equations of motion we were looking for.

Problem 3 Suppose the potential in a problem with one degree of freedom is linearly dependent on time, such that Hamiltonian has the form: $H = \frac{p^2}{2m} - mAtx$, where A is a constant. Solve the dynamical problem by means of Hamilton's principal function, under the initial conditions $t=0, x=0; p=mv_0$;

Let's write down HJ equations

$$p = \frac{\partial S}{\partial q}; \quad Q = \frac{\partial S}{\partial P}; \quad \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 - mAtq + \frac{\partial S}{\partial t} = 0$$

Let's try to read of mAtq term in the following way: assume Hamilton principal function is of the form

(4) $S(q,t) = -S_t(t) + S_q(q) + \frac{1}{2}mAt^2q$, then we get the following equation:

$$\frac{1}{2m}\left(S_q' + \frac{1}{2}mAt^2\right) - \dot{S}_t = 0 \quad \text{thus}$$

$S_q' = \sqrt{2m\dot{S}_t} - \frac{1}{2}mAt^2$ as l.h.s of this equation is q-dependent and r.h.s. is t-dependent, both of them should be constant: $S_q' = \alpha \Rightarrow S_q = q\alpha$; and $\sqrt{2m\dot{S}_t} - \frac{1}{2}mAt^2 = \alpha$

$\dot{S}_t = \frac{1}{2m}\left(\alpha + \frac{1}{2}mAt^2\right)^2 = \frac{1}{2m}\left(\alpha^2 + \alpha mAt^2 + \frac{1}{4}m^2A^2t^4\right)$ thus

$$S_t = \frac{mA^2}{40}t^5 + \frac{A\alpha}{6}t^3 + \frac{\alpha^2}{2m}t;$$

$$\boxed{S = \alpha q - \frac{1}{2m}\left(\alpha^2 t + \frac{1}{3}m\alpha At^3 + \frac{1}{20}m^2A^2t^5\right) + \frac{1}{2}mAt^2q;}$$

Then $p = \frac{\partial S}{\partial q} = \alpha + \frac{1}{2}mAt^2$. We take $P = \alpha$ and then

$Q = \frac{\partial S}{\partial P} = \frac{\partial S}{\partial \alpha} = q - \frac{\alpha t}{m} - \frac{1}{6}At^3 = \beta$. So general solution of system is the following:

$$\boxed{\begin{aligned} p &= \alpha + \tfrac{1}{2}mAt^2 \\ q &= \beta + \tfrac{\alpha t}{m} + \tfrac{1}{6}At^3 \end{aligned}}$$

Substituting initial condition $t=0, q=0, p=mv_0$; at $t=0$ $p=\alpha, q=\beta$, so we conclude that $\beta=0$; $\alpha = mv_0$; and thus

$$\boxed{\begin{aligned} p &= mv_0 + \tfrac{1}{2}mAt^2; \\ x &= v_0t + \tfrac{1}{6}At^3; \end{aligned}}$$

Problem 4 The Hamiltonian of the system has the form

$$H = \frac{1}{2}e^t\left(\frac{p^2}{2m} + Aq\right)$$

Find the general solution for $p(t)$ and $q(t)$ using Hamiltons principle function.

HJ equations for this system take form

$$p = \frac{\partial S}{\partial q}; \quad Q = \frac{\partial S}{\partial P}; \quad \frac{1}{2}e^t\left(\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + Aq\right) + \frac{\partial S}{\partial t} = 0$$

Let's assume $S = S_q(q) - S_t(t)$, and equation transforms to:

$\frac{1}{2m}\frac{\partial S_q}{\partial q} + Aq = 2e^{-t}\frac{\partial S_t}{\partial t}$ L.h.s. and r.h.s. of this equation depend on q and t respectively. So we take both them to be

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constant.

$2e^{-t}\frac{\partial S_t}{\partial t} = \alpha$ thus $\boxed{S_t = \frac{1}{2}e^{t}\alpha\,;}$ and

$\frac{\partial S_q}{\partial q} = \sqrt{2m(\alpha - Aq)} \Rightarrow \boxed{S_q = -\frac{\sqrt{2m}}{A}(\alpha - Aq)^{3/2}\cdot\frac{2}{3}}$ so that

principal function is given by:

$$\boxed{S = -\frac{\alpha}{2}e^{t} - \frac{2\sqrt{2m}}{3A}(\alpha - Aq)^{3/2}}$$

$p = \frac{\partial S}{\partial q} = \sqrt{2m(\alpha - Aq)}\,;\quad Q = \frac{\partial S}{\partial \alpha} = -\frac{1}{2}e^{t} - \frac{\sqrt{2m}}{A}\sqrt{\alpha - Aq} = \beta$

$$\boxed{\begin{array}{l} q = \frac{\alpha}{A} - \frac{A}{2m}\left(\beta + \frac{1}{2}e^{t}\right)^2; \\ p(t) = -A(\beta + e^{t}/2) \end{array}}$$

as $p = \sqrt{2m(\alpha - Aq)}$ we get

→ this is general solution for system with given Hamiltonian

Seminar 7 (Hamilton-Jacobi equations)

Theory

Algorithm of solving Hamilton systems using Hamilton-Jacobi equations

(I) We introduce type-II generating function $S(q_i, P_i, t)$ which reduces Hamiltonian to the identically zero, so that it satisfies Hamilton-Jacobi equation (HJE)

$$\boxed{H\left(q_1, \dots, q_n; \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_n}; t\right) + \frac{\partial S}{\partial t} = 0}$$

(II) Next we use formulas for S partial derivatives:

$Q_i = \frac{\partial F_2}{\partial P_i}$ we take $P_i = \alpha_i$ - constants of integration of HJE.

and $Q_i = \beta_i = \text{const}$, so that we get set of equations

$$\boxed{\beta_i = \frac{\partial S(q_i, \alpha_i, t)}{\partial \alpha_i}}$$

(III) Inverting last equation and using equation

$p_i = \frac{\partial S(q_i, \alpha_i, t)}{\partial q_i}$ we can write down general solution for mechanical system.

Useful ansatz for time independent Hamiltonians is $S(q_i, t) = W(q_i) - at$, in this case $W(q_i)$ is called Hamiltons characteristic function

For many systems $W(q_i)$ is separable so that we can write $W(q_i)$ in the following form:

$$W(q_1, q_2, \dots, q_n) = W_1(q_1) + W_2(q_2) + \dots + W_n(q_n);$$

Problem 1 A mechanical system has two angle-like degrees of freedom, $0 \leq \theta \leq \pi$, $-\pi \leq \varphi \leq \pi$. The Lagrangian for the system is:

$$L = \frac{I}{2}\left(\dot{\theta}^2 + \dot{\varphi}^2 \cos^2\theta\right).$$

Find a complete solution of the Hamilton-Jacobi equation in terms of 1 dimensional integral.

First of all let's derive Hamiltonian of the system.

② * first we find canonical momentum

$p_\theta = \frac{\partial L}{\partial \dot\theta} = I\dot\theta$; $p_\varphi = \frac{\partial L}{\partial \dot\varphi} = I\dot\varphi\cos^2\theta$, so $\boxed{p_\theta = I\dot\theta;\ p_\varphi = I\dot\varphi\cos^2\theta;}$

* inverting this expression we get $\boxed{\dot\theta = \frac{p_\theta}{I};\ \dot\varphi = \frac{p_\varphi}{I\cos^2\theta};}$

* making Legendre transformation

$$H = p_\theta\cdot\dot\theta + p_\varphi\cdot\dot\varphi - L_1 = \frac{p_\theta^2}{I} + \frac{p_\varphi^2}{I\cos^2\theta} - \frac{p_\theta^2}{2I} - \frac{p_\varphi^2}{2I\cos^2\theta} \quad \text{so}$$

$$\boxed{H = \frac{p_\theta^2}{2I} + \frac{p_\varphi^2}{2I\cos^2\theta};}$$

Now stating $p_\theta = \frac{\partial S}{\partial \theta}$ and $p_\varphi = \frac{\partial S}{\partial \varphi}$; we can write down HJE:

$$\frac{1}{2I}\left\{\left(\frac{\partial S}{\partial\theta}\right)^2 + \frac{1}{\cos^2\theta}\left(\frac{\partial S}{\partial\varphi}\right)^2\right\} + \frac{\partial S}{\partial t} = 0;$$

As Hamiltonian is time independent we assume ansatz $S(\theta,\varphi,t) = W_\theta(\theta) + W_\varphi(\varphi) - \alpha_1 t$ – here we have already assumed that characteristic function is separable. Then HJE turns into

$$\frac{1}{2I}\left\{(W_\theta')^2 + \frac{1}{\cos^2\theta}(W_\varphi')^2\right\} - \alpha_1 = 0; \quad \text{thus}$$

$(W_\varphi')^2 = (2I\alpha_1 - (W_\theta')^2)\cos^2\theta$ Here l.h.s. contains only φ-dependence and r.h.s. – only θ-dependence, so we can put both of them to be some constant α_2^2:

$W_\varphi' = \alpha_2$; $W_\varphi = \alpha_2\varphi$; and $(2I\alpha_1 - (W_\theta')^2)\cos^2\theta = \alpha_2^2$ so that

$W_\theta' = \sqrt{2I\alpha_1 - \frac{\alpha_2^2}{\cos^2\theta}} \Rightarrow W_\theta(\theta) = \int d\theta\sqrt{2I\alpha_1 - \frac{\alpha_2^2}{\cos^2\theta}}$, and final answer for Hamilton's principle function then becomes:

$$\boxed{S(\theta,\varphi,t) = \int d\theta\left(2I\alpha_1 - \frac{\alpha_2^2}{\cos^2\theta}\right)^{\frac{1}{2}} + \alpha_2\varphi - \alpha_1 t;}$$

Problem 2 A Hamiltonian system with 2 degrees of freedom is defined by the Hamiltonian $H = \frac{1}{2}(p_1 q_2 + 2p_1 p_2 + q_1^2)$. Find a complete solution of the Hamilton-Jacobi equation. Find the general solution for $q_1(t)$ and $q_2(t)$.

We introduce Hamilton's principle function $S(q,t)$, replacing

③ canonical momentum in Hamiltonian with $p_i = \frac{\partial S}{\partial q_i}$, $i=1,2$.

Then we get following HJE:

$$\frac{1}{2}\left(\frac{\partial S}{\partial q_1} q_2 + \frac{\partial S}{\partial q_1}\cdot\frac{\partial S}{\partial q_2} + q_1^2\right) + \frac{\partial S}{\partial t} = 0;$$

Since H is time independent we can write down principal function in the following way:

$S(q_1, q_2, t) = W_1(q_1) + W_2(q_2) - \alpha_1 t$ – here we have assumed from begining that characteristic function is separable.

Then HJE turns into:

$$q_2 \frac{\partial W_1}{\partial q_1} + 2\frac{\partial W_1}{\partial q_1}\cdot\frac{\partial W_2}{\partial q_2} + q_1^2 = 2\alpha_1 \Rightarrow W_1'(q_1)\{q_2 + 2W_2'(q_2)\} + q_1^2 = 2\alpha_1$$

or, finally, $\underline{\frac{W_1'(q_1)}{2\alpha_1 - q_1^2} = \frac{1}{2W_2'(q_2)+q_2}}$; Here, again, as l.h.s. has only q_1-dependence and r.h.s. – q_2-dependence, we can put both sides to be some constant α_2:

* $W_1'(q_1) = 2\alpha_1\alpha_2 - \alpha_2 q_1^2 \Rightarrow dW_1 = \alpha_2(2\alpha_1 - q_1^2)\,dq_1$ and finally:

$$\boxed{W_1(q_1) = \alpha_2\left(2\alpha_1 q_1 - \frac{1}{3}q_1^3\right)}$$

* $2W_2'(q_2) + q_2 = \frac{1}{\alpha_2} \Rightarrow 2W_2'(q_2) = \frac{1}{\alpha_2} - q_2$ so that $2W_2 = \frac{q_2}{\alpha_2} - \frac{1}{2}q_2^2$

$\boxed{W_2(q_2) = \frac{q_2}{2\alpha_2} - \frac{1}{4}q_2^2}$ So the Hamilton principal function is

$$\boxed{S = 2\alpha_1\alpha_2 q_1 - \frac{\alpha_2}{3}q_1^3 + \frac{q_2}{2\alpha_2} - \frac{1}{4}q_2^2 - \alpha_1 t;}$$

Now to find general solution for system we will use second equation of function partial derivative:

$Q_i = \beta_i = \frac{\partial S}{\partial \alpha_i} = \text{const}$. So

$\beta_1 = \frac{\partial S}{\partial \alpha_1} = 2\alpha_2 q_1 - t \Rightarrow \boxed{q_1 = \frac{\beta_1 + t}{2\alpha_2}}$; $\beta_2 = \frac{\partial S}{\partial \alpha_2} = 2\alpha_1 q_1 - \frac{1}{3}q_1^3 - \frac{q_2}{2\alpha_2^2} \Rightarrow$

$\Rightarrow \beta_2 = \frac{\alpha_1}{\alpha_2}(\beta_1 + t) - \frac{(\beta_1+t)^3}{24\alpha_2^3} - \frac{q_2}{2\alpha_2^2}$, so finally:

$$\boxed{q_2(t) = -2\alpha_2^2\beta_2 + 2\alpha_1\alpha_2(\beta_1 + t) - \frac{1}{12\alpha_2}(\beta_1 + t)^3; \quad q_1(t) = \frac{\beta_1 + t}{2\alpha_2};}$$

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Problem 3 A Hamiltonian system with two degrees of freedom is defined by the Hamiltonian $H = e^t \frac{p_2+q_2}{p_1+q_1}$;

ⓐ Find a complete solution of the Hamilton-Jacobi equation

We can introduse Hamilton's principal function $S(q,t)$ and replace canonical momentum with $p_i = \frac{\partial S}{\partial q_i}$. So we will get following HJE:

$$e^t\left(\frac{\partial S}{\partial q_2} + q_2\right)\left(\frac{\partial S}{\partial q_1} + q_1\right)^{-1} + \frac{\partial S}{\partial t} = 0$$

Let's take principle function in the form $S(q_1, q_2, t) = W_1(q_1) + W_2(q_2) - S_t(t)$; i.e. we assume that principal function is separable. We get:

$\underbrace{(W_2' + q_2)(W_1' + q_1)^{-1}}_{\alpha_1} - \underbrace{e^{-t}\frac{\partial S_t}{\partial t}}_{\alpha_1} = 0$ as first term is the function of q_1 and q_2, while second - on t variable we can put both of them to be equal to some constant α_1, so that:

$\dot{S}_t = e^t \alpha_1 \Rightarrow \boxed{S_t = \alpha_1 e^t}$ and $W_2' + q_2 = \alpha_1(W_1' + q_1)$ here l.h.s. has only q_2-dependence and r.h.s only q_1 dependence so we assume both of them to be constant:

$W_1' + q_1 = \alpha_2 \Rightarrow W_1' = \alpha_2 - q_1 \Rightarrow W_1 = \alpha_2 q_1 - \frac{1}{2}q_1^2$;

$W_2' + q_2 = \alpha_1\alpha_2 \Rightarrow \quad W_2 = \alpha_1\alpha_2 q_2 - \frac{1}{2}q_2^2$;

and then finally we can write down solution of HJE: $\boxed{S(q_1, q_2, t) = \alpha_2 q_1 + \alpha_1\alpha_2 q_2 - \frac{1}{2}q_1^2 - \frac{1}{2}q_2^2 - \alpha_1 e^t;}$

ⓑ Find the general solution for $q_1(t)$ and $q_2(t)$.

Now we use formulas for partial derivatives of principle function: $Q_i = const = \beta_i = \frac{\partial S}{\partial \alpha_i}$, so we can find $\beta_1 = \alpha_2 q_2 - e^t$; $\beta_2 = q_1 + \alpha_1 q_2$; From the first equation we get:

$\boxed{q_2(t) = \frac{\beta_1}{\alpha_2} + \frac{1}{\alpha_2}e^t;}$ substituting this into second we get:

$\boxed{q_1(t) = \beta_2 - \frac{\beta_1\alpha_1}{\alpha_2} - \frac{\alpha_1}{\alpha_2}e^t;}$

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Problem 4

A 2dimensional mechanics is defined by the Lagrangian $L = \frac{\dot{x}^2}{2y^2} + \frac{\dot{y}^2}{2} - \alpha x^2 y^2$; Find a complete solution of the Hamilton - Jacobi equation in terms of 1dimensional integrals.

First let's write down Hamiltonian of the system

* define canonical momentum

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{y^2}; \quad p_y = \frac{\partial L}{\partial \dot{y}} = \dot{y};$$

* invert relations: $\dot{x} = y^2 p_x$; $\dot{y} = p_y$;

* make Legendre transformation:

$$H = p_x \dot{x} + p_y \dot{y} - L = \frac{1}{2} y^2 p_x^2 + \frac{1}{2} p_y^2 + \alpha x^2 y^2$$

$$\boxed{H = \frac{1}{2} y^2 p_x^2 + \frac{1}{2} p_y^2 + \alpha x^2 y^2;}$$

Now we can write down HJE replasing canonical momentum in Hamiltonian with $p_i = \frac{\partial S}{\partial q_i}$;

$$\frac{1}{2} y^2 \left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial S}{\partial y}\right)^2 + \alpha x^2 y^2 + \frac{\partial S}{\partial t} = 0$$

As Hamiltonian doesn't depend on time explicitely we can look for the principle function of following form

$S(x, y, t) = S_x(x) + S_y(y) - \alpha_1 t$, assuming here that principal function is separable. Then HJE goes to

$$\frac{1}{2} y^2 (S_x')^2 + \frac{1}{2}(S_y')^2 + \alpha x^2 y^2 - \alpha_1 = 0 \Rightarrow y^2 \left\{\frac{1}{2}(S_x')^2 + \alpha x^2\right\} + \frac{1}{2}(S_y')^2 - \alpha_1 = 0$$

If we say that

$\frac{1}{2}(S_x')^2 + \alpha x^2 = \alpha_2 \Rightarrow S_x' = \sqrt{2(\alpha_2 - \alpha x^2)}$ and finally

$S_x = \int dx \sqrt{2(\alpha_2 - \alpha x^2)}$ coming back to HJE we get:

$(S_y')^2 - 2\alpha_1 + 2y^2 \alpha_2 = 0 \Rightarrow S_y' = \sqrt{2(\alpha_1 - \alpha_2 y^2)}$; thus

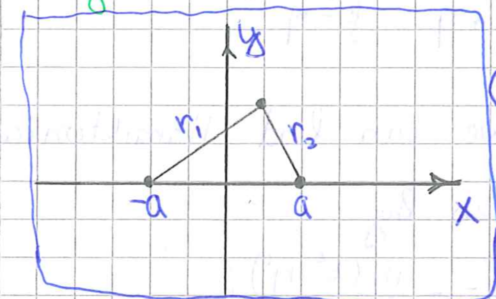
$\boxed{S_y(y) = \int dy \sqrt{2(\alpha_1 - \alpha_2 y^2)}}$ So Hamilton principal function is given by

$$\boxed{S = \int dx \sqrt{2(\alpha_2 - \alpha x^2)} + \int dy \sqrt{2(\alpha_1 - \alpha_2 y^2)} - \alpha_1 t;}$$

①

Seminar 8 (HJ, action-angle variables)

Problem 1 A particle of mass m is moving in the presence of two attracting centres placed at the points $x=-a, y=0$ and $x=+a, y=0$. The potentials are $-\frac{A}{r_1}$, $-\frac{B}{r_2}$ respectively. Find the principal function of Hamilton in terms of one-dimensional integrals. Use the coordinates $\xi = \frac{r_1+r_2}{2}$ and $\eta = \frac{r_1-r_2}{2}$.



ⓐ First let's write down Lagrangian in terms of variables ξ and η

$$\begin{cases} \xi = \frac{r_1+r_2}{2} \\ \eta = \frac{r_1-r_2}{2} \end{cases} \Rightarrow \begin{cases} r_1 = \xi+\eta; \\ r_2 = \xi-\eta; \end{cases}$$

Substituting this into potential $V = -\frac{A}{r_1} - \frac{B}{r_2} = -\frac{A}{\xi+\eta} - \frac{B}{\xi-\eta}$;

Now a little more difficult goal - to find kinetic energy $T = \frac{m}{2}(\dot{x}^2+\dot{y}^2)$ Let's find how to go from (x,y) to (ξ,η) variables. First we start with:

$$\begin{matrix} r_1^2 = (x+a)^2+y^2 \\ r_2^2 = (x-a)^2+y^2 \end{matrix} \Rightarrow \begin{matrix} r_1^2 = (x+a)^2-(x-a)^2+r_2^2, \text{ so that} \\ x = \frac{1}{4a}(r_1^2-r_2^2) = \frac{\xi\eta}{a}; \end{matrix}$$

substituting this into second equation we get:

$$r_2^2 = \left(\frac{\xi\eta}{a}-a\right)^2+y^2 \Rightarrow y^2 = (\xi-\eta)^2 - \left(\frac{\xi\eta}{a}-a\right)^2 = \xi^2+\eta^2-\frac{1}{a^2}\xi^2\eta^2-a^2 =$$

$$= \xi^2\left(1-\frac{\eta^2}{a^2}\right) - a^2\left(1-\frac{\eta^2}{a^2}\right) = (\xi^2-a^2)\left(1-\frac{\eta^2}{a^2}\right) \text{ so that:}$$

$$\boxed{x = \frac{\xi\eta}{a};\ y = \sqrt{(\xi^2-a^2)(1-\eta^2/a^2)};}$$ Then

$$\dot{x} = \frac{1}{a}(\dot{\xi}\eta+\xi\dot{\eta});\quad \dot{y} = \xi\dot{\xi}\sqrt{\frac{1-\eta^2/a^2}{\xi^2-a^2}} - \frac{1}{a^2}\eta\dot{\eta}\sqrt{\frac{\xi^2-a^2}{1-\eta^2/a^2}};$$

the kinetic energy is given by:

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m\left\{\frac{1}{a^2}\dot{\xi}^2\eta^2+\frac{1}{a^2}\dot{\eta}^2\xi^2+\frac{2}{a}\xi\dot{\xi}\eta\dot{\eta}+\xi^2\dot{\xi}^2\frac{1-\eta^2/a^2}{\xi^2-a^2}+\right.$$

$$\left.+\frac{1}{a^4}\eta^2\dot{\eta}^2\frac{\xi^2-a^2}{1-\eta^2/a^2}-\frac{2}{a^2}\xi\dot{\xi}\eta\dot{\eta}\right\} = \frac{1}{2}m\left(\dot{\xi}^2(\xi^2-a^2)^{-1}(\xi^2-\eta^2)+\right.$$

$$\left.+\dot{\eta}^2(a^2-\eta^2)^{-1}(\xi^2-\eta^2)\right\};$$

(2) or finally:

$$\boxed{T=\frac{m}{2}(\xi^2-\eta^2)\left\{\frac{\dot{\xi}^2}{(\xi^2-a^2)}-\frac{\dot{\eta}^2}{(\eta^2-a^2)}\right\};}$$

Now we can write down Lagrangian of system:

$L=T-V;$

$$\boxed{L=\frac{m}{2}(\xi^2-\eta^2)\left\{\frac{\dot{\xi}^2}{\xi^2-a^2}-\frac{\dot{\eta}^2}{\eta^2-a^2}\right\}+\frac{A}{\xi+\eta}+\frac{B}{\xi-\eta};}$$

(b) Now from Lagrangian we can find Hamiltonian:

* canonical momenta are given by

$$p_\xi=\frac{\partial L}{\partial\dot{\xi}}=\frac{m(\xi^2-\eta^2)}{\xi^2-a^2}\dot{\xi}\ ;\quad p_\eta=\frac{\partial L}{\partial\dot{\eta}}=-\frac{m(\xi^2-\eta^2)}{\xi^2-a^2}\dot{\eta}$$

* now we can inverse this equations:

$$\dot{\xi}=\frac{\xi^2-a^2}{m(\xi^2-\eta^2)}p_\xi\ ;\quad \dot{\eta}=-\frac{\xi^2-a^2}{m(\xi^2-\eta^2)}p_\eta\ ;$$

* finally we can make Legendre transformation

$$H=p_\xi\dot{\xi}+p_\eta\dot{\eta}-L=\frac{\xi^2-a^2}{2m(\xi^2-\eta^2)}p_\xi^2-\frac{\eta^2-a^2}{2m(\xi^2-\eta^2)}p_\eta^2-\frac{A}{\xi+\eta}-\frac{B}{\xi-\eta};\ \text{So}$$

$$\boxed{H=\frac{1}{2m(\xi^2-\eta^2)}\left\{(\xi^2-a^2)p_\xi^2-(\eta^2-a^2)p_\eta^2\right\}-\frac{A}{\xi+\eta}-\frac{B}{\xi-\eta};}$$

Now we can introduce Hamilton's principal function, $S(\xi,\eta,t)$ them $p_\xi=\frac{\partial S}{\partial\xi}$; $p_\eta=\frac{\partial S}{\partial\eta}$; and Hamilto-Jacobi is then:

$$\frac{1}{2m(\xi^2-\eta^2)}\left\{(\xi^2-a^2)\left(\frac{\partial S}{\partial\xi}\right)^2-(\eta^2-a^2)\left(\frac{\partial S}{\partial\eta}\right)^2\right\}-\frac{A}{\xi+\eta}-\frac{B}{\xi-\eta}+\frac{\partial S}{\partial t}=0$$

As usually in time-independent Hamiltonian case we assume it is separable and write down ansatz:

$S(\xi,\eta,t)=W_\xi(\xi)+W_\eta(\eta)-\alpha_1 t$, then we get:

$$\underbrace{\left[(\xi^2-a^2)(W_\xi')^2-2m(A+B)\xi-2m\alpha_1\xi^2\right]}_{2m\alpha_2}-\underbrace{\left[(\eta^2-a^2)(W_\eta')^2-2m(A-B)\eta-2m\alpha_1\eta^2\right]}_{2m\alpha_2}=0$$

First square brackets contain only ξ-dependency and second brackets contain only η-dependence we can put

③ Both this terms to be some constant so that

$$(\xi^2-a^2)(W_\xi')^2-2m(A+B)\xi-2m\alpha_1\xi^2=2m\alpha_2 \quad \text{so that:}$$

$$(W_\xi')^2=\frac{2m\alpha_1\xi^2+2m(A+B)\xi+2m\alpha_2}{\xi^2-a^2}, \quad \text{thus:}$$

$$W_\xi=\sqrt{2m}\int d\xi\sqrt{\frac{\alpha_1\xi^2+(A+B)\xi+\alpha_2}{\xi^2-a^2}}; \quad \text{and in the same way}$$

$$(\eta^2-a^2)(W_\eta')^2-2m(A-B)\eta-2m\alpha_1\eta^2=2m\alpha_2 \quad \text{and:}$$

$$W_\eta=\sqrt{2m}\int d\eta\sqrt{\frac{\alpha_1\eta^2+(A-B)\eta+\alpha_2}{\eta^2-a^2}}; \quad \text{So Hamilton principal function is given by}$$

$$\boxed{S(\xi,\eta,t)=\sqrt{2m}\int d\xi\sqrt{\frac{\alpha_1\xi^2+(A+B)\xi+\alpha_2}{\xi^2-a^2}}+\sqrt{2m}\int d\eta\sqrt{\frac{\alpha_1\eta^2+(A-B)\eta+\alpha_2}{\eta^2-a^2}}-\alpha_1 t;}$$

Problem 2 A Hamiltonian system with two degrees of freedom is defined by the Hamiltonian

$$H=\frac{1}{2}p_1^2+\frac{1}{2}\left(\frac{1}{2}p_2^2+\frac{1}{2}q_2^2\right)^2q_1^2;$$

ⓐ Find a complete solution of the HJE in terms of 1d integrals

Let's introduce Hamiltons principal function $S(q_1,q_2,t)$. HJE for this system is given by

$$\frac{1}{2}\left(\frac{\partial S}{\partial q_1}\right)^2+\frac{1}{2}\left(\frac{1}{2}\left(\frac{\partial S}{\partial q_2}\right)^2+\frac{1}{2}q_2^2\right)^2q_1^2+\frac{\partial S}{\partial t}=0;$$

Hamiltonian is time-independent so we as usually make ansatz: $S(q_1,q_2,t)=W_1(q_1)+W_2(q_2)-\alpha_1 t$, so that

$$\frac{1}{2}(W_1')^2+\frac{1}{2}\left(\frac{1}{2}(W_2')^2+\frac{1}{2}q_2^2\right)^2q_1^2-\alpha_1=0, \quad \text{so we get:}$$

$$\underbrace{\frac{1}{4}\left((W_2')^2+q_2^2\right)^2}_{\alpha_2^2}=\underbrace{\frac{1}{q_1^2}\left(2\alpha_1-(W_1')^2\right)}_{\alpha_2^2}$$

- here as usually l.h.s. depends on q_2 while r.h.s. - on q_1 so we put both of them to be constant α_2^2, so that

$$(W_2')^2=2\alpha_2-q_2^2 \Rightarrow W_2(q_2)=\int\sqrt{2\alpha_2-q_2^2}\,dq_2;$$

$$(W_1')^2=2\alpha_1-\alpha_2^2q_1^2 \Rightarrow W_1(q_1)=\int\sqrt{2\alpha_1-\alpha_2^2q_1^2}\,dq_1;$$

④ So principal function of this Hamiltonian system is

$$\boxed{S(q_1, q_2, t) = \int \sqrt{2\alpha_2 - q_2^2}\, dq_2 + \int \sqrt{2\alpha_1 - \alpha_2^2 q_1^2}\, dq_1 - \alpha_1 t;}$$

ⓑ Draw the phase portraits on the phase planes (p_1, q_1) and (p_2, q_2)

Using identities for principal function's partial derivative $p_i = \frac{\partial S}{\partial q_i}$ we can find out

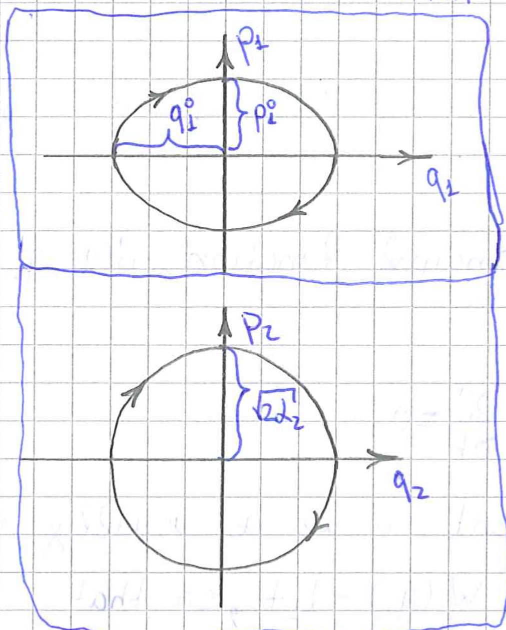
$$p_1 = \frac{\partial S}{\partial q_1} = \sqrt{2\alpha_1 - \alpha_2^2 q_1^2}; \quad p_2 = \frac{\partial S}{\partial q_2} = \sqrt{2\alpha_2 - q_2^2};$$

So that we get the following relations between canonical momentum and coordinates:

$$\boxed{\begin{aligned} p_1^2 + \alpha_2^2 q_2^2 &= 2\alpha_1 \\ p_2^2 + q_2^2 &= 2\alpha_2 \end{aligned}}$$

* phase portrait in (p_1, q_1)-plane

$p_1^2 + \alpha_2^2 q_2^2 = 2\alpha_1$ is equation for ellipse with axes length $p_1^0 = \sqrt{2\alpha_1}$; $q_1^0 = \frac{\sqrt{2\alpha_1}}{\alpha_2}$;



* phase portrait in (p_2, q_2)-plane

$p_2^2 + q_2^2 = 2\alpha_2$ - equation for the circle of radius of $\sqrt{2\alpha_2}$.

ⓒ Introduce action angle variables and express the Hamiltonian as a function of action variables J_1 and J_2. Find frequencies ν_1 and ν_2 as functions of the action variables.

Action variable is defined for systems with periodic motion as follows $J_i = \oint p_i\, dq_i$ where integration is done over full period of motion. For time-independent integral

(5) we can say that: $\alpha_1 = const = H(J)$ and J is function of α_1. Hamiltonian characteristic function can then be chosen to be $W = W(q, J)$. Then the generalized coordinate conjugate to J is $w = \frac{\partial W}{\partial J}$ and equation of motion for w is $\dot{w} = \nu(J) = \frac{\partial H}{\partial J}$. Physically $\nu(J)$ is frequency of oscillations (i.e. periodic motion - it can be either libration (oscillations) or rotation).

In our case:

$$J_1 = \oint p_1 dq_1 = \oint \sqrt{2\alpha_1 - \alpha_2^2 q_1^2}\, dq_1 = \sqrt{2\alpha_1} \oint \sqrt{1 - \frac{\alpha_2^2}{2\alpha_1} q_1^2}\, dq_1$$

we here make change of variables $\frac{\alpha_2^2}{\alpha_1} q_1^2 = \sin^2\theta_1$, then:

$$\sqrt{1 - \frac{\alpha_2^2}{2\alpha_1} q_1^2} = \cos\theta_1 \,;\ dq_1 = \frac{\sqrt{2\alpha_1}}{\alpha_2} \cos\theta_1 \, d\theta_1 \quad \text{and}$$

$$J_1 = \frac{2\alpha_1}{\alpha_2} \oint \cos^2\theta_1 \, d\theta_1 = \frac{\alpha_1}{\alpha_2} \oint (1 - \sin 2\theta_1) d\theta_1 = \frac{2\alpha_1 \pi}{\alpha_2} \,; \quad J_1 = \frac{2\alpha_1 \pi}{\alpha_2};$$

and $J_2 = \oint \sqrt{2\alpha_2 - q_2^2}\, dq_2 = \sqrt{2\alpha_2} \oint \sqrt{1 - \frac{q_2^2}{2\alpha_2}}\, dq_2$; $q_2 = \sqrt{2\alpha_2} \sin\theta$ is our change of variables. $J_2 = \oint \cos^2\theta_2 \, d\theta_2 2\alpha_2 = \frac{2\alpha_2}{2} \cdot 2\pi = 2\pi\alpha_2$; $J_2 = 2\pi\alpha_2$;

So $\boxed{J_1 = \frac{2\alpha_1 \pi}{\alpha_2} \,;\ J_2 = 2\pi\alpha_2;}$ From here we find that

$\alpha_1 = \frac{\alpha_2}{2\pi} J_1 = \frac{J_1 J_2}{4\pi^2}$, so that Hamiltonian written in terms of action variables looks like $\boxed{H = \frac{J_1 J_2}{4\pi^2};}$

And finally we are able to find frequencies of periodic motion: $\nu_1 = \frac{\partial H}{\partial J_1} = \frac{1}{4\pi^2} J_2$; $\nu_2 = \frac{\partial H}{\partial J_2} = \frac{1}{4\pi^2} J_1$ so that:

$$\boxed{\nu_1 = \frac{J_2}{4\pi^2} \,;\ \nu_2 = \frac{J_1}{4\pi^2};}$$

① Seminar 9 (adiabatic invariants + perturbation theory)

Theory

Adiabatic Invariants

Assume we perturb Hamiltonian of the system in the following manner:

$H_0(J, w) = E(J) \longrightarrow H(J, w) = H_0(J, w) + \varepsilon h(J, w)$, here (J, w) - are action-angle variables, expanding $h(J, w)$ in Fourier sum:

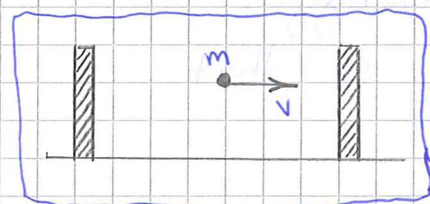
$h(J, w) = \sum_{n \in \mathbb{Z}} h_n e^{inw}$, canonical equation for J reads

$\dot{J} = -\frac{\partial H}{\partial w} = -i\varepsilon \sum_{n \in \mathbb{Z}} h_n \cdot n \cdot e^{inw}$ integration gives:

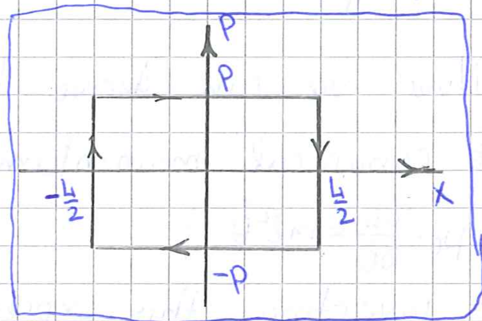
$J(T) - J(0) = -i\varepsilon \sum_{n \in \mathbb{Z}} n \cdot h_n \underbrace{\int_0^T dt\, e^{in(\nu t + \delta)}}_{0} = 0 + O(\varepsilon^2)$ - result is valid up to ε^2 as we have used zero order expansion term for w - angle variable. So in first order in perturbation action variable doesn't change. So action variable is an adiabatic invariant. This can be used in qualitative description of system.

Problem 1 An elastic ball is oscillating between two very slowly moving walls

ⓐ Find an action variable in this system.



First let's assume that walls don't move then the phase portrait look like on the picture below. By definition



action variable is given by:

$J = \oint p\,dx = \int_{-\frac{L}{2}}^{\frac{L}{2}} p\,dx + \int_{\frac{L}{2}}^{-\frac{L}{2}} (-p)\,dx =$

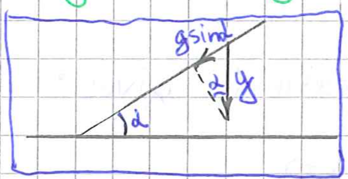
$= 2\int_{-\frac{L}{2}}^{\frac{L}{2}} p\,dx = 2pL;$ $\boxed{J = 2pL;}$

② ⓑ How the velocity of the ball depends on the distance $L(t)$ between walls.

as action variable is adiabatic invariant we can find

$v = \frac{p}{m} = \frac{J}{2mL}$ as J is constant we get $\boxed{v \sim \frac{1}{L(t)};}$

Problem 2 A plane pendulum of small amplitude is constrained to move on an inclined plane. How does the amplitude change when the inclination angle α of the plane is changed slowly?



Potential energy of pendulum is given by $V = mgl\sin\alpha(1-\cos\theta)$, where θ is the angle of deviation from equilibrium position. Then Lagrangian of pendulum is given by $\boxed{L = \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}mgl\sin\alpha\cdot\theta^2}$ Euler-Lagrange equations for this Lagrangian give

$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = ml^2\ddot{\theta} + mgl\sin\alpha\cdot\theta = 0$ so that $\ddot{\theta} = -\frac{g}{l}\sin\alpha\cdot\theta$

This is oscillation equation and it's solution is given by $\theta = A\cos(\omega t+\delta)$ where A is amplitude of oscillations. To find it we assume $\dot{\theta} = 0$ so that kinetic energy vanishes and all energy of the system equals it's potential energy:

$\frac{1}{2}mgl\sin\alpha\cdot A^2 = E \Rightarrow A = \sqrt{\frac{2E}{mgl\sin\alpha}}$ so that

$$\boxed{\theta = \sqrt{\frac{2E}{mgl\sin\alpha}}\cos(\omega t+\delta);}$$

Now we can derive Hamiltonian of the system:

* canonical momentum is given by

$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta}$;

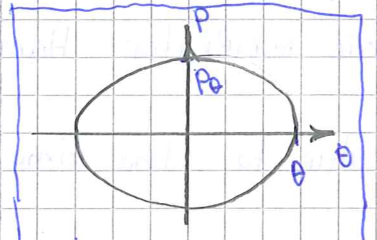
* inverting this expression we get: $\dot{\theta} = \frac{p_\theta}{ml^2}$;

(3) * Legendre transformation gives $H = p_\theta \cdot \dot{\theta} - L \Rightarrow$

$$\Rightarrow \boxed{H = \frac{p_\theta^2}{2ml^2} + \frac{1}{2} mgl \sin\alpha \cdot \theta^2;}$$

as Hamiltonian is time-independent energy is conserved and given by

$E = \frac{p_\theta^2}{2ml^2} + \frac{1}{2} mgl \cdot \sin\alpha \cdot \theta^2$, so that phase portrait is ellipse with axes $\theta_0 = \sqrt{\frac{2E}{mgl\sin\alpha}}$; $p_0 = l\sqrt{2mE}$;



Phase portrait

Action variable by definition is given by

$$J = \oint p_\theta d\theta = \sqrt{2ml^2E} \oint \sqrt{1 - \frac{mgl\sin\alpha}{2E}\theta^2}\, d\theta =$$

$$= \sqrt{2ml^2E} \cdot \sqrt{\frac{2E}{mgl\sin\alpha}} \oint \cos^2\varphi\, d\varphi = 2\pi E \sqrt{\frac{l}{g\sin\alpha}}$$ so

$\boxed{J = 2\pi E \sqrt{\frac{l}{g\sin\alpha}};}$ so we conclude that $E = \frac{J}{2\pi}\sqrt{\frac{g\sin\alpha}{l}}$.

Substituting this value of energy into expression of amplitude we get: $A = \sqrt{\frac{2E}{mgl \cdot \sin\alpha}} = \sqrt{\frac{J}{m\pi}} \frac{1}{(gl^3\sin\alpha)^{1/4}}$, so

$\boxed{A(t) \sim \sin^{-\frac{1}{4}}\alpha(t);}$

Theory

Time-independent perturbation theory.

Assume we perturbe Hamiltonian in the following way $H(w_0, J_0) \to H(w_0, J_0, \varepsilon)$ $H_0(w_0, J_0) + \varepsilon H_1(w_0, J_0) + \ldots$ we say that energy in this case is perturbed in similar way: $E(J_0; \varepsilon) = E_0(J) + \varepsilon E_1(J) + \ldots$, so perturbed frequencies of periodic motion are given by:

$\nu = \nu_0 + \varepsilon \frac{\partial E_1}{\partial J} + \ldots$. Correction to energy $E_1(J)$ can be easily found in the following way:

$E_1(J) = \overline{H_1(w_0, J)}$ where bar denotes an average over the periods of all angle variables w_0.

Problem 3 To the lowest order in correction terms, the relativistic Hamiltonian for 1d harmonic oscillator has the form

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) - \frac{p^4}{8m^3c^4};$$

Calculate the lowest order relativistic correction to the frequency of the harmonic oscillator.

So unperturbed Hamiltonian is given by

$H_0 = \frac{1}{2m}(p^2 + m^2\omega^2 q^2)$ - usual harmonic oscillator Hamiltonian.

For unperturbed system we get due to the time independence of hamiltonian

$\frac{1}{2m}(p^2 + m^2\omega^2 q^2) = E_0$ so that $p = \sqrt{2mE_0 - m^2\omega^2 q^2}$;

and thus angle variable by definition is given by

$$J = \oint p\,dq = \sqrt{2mE_0}\oint\sqrt{1 - \frac{m\omega^2}{2E_0}q^2}\,dq = \frac{2E_0}{\omega}\int_0^{2\pi}\cos^2\phi\,d\phi = \frac{2\pi E_0}{\omega}$$

So action variable is given by $\boxed{J = \frac{2\pi E}{\omega};\ H_0(J) = \frac{\omega J}{2\pi}}$

Solution for oscillator is given by: $p = p_0 \sin(\omega t + \delta)$

with the amplitude $p_0 = \sqrt{2mE_0} = \sqrt{\frac{m\omega J}{\pi}}$

Angle variable is given by:

$\dot{w} = \frac{\partial H}{\partial J} = \frac{\partial E}{\partial J} = \frac{\omega}{2\pi} = \nu$ so that choosing integration constant in apropriate way we get.

$p = \sqrt{\frac{m\omega J}{\pi}}\sin(2\pi w)$ and thus Hamiltonian corrections in terms of action-angle variables takes form:

$$\Delta H = -\left(\frac{m\omega J}{\pi}\right)^2 \frac{\sin^4(2\pi w)}{8m^3c^2} \Rightarrow \boxed{\Delta H = -\frac{J^2\omega^2}{8\pi^2 mc^2}\sin^4(2\pi w)}$$

Now we are able to find E_1 averaging ΔH over a period of angle variable $w \in [0;1)$; $E_1 = \overline{\Delta H} \Rightarrow$

$$\Rightarrow E_1 = -\frac{J^2\omega^2}{8\pi^2 mc^2}\int_0^1 \sin^4(2\pi w)\,dw = -\frac{J^2\omega^2}{8\pi^2 mc^2}\int_0^1 \frac{1}{4}(1 - \cos(4\pi w))^2\,dw =$$

$$= -\frac{J^2\omega^2}{8\pi^2 mc^2}\cdot\frac{1}{4}\int_0^1 \Big(1 \underbrace{- 2\cos(4\pi w)} + \frac{1}{2} + \underbrace{\frac{1}{2}\cos(8\pi w)}\Big)\,dw$$ so we get

give "0" after integration

(5)

$$\boxed{E_1 = -\frac{3}{64}\frac{J^2\omega^2}{\pi^2 m c^2}};$$ and then the shift in frequency is given by:

$$\Delta\nu = \frac{\partial E_1}{\partial J} = -\frac{3J\omega^2}{32\pi^2 c^2 m}; \quad \boxed{\Delta\nu = -\frac{3J\omega^2}{32\pi^2 c^2 m};}$$

Problem 4 A plane isotropic harmonic oscillator is perturbed by a change in the Hamiltonian of the form $\Delta H = \varepsilon p_x^2 p_y^2$. Find the shifts in frequencies in the first order in ε.

Unperturbed Hamiltonian is given by

$H_0 = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega_0(x^2+y^2)$ and we get formulas similar with previous problem

$$J_x = \frac{2\pi E_x}{\omega}; \quad J_y = \frac{2\pi E_y}{\omega}; \quad E_x + E_y = E = \text{const}.$$

the problem now is with angle variables

$\dot{w}_x = \frac{\partial H_0}{\partial J_x} = \dot{w}_y = \frac{\omega}{2\pi}$ so frequencies in x and y directions are the same and so are w_x and w_y looks the same. Way to avoid it is to make point transformation such that $w_0 = 0$ and $w_1 \neq 0 = \omega t + \delta$ so that

$p_x = \sqrt{\frac{mJ_x\omega_0}{\pi}}\cos(2\pi w_1)$; $p_y = \sqrt{\frac{mJ_y\omega_0}{\pi}}\cos(2\pi w_1)$ Then perturbation Hamiltonian is given by

$\Delta H = \varepsilon\frac{m^2\omega_0^2 J_x J_y}{\pi^2}\cos^4(2\pi w_1)$ as we have seen in previous problem $\overline{\cos^4(2\pi w_1)} = \frac{3}{8}$, so $E_1 = \overline{\Delta H} = \frac{3\varepsilon}{8\pi^2}m^2\omega_0^2 J_x J_y$ so that shifts in frequencies are given by:

$\Delta\nu_x = \frac{\partial E_1}{\partial J_x} = \frac{3\varepsilon}{8\pi^2}m^2\omega_0^2 J_y$; $\Delta\nu_y = \frac{\partial E_1}{\partial J_y} = \frac{3\varepsilon}{8\pi^2}m^2\omega_0^2 J_x$; so the answer is given by

$$\boxed{\begin{aligned}\Delta\nu_x &= \frac{3\varepsilon}{8\pi^2}m^2\omega_0^2 J_y;\\ \Delta\nu_y &= \frac{3\varepsilon}{8\pi^2}m^2\omega_0^2 J_x;\end{aligned}}$$

1

Seminar 11 (exam 2010)

Problem 11 $\lambda_R = 7{,}5\cdot 10^{-7}\,m$; $\lambda_g = 5\cdot 10^{-7}\,m$; $\Delta t = 14\,s$.

velocity of police cruiser is $u = \frac{12}{13}c$.

ⓐ How fast was the space-ship traveling with respect to the rest frame of the red-light?

Here we use Dopler effect formula

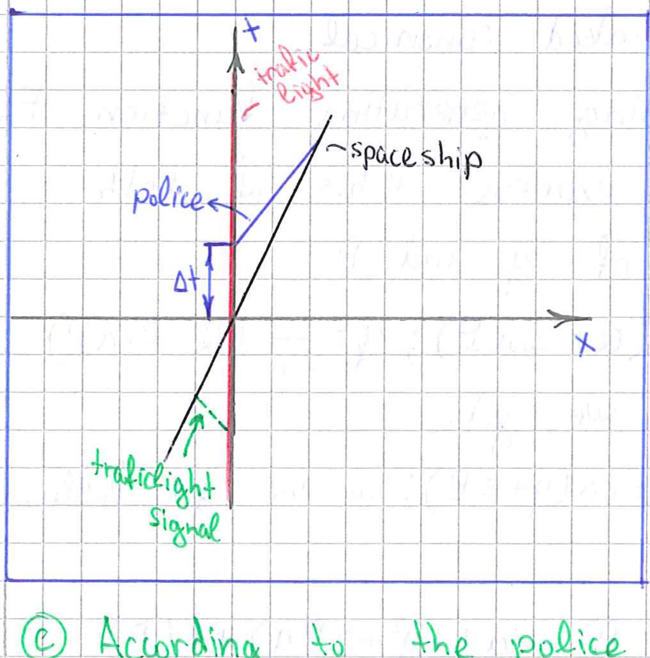
$\frac{\nu'}{\nu} = \sqrt{\frac{c+v}{c-v}}$ for observer moving towards source of light.

$\frac{\nu'}{\nu} = \frac{\lambda_R}{\lambda_g} = \sqrt{\frac{c+v}{c-v}}$ (we used here the fact that $\nu \sim \frac{1}{\lambda}$)

$\frac{\lambda_R^2}{\lambda_g^2} = \frac{c+v}{c-v} \Rightarrow \frac{1+\beta}{1-\beta} = \frac{\lambda_R^2}{\lambda_g^2}$ and finally $v = c\,\frac{\lambda_R^2 - \lambda_g^2}{\lambda_g^2 - \lambda_R^2}$ or

$$v = c\,\frac{(\lambda_R/\lambda_g)^2 - 1}{(\lambda_R/\lambda_g)^2 + 1} = c\,\frac{\frac{9}{4} - 1}{\frac{9}{4} + 1} \Rightarrow \boxed{v = \frac{5}{13}c;}$$

ⓑ Draw a space-time diagram that includes the world-lines of the red-light, the spaceship and the police cruiser. Also include a world-line for a light ray traveling from the red-light to the spaceship.



ⓒ According to the police cruiser's clock, how long did it take the police cruiser to catch the spce-craft once it started chasing it?

Police cruiser starts chasing space-craft after Δt period

(2) of time when space-craf is $\Delta x = V\Delta t = \frac{5}{13} c \Delta t$ away from traffic-light in it's rest frame.

In this frame it takes $t = \frac{V\Delta t}{u-V} = \frac{\frac{5}{13}c\Delta t}{\left(\frac{12}{13}-\frac{5}{13}\right)c} = \frac{5}{7}\Delta t$

$t = 10s$. Finally in police cruiser's rest frame due to time-dilation we get $\tau = \frac{t}{\gamma(u)} = \frac{5t}{13} = \frac{50}{13}$ s. So

$\boxed{\tau = \frac{50}{13} s.}$

Problem 2.1 Coordinate transformation is given by

$Q = p + iaq$; $P = \frac{1}{2ia}(p - iaq)$ where "a" is constant

Show that this transformation is canonical and find it's generating function. Apply this transformation to harmonic oscillator $H(q,p)$ and use it to find equation of motion $q(t)$, $p(t)$. You may choose the constant a to have a convinient value.

First of all let's show that transformation is indeed canonical looking on fundamental Poisson Brackets $\{Q, P\} = \frac{1}{2ia}\{p+iaq; p-iaq\} = \frac{1}{2}(\{q,p\} - \{p,q\}) = 1$ so transformation is indeed canonical.

Let's find corresponding generating function $F(q, P)$

$p = \frac{\partial F}{\partial q}$; $Q = \frac{\partial F}{\partial P}$; Let's express l.h.s of both equations in terms of q and P

First of all $p = \frac{1}{2}(Q + 2iaP)$; $q = \frac{1}{2ia}(Q - 2iaP)$

from second equation we get:

$Q = 2ia(q + P)$; so $p = ia(q + 2P)$; so we get following differential equations

$ia(q + 2P) = \frac{\partial F}{\partial q} \Rightarrow F(q, P) = ia\left(\frac{1}{2}q^2 + 2Pq\right) + f(P)$

Then second equation turns to be

$Q = \frac{\partial F}{\partial P} = \cancel{2iaq} + f'(P) = \cancel{2iaq} + 2iaP$ so that $f(P) = iaP^2$

③ So we get $\boxed{F(P,q) = ia\left(\frac{1}{2}q^2 + 2Pq + P^2\right);}$

So transformation is time-independent so Hamiltonian can be simply obtained by the change of variables so that $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 = \frac{1}{8m}(Q + 2iaP)^2 - \frac{m\omega^2}{8a^2}(Q - 2iaP)^2$

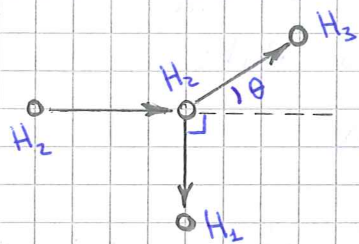
Convinient choise of constant in this case is $\frac{m\omega^2}{a^2} = \frac{1}{m}$ so that $a = m\omega$, so that $H = \frac{1}{8m} 8iaQP = i\omega QP$ so that we get $H = i\omega QP$ and canonical equations are given by

$\dot{Q} = \frac{\partial K}{\partial P} = i\omega Q$; $\dot{P} = -i\omega P$; Solutions of this equations are given by:

$Q = C_1 e^{i\omega t}$; $P = C_2 e^{-i\omega t}$; finally going to old coordinates

$$\boxed{\begin{aligned} p(t) &= \frac{1}{2}\left(C_1 e^{i\omega t} + 2iaC_2 e^{-i\omega t}\right); \\ q(t) &= \frac{1}{2ia}\left(C_1 e^{i\omega t} - 2iaC_2 e^{-i\omega t}\right); \end{aligned}}$$

Problem 1.2



① H_2 – deutron, rest energy $m_2c^2 = 1876{,}1$ MeV

$T_2 = 1{,}8$ MeV – kinetic energy.

② H_2 – second deutron in rest

③ after reaction take place

H_1 – hydrogen nucleus, rest energy $m_1c^2 = 938{,}8$ MeV kinetic energy $T_1 = 3{,}5$ MeV at $90°$ angle with initial deutron motion

④ H_3 – tritium nucleus going at angle θ

ⓐ Find rest energy of H_3

Let's write down 4-momentum conservation for this reaction

$$\underset{H_2}{p_1^\mu} + \underset{H_2}{p_2^\mu} = \underset{H_1}{p_3^\mu} + \underset{H_3}{p_4^\mu}$$

$H_2 + H_2 \rightarrow H_1 + H_3$

we know that tritium rest energy is given by $p_4^2 = m_3^2c^2$ so what we should do is rewrite 4-momentum conservation in the

④ following form $p_4^\mu = p_1^\mu + p_2^\mu - p_3^\mu$ and square it so that:

$$p_4^2 = m_3^2c^2 = (p_1+p_2-p_3)^2 = \underbrace{p_1^2}_{m_1^2c^2} + \underbrace{p_2^2}_{m_2^2c^2} + \underbrace{p_3^2}_{m_1^2c^2} + 2p_1p_2 - 2p_1p_3 - 2p_2p_3;$$

We can write down all momentum:

$p_1^\mu = (m_2c + \frac{T_2}{c}, p_1, 0, 0)$; $p_2^\mu = (m_2c, 0, 0, 0)$; $p_3^\mu = (m_1c + \frac{T_1}{c}, 0, p_3, 0)$;

$p_4^\mu = (m_3c + \frac{T_3}{c}, p_4\cos\theta, p_4\sin\theta, 0, 0)$; so that

$p_1 \cdot p_2 = m_2^2c^2 + m_2T_2$; $p_1p_3 = m_1m_2c^2 + m_2T_1 + m_1T_2 + \frac{T_1T_2}{c^2}$;

$p_2 \cdot p_3 = m_1m_2c^2 + m_2T_1$; substituting all this we get:

$$m_3^2c^2 = 2m_2^2c^2 + m_1^2c^2 + 2m_2^2c^2 + 2m_2T_2 - 2m_1m_2c^2 - 2m_2T_1 - 2m_1T_2 - 2\frac{T_1T_2}{c^2} - 2m_1m_2c^2 - 2m_2T_1$$, thus

$$\boxed{m_3c^2 = \left\{(2m_2c^2 - m_1c^2)^2 + 2m_2c^2T_2 - 4m_2c^2T_1 - 2m_1c^2T_2 - 2\frac{T_1T_2}{c}\right\}^{\frac{1}{2}}}$$

then $m_3c^2 = 2809,3\,MeV$

ⓑ Find the kinetic energy of H_3

Kinetic energy can be found from energy conservation law:

$2m_2c^2 + T_2 = T_3 + m_3c^2 + m_1c^2 + T_1$ so that:

$$\boxed{T_3 = 2m_2c^2 + T_2 - m_1c^2 - T_1 - m_3c^2 = 2{,}4\,MeV}$$

Problem 2.2. A spherical pendulum is a particle of mass "m" which is constrained to move on the surface of a sphere under the influence of the usual gravitational attraction from the earth.

* Write down the Lagrangian in spherical coordinates

Lagrangian of the particle is given by:

$x = r\sin\theta \cdot \cos\varphi$ $\quad L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$;

$y = r\sin\theta \cdot \sin\varphi$ $\quad$ so $\dot{x} = r\cos\theta \cdot \cos\varphi \cdot \dot{\theta} - r\sin\theta \cdot \sin\varphi \cdot \dot{\varphi}$;

$z = r \cdot \cos\theta$ $\quad \dot{y} = r\cos\theta \cdot \sin\varphi \cdot \dot{\theta} + r\sin\theta \cdot \cos\varphi \cdot \dot{\varphi}$;

$\dot{z} = -r\sin\theta \cdot \dot{\theta}$;

(5) So $\dot{x}^2+\dot{y}^2+\dot{z}^2 = r^2\cos^2\theta\cdot\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2 + r^2\sin^2\theta\cdot\dot{\theta}^2 = r^2\dot{\theta}^2 + r^2\sin^2\theta\cdot\dot{\varphi}^2$

and Lagrangian

$$\boxed{L = \frac{mr^2\dot{\theta}^2}{2} + \frac{mr^2\sin^2\theta\cdot\dot{\varphi}^2}{2} + mgr\cos\theta;}$$

Now we can derive Hamiltonian of the system

* $p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mr^2\dot{\theta}$; $p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = mr^2\sin^2\theta\,\dot{\varphi}$;

* inverting this expressions we get $\dot{\theta} = \frac{p_\theta}{mr^2}$; $\dot{\varphi} = \frac{p_\varphi}{mr^2\sin^2\theta}$;

* now we can make Legendre transformation

$$H = p_\theta\dot{\theta} + p_\varphi\dot{\varphi} - L = \frac{p_\theta^2}{2mr^2} + \frac{p_\varphi^2}{2mr^2\sin^2\theta} - mgr\cos\theta; \text{ so}$$

$$\boxed{H = \frac{p_\theta^2}{2mr^2} + \frac{p_\varphi^2}{2mr^2\sin^2\theta} - mgr\cos\theta;}$$

φ variable is cyclic so we immedeately conclude that $p_\varphi = \text{const} = j$ and thus:

$$H = \frac{p_\theta^2}{2mr^2} + \frac{j^2}{2mr^2\sin^2\theta} - mgr\cos\theta;$$ is effective Hamiltonian

Now we introduce Hamilton's principal function $S(\theta,t) = W_\theta(\theta) - \alpha_1 t$ because Hamiltonian is time-independent.

And thus $$\frac{1}{2mr^2}(W_\theta')^2 + \frac{\dot{j}^2}{2mr^2\sin^2\theta} - mgr\cos\theta - \alpha_1 = 0;$$

So that $$W_\theta' = \sqrt{2mR^2\alpha_1 - \dot{j}^2/\sin^2\theta + 2m^2gR^3\cos\theta}$$

$$\boxed{S = \int\sqrt{2mR^2\alpha_1 - \frac{\dot{j}^2}{\sin^2\theta} + 2m^2gR^3\cos\theta}\,d\theta - \alpha_1 t + j\varphi}$$

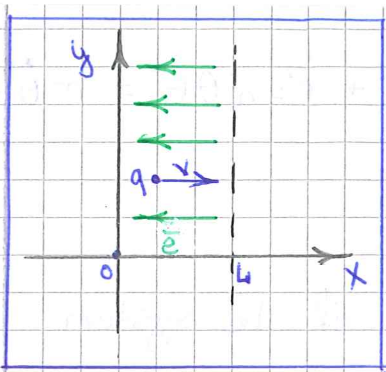
j is angular momentum of rotation in xy-plane.

Problem 1.3 Electric field is given by

$\bar{e} = -E\hat{x}$, $0 \leq x \leq L$; $\bar{e} = 0$ $x<0$ or $x>L$; as seen by observer in S. Furthe suppose there is a particle with charge q and relativistic velocity $\bar{V} = V\cdot\hat{x}$ as measured by the observer in S that enters into the field at $x=0$.

ⓐ Find the force acting on the particle as seen by the observer in S when the particle is in the electric field.

⑥



Force is given by Lorenz formula

$\overline{F} = q\overline{E} = -qE\hat{x}$; $\boxed{\overline{F} = -qE\hat{x};}$

ⓑ Find the force on the particle as seen by an observer in the rest frame of the particle when it is in the electric field.

Boost is along x-axis. Lorentz transformation for the components of the electric field looks like

$E'_x = E_x$; $E'_y = \gamma(E_y - \frac{v}{c}B_z)$; $E'_z = \gamma(E_z + \frac{v}{c}B_y)$;

$B'_x = B_x$; $B'_y = \gamma(B_y + \frac{v}{c}E_z)$; $B'_z = \gamma(B_z - \frac{v}{c}E_y)$;

So we see that electromagnetic field is not changed after Lorentz transformation. So that $\bar{e}' = -Ex'$ and force is just the same $\overline{F} = -qE\hat{x}$;

ⓒ Find the proper acceleration of the particle.

In particle instanteneous frame as we have seen force is given by $\overline{F} = -qE\hat{x}$; so that proper acceleration is given by $\bar{a} = \frac{1}{m}\bar{f} = -\frac{qE}{m}\hat{x} \Rightarrow \boxed{\bar{a} = -\frac{qE}{m}\hat{x};}$

ⓓ Find the velocity of the particle as measured by the observer in S when the particle exits electric field at $x = L$

Here we will use the fact that electromagnetic (Lorentz) force is pure force $u_\mu F^\mu = \gamma^2\left(\frac{dE}{dt} - \bar{v}\cdot\bar{f}\right)$ so that

$\frac{d}{dt} = \bar{v}\cdot\bar{f}$, where $\mathcal{E}$ is the energy of the particle

so that $\frac{d\mathcal{E}}{dt} = -qvE \Rightarrow \Delta\mathcal{E} = \int dt(-qvE) = -\int_0^L dx\, qE = -qEL$

so $\Delta E = -qEL$ is the change in particles energy after going through electric field. So that $\Delta E = m\gamma_1 c^2 - m\gamma c^2 = -qLE \Rightarrow$

$\Rightarrow \gamma_1 = \gamma - \frac{qLE}{mc^2}$; $\gamma_1 = \frac{1}{\sqrt{1 - \frac{v_1^2}{c^2}}}$ – particles γ-factor after going through electric fielv and $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ – before, so that we finally get: $\frac{1}{\gamma_1^2} = \frac{1}{\gamma^2\left(1 - \frac{qLE}{\gamma mc^2}\right)^2}$; $\Rightarrow \boxed{v_1 = c\sqrt{1 - \frac{1 - \frac{v^2}{c^2}}{\left(1 - \frac{qLE}{\gamma mc^2}\right)^2}}}$

(7) A particle of mass m moves in a plane in a square well potential

Problem 2.3

$$V(r)=\begin{cases}-V_0 & , 0<r<r_0\\ 0 & , r>r_0\end{cases}$$

Under what initial conditions can the method of action-angle variables be applied?

Assuming these conditions are met, use the method of action-angle variables to find the frequencies of motion.

This problem should be considered in polar coordinates $\begin{cases}x=r\cos\varphi\\ y=r\sin\varphi\end{cases}$; $T=\frac{m}{2}(\dot{x}^2+\dot{y}^2)=\frac{m}{2}(\dot{r}^2+r^2\dot{\varphi}^2)$ so Lagrangian of the system is:

$$L=T-V=\frac{m}{2}(\dot{r}^2+r^2\dot{\varphi}^2)-V(r)$$

(I) Let's first derive Hamiltonian

* canonical momentum is given by

$$\begin{aligned}p_r&=\frac{\partial L}{\partial \dot{r}}=m\dot{r}\\ p_\varphi&=\frac{\partial L}{\partial \dot{\varphi}}=mr^2\dot{\varphi}\end{aligned} \Rightarrow \begin{aligned}\dot{r}&=\frac{p_r}{m};\\ \dot{\varphi}&=\frac{p_\varphi}{mr^2};\end{aligned}$$

* Legendre transformation gives

$H=p_r\cdot\dot{r}+p_\varphi\cdot\dot{\varphi}-L=\frac{p_r^2}{2m}+\frac{p_\varphi^2}{2mr^2}+V(r)$, so Hamiltonian we will be working with is

$$\boxed{H=\frac{p_r^2}{2m}+\frac{p_\varphi^2}{2mr^2}+V(r);}$$

For action-angle variables to be applicable motion of the particle should be confined in $r<r_0$ area. For this to happen we need $T_0<V_0$ where $T_0=\frac{p_r^2(0)}{2m}+\frac{p_\varphi^2(0)}{2mr_0^2}$ is the kinetic energy in the initial moment of time $t=0$; Assuming this condition is satisfied we can proceed with Hamilton-Jacobi procedure.

⑧ * First we introduce Hamiltons principal function

$p_r = \frac{\partial S}{\partial r}$; $p_\varphi = \frac{\partial S}{\partial \varphi}$

* HJE then takes form:

$$\frac{1}{2m}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 + V(r) + \frac{\partial S}{\partial t} = 0;$$

* ansatz takes form

$$S(r,\varphi,t) = W_r(r) + \alpha_2\varphi - \alpha_1 t$$

- Here φ-depend. is due to φ being cyclic variable
- t-dependence of S is due to time-independence of Hamiltonian.

Then we get $\frac{1}{2m}(W_r')^2 + \frac{\alpha_2}{2mr^2} + V(r) - \alpha_1 = 0$

as $r<r_0$, $V=-V_0$ and finally $W_r = \int dr\sqrt{2m(V_0+\alpha_1) - \frac{\alpha_2}{r^2}}$, and

$$\boxed{S(r,\varphi,t) = \int dr\sqrt{2m(V_0+\alpha_1) - \frac{\alpha_2^2}{r^2}} + \alpha_2\varphi - \alpha_1 t;}$$

The canonical momentum is given by

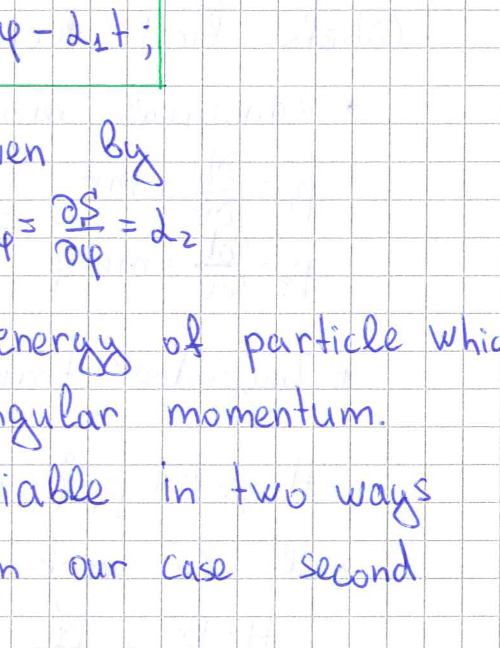
$p_r = \frac{\partial S}{\partial r} = \sqrt{2m(V_0+\alpha_1) - \frac{\alpha_2^2}{r^2}}$; $p_\varphi = \frac{\partial S}{\partial \varphi} = \alpha_2$

So we can see that $\alpha_1 = E$ is energy of particle which is conserved, and $\alpha_2 = p_\varphi$ is angular momentum.

We can introduce action variable in two ways

$J = \oint p_r dr$ or $J = \oint r dp_r$. In our case second definition is more convinient

$J = \oint \frac{\alpha_2}{\sqrt{2m(E+V_0) - p_r^2}} dp_r$; This integral is "drawn" on picture $-p_{max} \leq p_r \leq p_{max}$



this turning points are defined

as $p_{max} = \sqrt{2m(E+V_0) - \frac{\alpha_2^2}{r_0^2}}$;

$J = \int_{-p_r}^{p_r} \frac{\alpha_2}{\sqrt{2m(E+V_0) - p_r^2}} dp_r \Rightarrow$ [making change of variables $\sin\varphi = \frac{p_r}{\sqrt{2m(E+V_0)}}$ we get] $\Rightarrow J = \int_{-p_r}^{p_r} \alpha_2 d\varphi = \alpha_2 \arcsin \frac{p_r}{\sqrt{2m(E+V_0)}}\Bigg|_{-p_{max}}^{p_{max}}$

9

Thus $J = 2\alpha_2 \arcsin \frac{p_{max}}{\sqrt{2m(E+V_0)}}$

Then we get $\frac{p_{max}}{\sqrt{2m(E+V_0)}} = \sin \frac{J}{2\alpha_2}$; thus

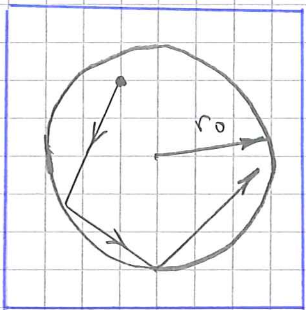
$\sqrt{1 - \frac{\alpha_2^2}{2mr_0^2(E+V_0)}} = \sin \frac{J}{2\alpha_2}$; thus

$H = E = -V_0 + \frac{\alpha_2^2}{2mr_0^2} \cdot \frac{1}{\cos^2 \frac{J}{2\alpha_2}}$; and the frequency is

then given by $\nu = \frac{\partial H}{\partial J} = \frac{\alpha_2}{2mr_0^2} \frac{\tan \frac{J}{2\alpha_2}}{\cos^2 \frac{J}{2\alpha_2}}$;

So we get $\boxed{\nu = \frac{\alpha_2}{2mr_0^2} \cdot \frac{\tan\left(\frac{J}{2\alpha_2}\right)}{\cos^2\left(\frac{J}{2\alpha_2}\right)}}$;

The motion of the particle inside well is free unless it reaches the wall, then it is reflected and goes back



1

Theoretical part (Introduction class)

Ground for relativistic mechanics is the pain of einstein postulates:

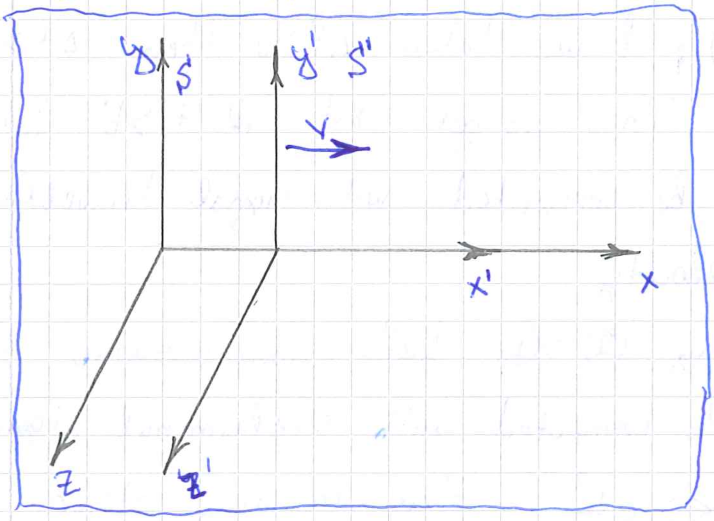
I The laws of physics are identical in any inertial frame

II The speed of light in a vacuum, c, is the same in any inertial frame.

Def An inertial frame is a reference frame with following properties:

* There is a universal time coordinate that can be synchronized.
* Euclidian spatial components
* A body with no forces acting on it will travel at constant velocity according to the clocks and measuring sticks in the inertial frame.

Starting from this 2 postulates we can derive transformation of time and coordinate in two inertial systems. Suppose S' is moving with velocity $\bar{v} = v\hat{x}$ with



respect to S. Then coordinates of some event can be related in this 2 systems in the following way

$$\boxed{\begin{aligned} &t' = \gamma\left(t - \frac{v}{c}x\right); \\ &x' = \gamma(x - vt);\ y' = y;\ z' = z; \end{aligned}}$$

This are the only basic formulas we need for this class

2

Problem 3

Prove that the temporal order of 2 events is the same in all inertial frames if and only if they can be joined in one inertial frame by a signal travelling at or below the speed of light. Illustrate this result on a spacetime diagram.

So, we have 2 events:

$P_1 = (t_1; x_1; y_1; z_1)$ and $P_2 = (t_2; x_2; y_2; z_2)$

and we have 2 statements about this events:

I $t_2' > t_1'$ for any inertial system S';

II P_1 and P_2 can be causally connected by a signal in S travelling with luminal or subluminal speed.

We should prove that I is satisfied if and only if II is satisfied, i.e. we should prove that I $\Rightarrow$ II and II $\Rightarrow$ I;

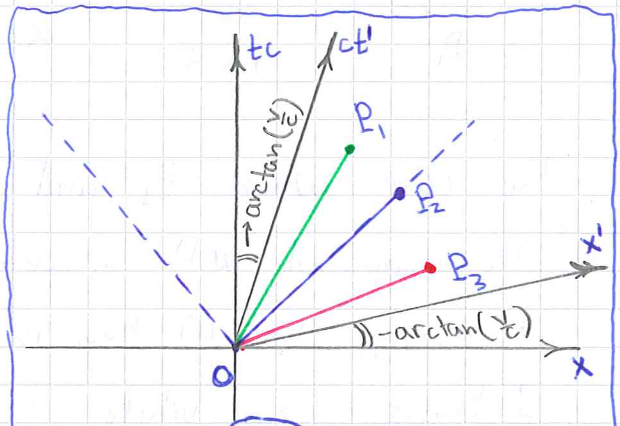
* proof in one way I $\Rightarrow$ II Let's assume $\Delta t' > 0$. If we use Lorentz transformation formula:

$\Delta t' = \gamma(\Delta t - \frac{v}{c^2}\Delta x)$; if we take $\Delta t' > 0$ then $\Delta t > \frac{v}{c^2}\Delta x$ and $\frac{\Delta x}{\Delta t} < \frac{c^2}{v} < c$. Thus we get that if $t_2' > t_1'$ for any S' then they can be connected with signal travelling with subluminal velocity.

* proof in another way II $\Rightarrow$ I Let's now assume that two events can be connected with subluminal signal, so that: $\frac{\Delta x}{\Delta t} < c < \frac{c^2}{v}$ thus in any other inertial system S' we get

$\Delta t' = \gamma(\Delta t - \frac{v}{c^2}\Delta x) = \gamma \Delta t \left(1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t}\right) > 0$, thus we have finally proved statements both way, q.e.d.

(3) Now let's illustrate our result on the diagram. Here it is reasonable to make step aside and explain what is space-time diagram. This is just 2d plot of time vs. one of the spatial coordinates. Trajectory of point picted on the plot is called world-line.



pic 2

Here on the picture we have drawn 3 trajectories OP_1, OP_2 and OP_3 wich can be classified withe the value of interval $\Delta S^2 = c^2\Delta t^2 - \Delta x^2$:

(I) if $\Delta S^2 = 0$ (OP_2) we call trajectory light-like

(II) if $\Delta S^2 > 0$ (OP_1) we call trajectory time-like. This type of trajectories corresponds to moving with subluminal velocities

(III) if $\Delta S^2 < 0$ (OP_3) we call trajectory space-like

Now it is reasonable to ask question:

How can we include axes of another inertial system S' on the same diagram?

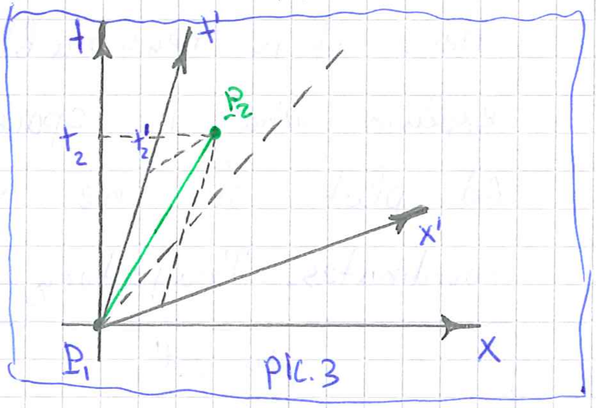
Times and coordinates of some event are related with Lorentz transformations:

$t' = \gamma(t - \frac{v}{c^2}x)$; $x' = \gamma(x - vt)$. Thus x' axis is given by $t' = 0$ or $t = \frac{v}{c^2}x$ and t' axis is given by $x' = 0$ thus $x = vt$ on the coordinate frame (ct, x) we get 2 straight lines $ct = x^0 = \frac{v}{c}x^1$; and $x^0 = \frac{c}{v}x^1$ (t'-axis). This axes are drawn on pic. 2.

Now let's come back to our problem:

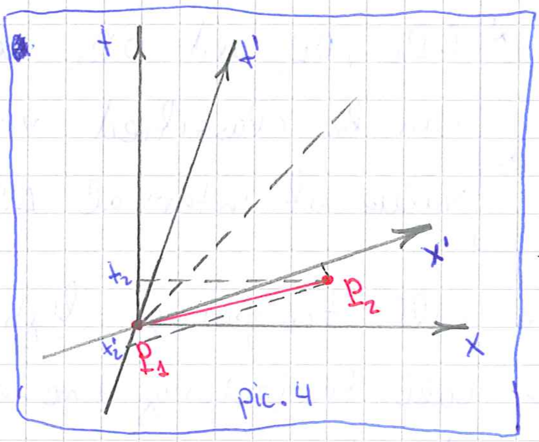
(4) * First let's look what will happen if P_1 and P_2 are connected with the time-like trajectory (pic. 3)

We see from this picture that in both inertial systems $t_2' > t_1'$



pic. 3

* Now let's consider the case when P_1 and P_2 are connected with space-like trajectory (pic. 4) now in S frame $t_2 > t_1$ while in S' frame $t_2 < t_1$ (and we can always find such system S' for any events that can be connected with superluminal signal)



pic. 4

Problem 5

If 2 events occur at the same time in some inertial frame S, prove that there are no limit on the time separation in other frames but that their space separation varies from infinity to a minimum wich is measured in S

S: $\Delta t = 0$; $\Delta x \neq 0$ Now using Lorentz transformation we can write down in some other system S':

$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) = -\frac{v\gamma}{c^2}\Delta x; \quad \Delta x' = \gamma(\Delta x - v\Delta t) = \gamma \Delta x$$

$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \geq 1$ minimum $\gamma = 1$ is obtained when $v = 0$, (wich corresponds to S frame) thus

$$\Delta t' = -\frac{v\gamma}{c^2}\Delta x \in [0; \infty) : \text{as } v \to 0 \ \ \Delta t' \to 0 \quad \text{as } v \to c ; \ \Delta t' \to \infty$$

$$\Delta x' = \gamma \Delta x \in [\Delta x, \infty) : \text{as } v \to 0, \ \Delta x' \to \Delta x, \quad \text{as } v \to c \ \ \Delta x' \to \infty$$

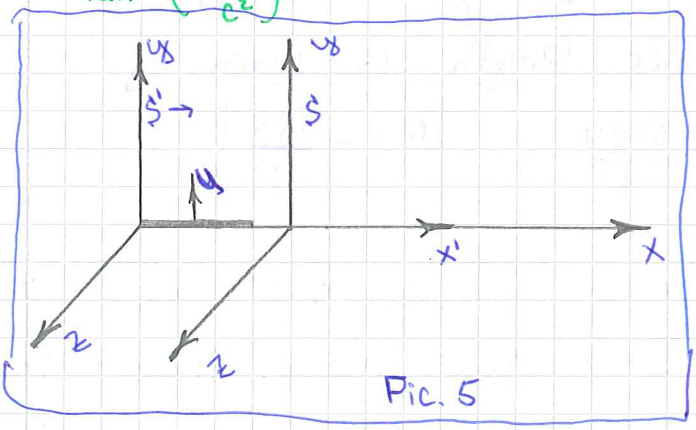
5 and that's exactely what we were asked to show.

Problem 6 In the inertial frame S' the standart lattice clocks all emit "flash" at noon. Prove that in S this flash occurs on a plane orthogonal to the x-axis and travelling in the positive x-direction. Let's $(\Delta t', \Delta x', \Delta y', \Delta z')$ be interval between 2 flashes in S'.
Let's assume $(\Delta t'; \Delta x'; \Delta y'; \Delta z')$ is interval between 2 flashes in S'. We take $\Delta t' = 0$ (flashes are simultaneous)
In S we get

$$\Delta x = \gamma(\Delta x' + v\Delta t') = \gamma \Delta x'; \quad \Delta t = \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) = \gamma \frac{v}{c^2}\Delta x';$$

thus $\frac{\Delta x}{\Delta t} = \frac{c^2}{v}$ at the same time as boost is made along x-direction $\Delta y' = \Delta y$ and $\Delta z' = \Delta z$
Thus indeed observer in S-frame sees flashing plane travelling with the speed $u = \frac{c^2}{v}$ along x-direction.
q.e.d.

Problem 8 In S' straight rod parallel to the x'-axis moves in the y'-direction with constant velocity u. Show that in S the rod is inclined to the x-axis at an angle $-\tan^{-1}\left(\frac{\gamma u v}{c^2}\right)$



Pic. 5

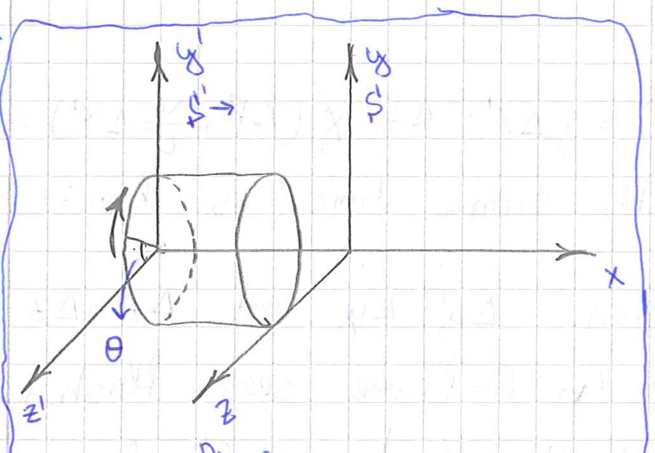
Here we should first imagine what is rigid body for us. The answer is that this is set of points observed simultaneously. And as you remember "simultaneously" can in principle mean different in different reference frames. In S' rod is moving with the speed $u = \frac{\Delta y'}{\Delta t'}$; and $\Delta y' = \Delta y$, as

(6) boost is made along x-axis. at the same time $\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$, $\Delta t = 0$ when we observe two points of rod simultaneously in S thus in S' this events differ by $\Delta t' = -\gamma\frac{v}{c^2}\Delta x$ time interval. angle in S is given by $\tan\theta = \frac{\Delta y}{\Delta x} = \frac{\Delta y'}{\Delta x} \Rightarrow$

$\Rightarrow \tan\theta = -\frac{uv\gamma\Delta t'}{c^2\Delta t'} = -\frac{uv\gamma}{c^2}$ and thus:

$\boxed{\theta = -\arctan\left(\frac{uv\gamma}{c^2}\right)}$, q.e.d.

Problem 9



Pic 6

Assume in S' cylinder is rotationg about x-axis with angular speed ω. Prove that in S cylinder is twisted with the twist per unit length $\frac{\gamma\omega v}{c^2}$

Simultaneous observation in S means $\Delta t = 0$ at the same time in S' we get $\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) = -\frac{\gamma v}{c^2}\Delta x$. in S' we have due to rotation $\Delta\theta' = \omega\Delta t'$ for all x. At the same time as S' is moving along x-axis y'z' is not transformed under this boost and we get $\Delta\theta' = \Delta\theta$;

Number of twists per length is given by:

$N = \frac{\Delta\theta}{\Delta x} = \frac{\omega\Delta t'}{\Delta x} = -\frac{\omega\gamma v}{c^2}$; $\boxed{N = -\frac{\omega\gamma v}{c^2}}$

Problem 10 Two photons travel along x-axis of S with constant distance L between them. Prove that in S' the distance between this photons is $L\sqrt{\frac{c+v}{c-v}}$

Solving this problem, we should be accurate with concept of simultanety again.

(7) * Let's first consider what we get in S system: positions of photons are given by
$x_1 = ct_1$; $x_2 = ct_2 + L$ and $t_2 = t_1$ - that what we mean by distance between photons - i.e. difference in their positions in the same moment of time.

* Now let's go to S' frame. Here we get:

$$t_1' = \gamma\left(t_1 - \frac{v}{c^2}x_1\right) = \gamma t_1\left(1 - \frac{v}{c}\right); \quad t_2' = \gamma\left(t_2 - \frac{v}{c^2}x_2\right) =$$

$$= \gamma t_2\left(1 - \frac{v}{c}\right) - \frac{\gamma v}{c^2}L$$

, now let's inverse this relations to express t_1 and t_2 through t_1' and t_2':

$$t_1 = \frac{ct_1'}{\gamma(c-v)}; \quad t_2 = \frac{ct_2'}{\gamma(c-v)} + \frac{vL}{c(c-v)};$$

Positions of photons in S' are given by Lorenz transform.:

$$x_1' = \gamma(x_1 - vt_1) = \gamma t_1(c-v) = ct_1'; \quad x_1' = ct_1'$$

$$x_2' = \gamma(x_2 - vt_2) = \gamma(c-v)t_2 + L\gamma = ct_2' + \frac{\gamma v}{c}L + \gamma L =$$

$$= ct_2' + L\sqrt{\frac{c+v}{c-v}}$$

, thus $x_2' = ct_2' + L\sqrt{\frac{c+v}{c-v}}$;

And distance between photons in S' is given by difference of simultaneous $(t_1' = t_2')$ positions of this photons in S':

$$L' = x_2'(t_2' = t') - x_1'(t_1' = t') = L\sqrt{\frac{c+v}{c-v}}; \quad \boxed{L' = L\sqrt{\frac{c+v}{c-v}}}$$

q.e.d.

Problem 14 Let's define alternative coordinates $\xi = ct + x$; $\eta = ct - x$, whose axes are $\pm 45°$ lines on diagram (pic 2). Prove that under Lorentz transformations the directions of these axes do not change, how do their calibration change?

After Lorentz transformation we get:

$$\xi' = ct' + x' = c\gamma\left(t - \frac{v}{c^2}x\right) + \gamma(x - vt) = \gamma t(c-v) + \gamma x\left(1 - \frac{v}{c}\right) =$$

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$$= \gamma\left(1-\frac{v}{c}\right)(ct+x) = \sqrt{\frac{c-v}{c+v}}\,\xi;$$

$$\eta' = ct' - x' = c\gamma\left(t - \frac{v}{c^2}x\right) - \gamma(x-vt) = \gamma ct\left(1+\frac{v}{c}\right) -$$

$$-\gamma x\left(1+\frac{v}{c}\right) = \gamma\left(1+\frac{v}{c}\right)(ct-x) = \sqrt{\frac{c+v}{c-v}}\,\eta;$$

Thus this coordinates, often called light-cone coordinates transform under Lorentz transformation in the following way:

$$\boxed{\xi' = \sqrt{\frac{c-v}{c+v}}\,\xi; \quad \eta' = \sqrt{\frac{c+v}{c-v}}\,\eta;}$$

As we can see they are not mixed by Lorentz transformation as it happens with (t,x)-coordinates, and thus, there is no rotation of axes. The only thing happening with this coordinates under Lorentz transformation is their rescaling. q.e.d.

Problem session N2 (Relativistic kimematics)

On this lecture we will consider some more effects related to frames.

I Length contraction Suppose we have bar of length L, in S'-frame it is stationary. What is it's length in frame S? When we measure length of rode in S we fix positions of it's endpoints simultaneously, thus $\Delta t = 0$. Now if we use Lorentz transformation

$$\Delta t = 0 = \gamma\left(\Delta t' + \frac{V}{c^2}\Delta x'\right) \Rightarrow \Delta t' = -\frac{V}{c^2}\Delta x';$$

$$\Delta x = \gamma(\Delta x' + V\Delta t') = \gamma\left(1 - \frac{V^2}{c^2}\right)\Delta x' = \frac{L_0}{\gamma}$$ thus we get

Lorentz contraction of length $\boxed{L = \frac{L_0}{\gamma}}$

II Time dilation Suppose clock at fixed point in S' measures time interval $\Delta t' = \Delta\tau$ (proper time of clock). What is time interval measured by observer in S. As $\Delta x' = 0$ Lorentz transformation immideately give $\boxed{\Delta t = \gamma\Delta t'}$ as $\gamma > 1$ observer in S measures a longer elapsed time than the proper time of clock.

III Velocity transformation Let's try to understand how does velocity transform when we go from one reference frame to other.

in S': $u_x' = \frac{\Delta x'}{\Delta t'}$; $u_y' = \frac{\Delta y'}{\Delta t'}$; $u_z' = \frac{\Delta z'}{\Delta t'}$;

in S: $u_x = \frac{\Delta x}{\Delta t}$; $u_y = \frac{\Delta y}{\Delta t}$; $u_z = \frac{\Delta z}{\Delta t}$;

now we use Lorentz transformation formulas

$$\Delta x = \gamma(\Delta x' + V\cdot\Delta t'); \quad \Delta t = \gamma\left(\Delta t' + \frac{V}{c^2}\Delta x'\right); \quad \Delta y' = \Delta y; \quad \Delta z' = \Delta z;$$

thus we get

$$u_x = \frac{\Delta x' + V\Delta t'}{\Delta t' + \frac{V}{c^2}\Delta x'} = \frac{u_x' + V}{1 + \frac{u_x' V}{c^2}}; \quad u_y = \frac{\Delta y'}{\gamma\left(\Delta t' + \frac{V}{c^2}\Delta x'\right)} = \frac{u_y'}{\gamma\left(1 + \frac{V u_x'}{c^2}\right)};$$

$$u_z = \frac{\Delta z'}{\gamma\left(\Delta t' + \frac{V}{c^2}\Delta x'\right)} = \frac{u_z'}{\gamma\left(1 + \frac{V u_x'}{c^2}\right)};$$

② $\boxed{u_x = \dfrac{u_x' + V}{1 + \frac{u_x' V}{c^2}} ; \quad u_y = \dfrac{u_y'}{\gamma\left(1 + \frac{u_x' V}{c^2}\right)} ; \quad u_z = \dfrac{u_z'}{\gamma\left(1 + \frac{u_x' V}{c^2}\right)} ;}$

It can be also showed that if $|\bar{u}'|<c$ then $|\bar{u}|<c$ (if $v<c$), i.e. speed of light can not be exceeded.

Ⅳ Acceleration We define proper acceleration ~~xxxx~~ α as acceleration in instantoneous rest frame. $S'(t)$ Then by definition $du' = \alpha dt'$; as $S'(t)$ is instantoneous rest frame we can use, $dt = \gamma_u dt'$. At the same time we assume that velocity in $S'(t)$ is du' and velocity in S is $u+du$. Using velocity transformation formula we get

$$du = \frac{du' + u}{1 + \frac{u\,du'}{c^2}} - u \approx (du'+u)\left(1 - \frac{u\,du'}{c^2}\right) - u \approx du' \cdot \left(1 - \frac{u^2}{c^2}\right)$$

thus acceleration in S is given by:

$$a = \frac{du}{dt} = \frac{du'}{dt'} \cdot \left(1 - \frac{u^2}{c^2}\right)^{3/2} = \frac{\alpha}{\gamma_u^3} ;$$

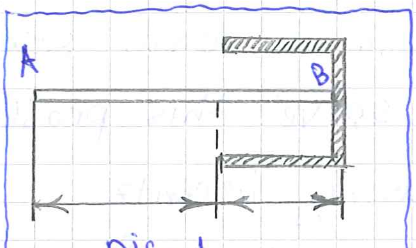
If we now say that proper acceleration is constant ($\alpha = \text{const}$) we can observe that $\frac{d}{dt}(\gamma_u u) = \alpha$, integrating this equation will give us trajectory of relativistic particle moving with constant proper acceleration α:

$$x = \frac{c^2}{\alpha}\sqrt{1 + \frac{\alpha^2 t^2}{c^2}} + x_0 , \quad \text{or} \quad \boxed{x^2 - (ct)^2 = \frac{c^4}{\alpha^2} ;}$$

(3)

Problem 1 In the pole-and-garage problem, what is the longest pole that can be run into a 12-foot garage at a speed v making $\gamma(v)=3$, assuming the elastic shock wave travels at the speed of light.

Let S be the rest frame of garage and S' - rest frame of pole.


pic. 1

Any phenomenon should have reasoning in any inertial frame, so it will be usefull to consider problem in both S and S'

* in S-frame Let x' be the proper length of the garage (12 ft.) and x - the proper length of pole. Then length of pole in S due to length contraction is $L=\frac{x}{\gamma}$. In S-frame rod hits the wall of the garage but the rear end of rod A is still moving as elastic shock wave travels with finite speed c and it takes $t=\frac{x'}{c}$ time for wave to propogate along rod so that it will meet A endpoint exactely where the rear wall of the garage is. This time should be equal to $\frac{L-x'}{v}$ (time A endpoint need to reach rear wall of the garage). Thus:

$\frac{x'}{c}=\frac{L}{v}-\frac{x'}{v}$ and $L=x'(1+\frac{v}{c}) \Rightarrow \boxed{x=\gamma x'(1+\frac{v}{c})=69,9 \text{ ft.}}$

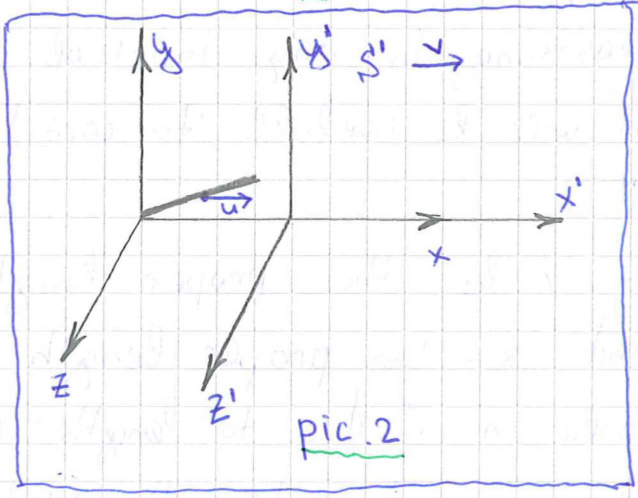
* in S'-frame in this frame length of the garage is $L'=\frac{x'}{\gamma}$. When wall of garage hits the rod deformation wave starts and should travel the whole longitude of rod x which takes time $t=\frac{x}{c}$

During this time front end of the garage should travel $(x-L')$ distance with velocity v

Thus $\frac{x}{c} = \frac{x-L'}{v}$; $x\left(\frac{1}{v}-\frac{1}{c}\right) = \frac{L'}{v} \Rightarrow x = x'\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$;

$\boxed{x = \gamma x'\left(1+\frac{v}{c}\right);}$ and we get the same result.

Problem 3 A rod having slope m relative to the x-axis of S, moves in the x-direction at speed u. What is the rod's slope in the usual second frame S'? (I) First let's solve this problem



pic.2

using Lorentz contraction formula

Let's introduce third reference frame S'' - the proper frame of rod. Assume one end of rode is in the origin of all 3 systems. Another point is in $(\Delta x, \Delta y, 0)$ (or the same point with primed Δx and Δy)

* in S'' we get $(\Delta x'', \Delta y'')$ point
* in S due to Lorentz contraction we get $\Delta x = \frac{\Delta x''}{\gamma(u)}$; $\Delta y = \Delta y''$; and thus for the slope we get $\boxed{m = \frac{\Delta y}{\Delta x} = \frac{\Delta y''}{\Delta x''}\gamma(u);}$
* in S' first of all rod's velocity in S' is given by addition law:

$u' = \frac{u-v}{1-\frac{uv}{c^2}}$; thus in S' we get:

$\Delta y' = \Delta y''$; $\Delta x' = \frac{1}{\gamma(u')}\Delta x''$. Let's calculate $\gamma(u')$

$$\gamma^{-2}(u') = 1 - \frac{(u-v)^2}{c^2\left(1-\frac{uv}{c^2}\right)^2} = \left\{c^2 - u^2 - v^2 + 2uv - 2uv + \frac{1}{c^2}u^2v^2\right\}\frac{1}{c^2\left(1-\frac{uv}{c^2}\right)^2}$$

(5) thus $\gamma^2(u') = \frac{1}{\left(1-\frac{uv}{c^2}\right)^2}\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)$; and thus

$\frac{1}{\gamma(u')} = \frac{1}{\gamma(u)}\cdot\frac{1}{\gamma(v)}\frac{1}{\left(1-\frac{uv}{c^2}\right)}$; finally $\boxed{\frac{\gamma(u')}{\gamma(u)} = \gamma(v)\left(1-\frac{uv}{c^2}\right)}$

for the slope of rod in S'-frame we get:

$$m' = \frac{\Delta y'}{\Delta x'} = \frac{\Delta y''}{\Delta x''}\gamma(u') = m\cdot\frac{\gamma(u')}{\gamma(u)} = m\gamma(v)\left(1-\frac{uv}{c^2}\right);$$

$$\boxed{m' = m\gamma(v)\left(1-\frac{uv}{c^2}\right);}$$

(II) Now let's consider the same problem in the style of previous Chapter. When we observe rod we measure positions of it's points simultaneously. Thus:

* in S-frame we get Δx and Δy in S' $\Delta t' = 0$ (simultaneous measurement). Using Lorentz transformation formulas $\Delta x' = \gamma(v)(\Delta x - v\Delta t)$; $\Delta t' = \gamma(v)(\Delta t - \frac{v}{c^2}\Delta x)$ thus $\Delta t = \frac{v}{c^2}\Delta x$. And thus while measurement is made rod is shifted with $u\Delta t$ distance:

$\Delta x = \Delta x_0 + u\Delta t$ thus we get:

$\Delta t = \frac{v}{c^2}\Delta x_0 + \frac{uv}{c^2}\Delta t \Rightarrow \boxed{\Delta t = \Delta x_0 \frac{v}{c^2}\frac{1}{1-\frac{uv}{c^2}}}$; then finally

$$\Delta x' = \gamma(v)\cdot\left\{\Delta x_0 + \frac{uv}{c^2}\frac{\Delta x_0}{\left(1-\frac{uv}{c^2}\right)} - \Delta x_0\frac{v^2}{c^2}\frac{1}{\left(1-\frac{uv}{c^2}\right)}\right\} =$$

$= \gamma(v)\cdot\Delta x_0\cdot\left\{\frac{1-\frac{v^2}{c^2}}{1-\frac{uv}{c^2}}\right\}$, thus finally $\Delta x' = \frac{\Delta x_0}{\gamma(v)\left(1-\frac{uv}{c^2}\right)}$

$m = \frac{\Delta y_0}{\Delta x_0}$ and $m' = \frac{\Delta y'}{\Delta x'} = \frac{\Delta y_0}{\Delta x_0}\gamma(v)\left(1-\frac{uv}{c^2}\right)$, thus finally we get the same answer:

$\boxed{m' = m\gamma(v)\left(1-\frac{uv}{c^2}\right);}$ as we see we get the same answer.

(6)

Problems 5 (i) 2 particles move along the x-axis of S' at velocities 0,8c and 0,9c., respectively, the faster one momentarily 1 m behind the slower one. How many seconds elapse before collision?

This problem is solved just as usual Newtonian mechanics problem. If we choose time origin $t=0$ so that at $t=0$ distance between particles is $h=1m$. Then coordinates of particles are given by $x_1 = V_1 t + h$; $x_2 = V_2 t$; At the moment of impact $x_1 = x_2$ and thus $\boxed{t = \frac{h}{V_2 - V_1} \approx 3{,}3 \cdot 10^{-8} s.}$

(ii) A rod of proper length 10 cm moves longitudinally along the x-axis of S at speed $\frac{1}{2}c$. How long (in S) does it take a particle, moving oppositely at the same speed, to pass the rod?

Length of rod is $L = \frac{L_0}{\gamma(\frac{c}{2})} = \frac{\sqrt{3}}{2} L_0$; Thus time it is take for particle to pass the rod is

$t = \frac{L}{c} = \frac{\sqrt{3}}{2} \frac{L_0}{c} = 2{,}89 \cdot 10^{-10} s$; $\boxed{t = \frac{\sqrt{3}}{2} \frac{L_0}{c} = 2{,}89 \cdot 10^{-10} s.}$

Problem 6 In a given inertial frame, 2 particles are shot out simultaneously from a given point, with equal speeds v, in orthogonal directions. What is the speed of each particle relative to the other?

in S-frame we have velocities of particles

$\bar{u}_1 = (v; 0; 0)$; $\bar{u}_2 = (0; v; 0)$

Let S' be frame following first particle (so it moves with velocity v along x-axis) then Lorentz transformation formulas give us

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$u'_{2x} = -v$; $u'_{2y} = \frac{v}{\gamma(v)}$; thus we get:

$\bar{u}'_1 = (0;0;0)$; $\bar{u}'_2 = (-v; \frac{v}{\gamma}; 0)$ and relative velocity is then given by

$u^2_{rel} = |\bar{u}'_2 - \bar{u}'_1|^2 = v^2\left(1 + \frac{1}{\gamma^2}\right) = v^2 \cdot \left(1 + 1 - \frac{v^2}{c^2}\right) =$
$= v^2\left(2 - \frac{v^2}{c^2}\right)$; $\boxed{u_{rel} = v\sqrt{2 - \frac{v^2}{c^2}};}$

Problem 9 The rapidity φ, of a particle moving with velocity u, is defined by $\varphi = \operatorname{arctanh}\left(\frac{u}{c}\right)$. Prove that collinear rapidities are additive, i.e. if A has rapidity φ relative to B, and B has rapidity ψ relative to C, then A has rapidity $\varphi + \psi$ relative to C.

Let velocity of B with respect to A be u, and C has velocity "v" with respect to B. thus $\varphi = \operatorname{arctanh}\left(\frac{u}{c}\right)$ and $\psi = \operatorname{arctanh}\left(\frac{v}{c}\right)$. Velocity of C with respect to A is given by velocities addition formula $w = \frac{u+v}{1+\frac{uv}{c^2}}$, Corresponding rapidity is then $\chi = \operatorname{arctanh}\left(\frac{w}{c}\right)$; Another more convinient way to write "arctanh" is $\operatorname{arctanh}\left(\frac{x}{y}\right) = \frac{1}{2}\ln\left(\frac{y+x}{y-x}\right)$;

Thus $\chi = \frac{1}{2}\ln\left\{\frac{c + \frac{u+v}{1+\frac{uv}{c^2}}}{c - \frac{u+v}{1+\frac{uv}{c^2}}}\right\} = \frac{1}{2}\ln\left\{\frac{1 + \frac{u}{c}\cdot\frac{v}{c} + \frac{u}{c} + \frac{v}{c}}{1 + \frac{u}{c}\cdot\frac{v}{c} - \frac{u}{c} - \frac{v}{c}}\right\} =$

$= \frac{1}{2}\ln\left\{\frac{\left(1+\frac{u}{c}\right)\left(1+\frac{v}{c}\right)}{\left(1-\frac{u}{c}\right)\left(1-\frac{v}{c}\right)}\right\} = \frac{1}{2}\ln\left(\frac{1+\frac{u}{c}}{1-\frac{u}{c}}\right) + \frac{1}{2}\ln\left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right) = \psi + \varphi,$

thus, indeed, we get additivity of rapidities

$\boxed{\chi = \varphi + \psi}$; q.e.d.

Problem 10 How many successive velocity increments of $\frac{1}{2}c$ from instantaneous rest frame are needed to produce resultant velocity of (i) 0,99c; (ii) 0,999c?

Hint: $\tanh 0{,}55 = 0{,}5$; $\tanh 2{,}65 = 0{,}99$; $\tanh 3{,}8 = 0{,}999$.

Here we will use results of previous problem and speak in terms of rapidity, which as we have show is additive. If one $\frac{1}{2}c$ increment corresponds to rapidity $\phi = \operatorname{arctanh}(\frac{1}{2})$ then to get rapidity $\psi = N\phi$ we need N increments:

$$N = \frac{\psi}{\phi}$$

(i) $\psi_1 = \operatorname{arctanh}(0{,}99) = 2{,}65$; $N_1 = \frac{2{,}65}{0{,}55} = 5$;

(ii) $\psi_2 = \operatorname{arctanh}(0{,}999) = 3{,}8$; $N_2 = \frac{3{,}8}{0{,}55} = 7$;

$\boxed{N_1 = 5;\ N_2 = 7}$

Problem 12 In S' a particle is momentarily at rest and has acceleration α in the y'-direction. What is the magnitude and direction of it's acceleration in S?

In a muon "storage ring" of radius 7m at the CERN laboratories in 1975, muons circled around at a speed $v = 0{,}9994c$. Find the magnitude of their proper acceleration.

First let's write down velocity transformation:

$$u_y = \frac{u_y'}{\gamma(v)} = \sqrt{1-\frac{v^2}{c^2}}\, u_y' \quad \text{and thus}$$

$$\frac{du_y}{dt'} = \frac{du_y'}{dt'}\sqrt{1-\frac{v^2}{c^2}}$$, and as we know from time dilation formulae $\frac{dt}{dt'} = \gamma(v)$ thus

$$\frac{du_y'}{dt'} = \frac{du}{dt}\gamma^2 \quad \text{thus} \quad \boxed{\frac{du}{dt} = \alpha \cdot \left(1-\frac{v^2}{c^2}\right)}$$

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Now we can consider muons in the storage ring. If they are moving with velocity v on the ring of radius r, they have acceleration $\alpha = \frac{v^2}{r}$ pointing to the center of ring i.e. transversal to velocity, and that's exacetely the example we have considered.

Thus in lab frame we have $\alpha = \frac{v^2}{r}$ and in muon in instaneous fram $\alpha' = \frac{\alpha}{(1-v^2/c^2)} = \frac{v^2}{r(1-\frac{v^2}{c^2})} \approx 10^{19} \frac{m}{s^2}$;

$$\boxed{\alpha' = \frac{v^2}{r(1-\frac{v^2}{c^2})} \approx 10^{19} \frac{m}{s^2}}$$

Problem 13 A certain piece of elastic breaks when it is stretched length. At time $t=0$, all points of it are accelerated longitudally with constant proper acceleration α, from rest in unstretched state. Prove that the elastic breaks at $t = \frac{\sqrt{3}c}{\alpha}$;

If we use equation of motion of motion of point particle moving with constant proper acceleration we, as we have seen, get $x = \frac{c^2}{\alpha}\sqrt{\frac{\alpha^2 t^2}{c^2}+1} + x_0$;

Thus in lab frame S length of the rod is always the same as it was before rod started acceleration (L_0). Thus in S'-frame following the rod we obtain, due to Lorentz contraction $L = \gamma L_0$, thus the rod is stretche, and when γ becomes $\gamma = 2$ it breaks. From equations of motion we know that $\alpha t = \gamma(v) \cdot v$ thus $t = \frac{\gamma \cdot v}{\alpha}$. As critical $\gamma = 2$, critical velocity is $v = \frac{\sqrt{3}}{2}$ and time when rod breaks down is $\boxed{t = \frac{\sqrt{3}c}{\alpha}}$, q.e.d.

(10)

Problem 16 Rocket moves from rest in an inertial frame S with constant proper acceleration $g = 9{,}8\ m/s^2$

* Verify that in units of years and light years $g \simeq 1$;

$1\,yr = 3{,}2\cdot 10^7\,s$; $1\,l.y. = 3{,}2\cdot 10^7\,s \cdot 3\cdot 10^8\,\frac{m}{s} = 9{,}6\cdot 10^{15}\,m$;

then $g = 9{,}8\,\frac{m}{s^2} = 9{,}8\cdot\frac{(3{,}2\cdot 10^7)^2}{3{,}2\cdot 10^7\cdot 3\cdot 10^8} = 9{,}8\,\frac{3{,}2\cdot 10^7}{3\cdot 10^8} = 1{,}05\,\frac{ly}{yr^2}$

So indeed $\boxed{g = 1{,}05\,\frac{ly}{yr^2};}$

* Find it's Lorentz factor relative to S when it's own clock indicates times $\tau = 1\,day, 1\,year, 10\,years$. Find also corresponding distances and times traveled in S.

We know equation of motion for accelerated particle: $\alpha t = \gamma(u)\cdot u$, where α is proper acceleration. t is time elapsed in S frame then we get:

$\alpha t = \frac{u}{\sqrt{1-\frac{u^2}{c^2}}}$ and $g^2t^2 - \frac{u^2g^2t^2}{c^2} = u^2 \Rightarrow u = \frac{gt}{\sqrt{1+\frac{g^2t^2}{c^2}}}$; and

$\gamma(u) = \sqrt{1+\frac{g^2t^2}{c^2}}$; Now we should relate S-frame time t with the rocket proper time τ; as we have seen $d\tau = \frac{dt}{\gamma(u)} = \frac{dt}{\sqrt{1+\frac{g^2t^2}{c^2}}}$ and

$\tau = \int_0^t \frac{dt'}{\sqrt{1+\frac{g^2t'^2}{c^2}}} = \frac{c}{g}\,\text{arcsinh}\,\frac{gt}{c}$ and then

$t = \frac{c}{g}\sinh\frac{g\tau}{c}$; $\gamma(u) = \sqrt{1+\frac{g^2t^2}{c^2}} = \cosh\frac{g\tau}{c}$; finally we want to know distance passed in S-frame.

$u = \frac{dx}{dt} = \frac{gt}{\sqrt{1+\frac{g^2t^2}{c^2}}}$ thus $x = \int_0^t \frac{dt'\cdot gt'}{\sqrt{1+\frac{g^2t'^2}{c^2}}} = \frac{c^2}{g}\left(\sqrt{1+\frac{g^2t^2}{c^2}} - 1\right) =$

$= \frac{c^2}{g}\left(\cosh\frac{g\tau}{c} - 1\right)$ So we have obtained

$$\boxed{t = \frac{c}{g}\sinh\frac{g\tau}{c};\quad \gamma(u) = \cosh\frac{g\tau}{c};\quad x = \frac{c^2}{g}\left(\cosh\frac{g\tau}{c} - 1\right)}$$

(11) If we substitute in numerics $\tau = \frac{1}{365}, 1, 10$ and $g = c = 1$ we get:

τ	$\frac{1}{365}$	1	10
γ	1,0000036	γ=1,5431	11013
x	0,0000038	0,5431	11012
t	0,0027	1,1752	11013

* If the rocket accelerates $\tau = 10$ yr of proper time and then decelerates $\tau = 10$ years, and then repeats the whole manoeuvre in the reverse direction, what is total time elapsed in S during rocket absence?

The time spent on each accelertion and deceleraton as we have seen is 11013 years. Thus in total the journey will take $4 \cdot 11013$ yr. = 4405 years

$\boxed{t = 44052 \text{ years}}$

1

Problem session N3 (Relativistic Optics)

Theory There are two effects that we will consider on this session: Doppler effect and aberration.

* Doppler effect:

Light source P travels in S-frame with velocity $\bar{u}$ and radial velocity u_r relative to O - origin of S. Let S' be the frame of moving light-source. Let the time between 2 pulses in S' be dt_0 and therefore in S time between pulses is $dt_0 \gamma(u)$. In that time, the source has increased it's distance from O by $dt_0 \gamma(u) u_r$. Thus interval between pulses arriving to O is $dt_0 \gamma(u)\left(1+\frac{u_r}{c}\right)$. Interval between pulses is inversly proportional to corresponding frequencies. Thus

$$\frac{\nu_0}{\nu} = \frac{dt}{dt_0} = \frac{1+u_r/c}{\sqrt{1-\frac{u^2}{c^2}}} \; ; \quad \boxed{\frac{\nu_0}{\nu} = \frac{1+\frac{u_r}{c}}{\sqrt{1-\frac{u^2}{c^2}}}}$$

if movement, for example, is radial we get:

$$\frac{\nu_0}{\nu} = \sqrt{\frac{1+u/c}{1-u/c}}$$

; One more thing we can derive is relation between the frequencies ν and ν' ascribed by two observers O and O' to an incoming ray. α is the angle that negative direction of ray makes with x-axis. If we assume that ray originates from the source being in rest in S', then we should make following notations $\nu_0 = \nu'$, $u = v$, $u_r = v \cdot \cos\alpha$; Thus

$$\boxed{\frac{\nu'}{\nu} = \frac{1+\frac{v}{c}\cos\alpha}{\sqrt{1-\frac{v^2}{c^2}}}}$$

* Aberration Let's now consider incoming light signal whose negative direction makes angles α and α' with the x-axes of the usual frames S and S'. Let's say that $u_1 = -c\cdot\cos\alpha$ and $u_1' = -c\cdot\cos\alpha'$;

(2) now we can use transformation of velocities formula:

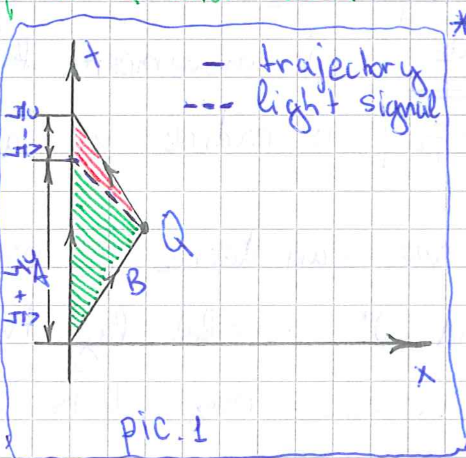
$$u_1' = \frac{u_1 - v}{1 - \frac{u_1 v}{c^2}}$$ and thus $$\boxed{\cos\alpha' = \frac{\cos\alpha + v/c}{1 + \frac{v}{c}\cos\alpha};}$$ and

in analogy we can obtain: $$\boxed{\sin\alpha' = \frac{\sin\alpha}{\gamma\left[1 + \frac{v}{c}\cdot\cos\alpha\right]};}$$

using trigonometric identity $\tan\frac{1}{2}\alpha' = \frac{\sin\alpha'}{1+\cos\alpha'}$ and obtain:

$$\boxed{\tan\frac{1}{2}\alpha' = \sqrt{\frac{c-v}{c+v}}\tan\frac{1}{2}\alpha;}$$ which will be use full for us.

Problem 1 If the twins A and B, in the twin paradox "experiment" discussed on p.30 [Rindler] visually observe the regular ticking of each other's standard clocks, describe quantitatively what each sees as B travels to a distant point Q and back.



pic. 1

* First let's consider ticks that A recievs from B. Let L be the distance from the start point to Q. It takes $\frac{L}{v}$ time to travel to this point for B and light signal of tick emmited by B at Q will go back to A for $\frac{L}{c}$ time. So

for $t_1 = \frac{L}{v} + \frac{L}{c}$ A recievs signals from receding B, and it recievs ticks with shift $\nu_1 = \nu_0 \cdot \sqrt{\frac{1-v/c}{1+v/c}}$. Total number of ticks recieved from B going towards Q is

$$N_{1A} = \nu_1 \cdot t_1 = \frac{L_0 \nu_0}{v}\sqrt{1-\frac{v^2}{c^2}} = \frac{L_0 \nu}{v\gamma(v)}; \quad \boxed{N_{1A} = \frac{L_0 \nu}{v\gamma(v)};}$$

For remaining time $t_2 = \frac{L}{v} - \frac{L}{c}$ A recievs signals from impedent source B and thus total number recieved is $N_{2A} = t_2 \cdot \nu_2$ where $\nu_2 = \nu_0\sqrt{\frac{1+v/c}{1-v/c}}$ and

(3) $N_{2A} = \frac{L\nu_0}{v}\sqrt{1-\frac{v^2}{c^2}} \Rightarrow N_{2A} = \frac{L\nu_0}{v\gamma(v)}$ thus total number of ticks recieved by A is $N_A = N_{1A} + N_{2A} = \frac{2L\nu_0}{v\gamma(v)}$

$\boxed{N_A = \frac{2L\nu_0}{v\cdot\gamma(v)}}$; * Now let's consider ticks recieved by B from A. One-way journey in proper frame of B is, due to time dilation, takes $\tau_1 = \tau_2 = \tau = \frac{L}{v\gamma}$. For the first part of trip B is gowing away from A and thus it recievs ticks with the frequency $\nu_1 = \nu_0\sqrt{\frac{1-v/c}{1+v/c}}$. For the second part of jorney situation is opposite and frequency of recieved ticks is $\nu_2 = \nu_0\sqrt{\frac{1+v/c}{1-v/c}}$; thus total number of ticks is $N_B = (\nu_1+\nu_2)\tau = \frac{\nu_0 L}{v\gamma(v)}\left(\sqrt{\frac{1+v/c}{1-v/c}} + \sqrt{\frac{1-v/c}{1+v/c}}\right) = \frac{2\nu_0 L}{v\gamma(v)}\cdot\gamma(v) = \frac{2\nu_0 L}{v}$; $\boxed{N_B = \frac{2\nu_0 L}{v}}$

As $\gamma(v) \geq 1$ we conclude that $N_B > N_A$, so that A has recieved less ticks from B then B from A, and that means that B is indeed younger, as we know it from other justifications of twin paradox.

Problem 5 A source of monochromatic light of frequency ν_0 is fixed at the origin of a frame S. An observer travels through S with instantoneous velocity u and radial velocity u_r relative to the origin, when he observes the source. What Doppler shift ν/ν_0 does he observe?

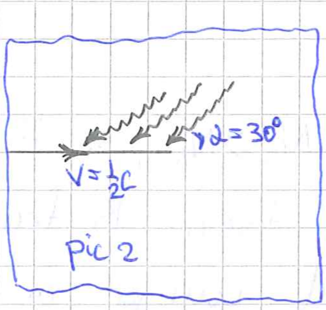
The situation in this problem is exactely opposite to the one considered in theory part of Doppler effect. I.e. now observer moves while source of light stays on the place. This corresponds to the change of variables $u \to -u$; $u_r \to u_r$. Note here

④ that radial velocity hasn't changed the sign because source and observer a still either coming closer or away from each other. Now if we apply formula we get:

$$\boxed{\frac{\nu_0}{\nu} = \frac{1+\frac{u_r}{c}}{\sqrt{1-\frac{u^2}{c^2}}}}$$

Note: So to make picture clear go to the proper frame of observer. What you get there is source moving with velocity $-u$ and radial velocity u_r.

Problem 6 An observer moves into parallel radiation of frequency ν at the angle of $30°$ with the radiation and at speed $\frac{1}{2}c$. What frequency does he observe?



Let's use already observed formula for relation of observed frequencies in standart frames S and S':

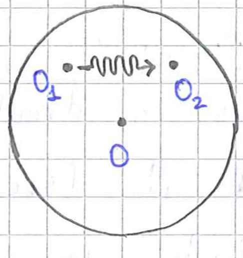
$$\frac{\nu'}{\nu} = \frac{1+\frac{V}{c}\cos\alpha}{\sqrt{1-\frac{V^2}{c^2}}}$$

here, just to remined, ν' and ν are frequencies in S' and S respectively and α is the angle betwee velocity of S' $\bar{V}$ and negative direction of incoming light frame ($30°$ in our case) $V = \frac{1}{2}c$ in our case. So if we substitute everything into this formula we get $\frac{\nu'}{\nu} = \frac{1+\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2}}$. Thus observer will observe radiation frequency equal to

$$\nu' = \nu \cdot \frac{4+\sqrt{3}}{2\sqrt{3}} = \left(\frac{2}{\sqrt{3}}+\frac{1}{2}\right)\nu \quad ; \text{thus} \quad \boxed{\nu' = \left(\frac{2}{\sqrt{3}}+\frac{1}{2}\right)\nu}$$

(5)

Problem 7 A large disc rotates at uniform angular speed ω in an inertial frame S. Two observers O_1 and O_2 ride on the disc at radial distances r_1 and r_2 from the centre (not necessarily on the same radius). They carry clocks C_1 and C_2 which they adjust so that they keep time with the clocks in S, i.e. they speed up their natural rates by the Lorentz factors $\gamma_1 = \left(1 - \frac{r_1^2\omega^2}{c^2}\right)^{-\frac{1}{2}}$, $\gamma_2 = \left(1 - \frac{r_2^2\omega^2}{c^2}\right)^{-\frac{1}{2}}$, respectively. Deduce that, when O_2 sends a light-signal to O_1, this signal suffers a Doppler shift $\frac{\nu_2}{\nu_1} = \frac{\gamma_2}{\gamma_1}$; There is no Doppler shift between any two observers equidistant from the centre.



Pic. 3

Assume in the rest frame of disc center (O) frequency of light signal is ν. To go to O_1 rest frame we can use formula for relation between frequencies in different reference frames and you can check result yourself, but we will do simplier thing. All points are on the solid disk so that distance betwee all points remain constant, so that there is no relative radial velocity between the points.
Thus only effect of changing frequency is due to time dilation. I.e. if in S time interval between 2 pulses is Δt_0 then in S_1 frame (proper frame O_1) we get $\Delta\tau_1 = \frac{\Delta t_0}{\gamma_1}$ and in S_2-frame $\Delta\tau_2 = \frac{\Delta t_0}{\gamma_2}$ where $\gamma_{1,2} = \left(1 - \frac{r_i^2\omega^2}{c^2}\right)^{-\frac{1}{2}}$; and then $\boxed{\frac{\nu_2}{\nu_1} = \frac{\Delta t_1}{\Delta t_2} = \frac{\gamma_2}{\gamma_1}}$; q.e.d.

(6) now if we take a look on this formula we get that if $r_1 = r_2$ then $\nu_1 = \nu_2$, q.e.d.

Problem 9 From (17.3) and (18.2) [Rindler] derive the following relation between Doppler shift and abberation:

$$\frac{\nu'}{\nu} = \frac{\sin\alpha}{\sin\alpha'};$$

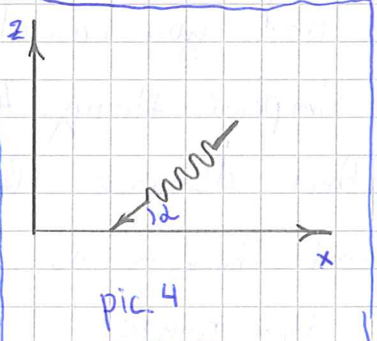
We have two equations:

$$\frac{\nu'}{\nu} = \frac{1+\frac{v}{c}\cos\alpha}{\sqrt{1-\frac{v^2}{c^2}}}; \qquad \sin\alpha' = \frac{\sin\alpha}{\gamma\left[1+\frac{v}{c}\cos\alpha\right]}$$

as we see $\frac{\nu'}{\nu} = \gamma(v)\left(1+\frac{v}{c}\cos\alpha\right) = \frac{\sin\alpha}{\sin\alpha'}$; $\boxed{\frac{\nu'}{\nu} = \frac{\sin\alpha}{\sin\alpha'}}$, q.e.d

Problem 10 Let Δt and $\Delta t'$ be the time separations in the usual two frames S and S' between two events occuring at a freely moving photon. If the photon has frequencies ν and ν' in these frames, prove that $\frac{\nu}{\nu'} = \frac{\Delta t}{\Delta t'}$

Let's assume that photon moves in direction making α angle with x-axis of S. (see pic 4)



pic. 4

So that it's velocity is given by $\bar{V} = (-c\cdot\cos\alpha; -c\cdot\sin\alpha; 0)$; and let the time separation between 2 events be Δt. In this case space separation in x-coordinate is $\Delta x = -c\cdot\cos\alpha\cdot\Delta t$. Using Lorentz transformation formulas we get in S' frame the following time separation between this two events:

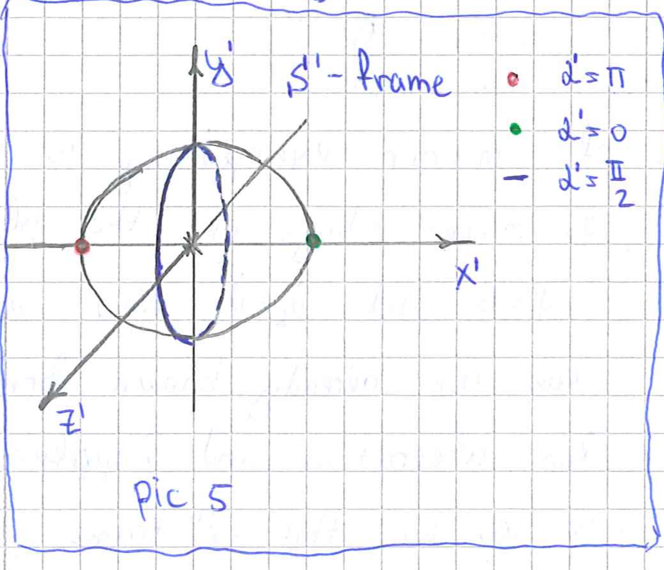
$$\Delta t' = \gamma(v)\left(\Delta t - \frac{v}{c^2}\Delta x\right) = \gamma(v)\,\Delta t\left(1+\frac{v}{c}\cdot\cos\alpha\right)$$ and due to formula (17.3) from [Rindler] $\gamma(v)\cdot\left(1+\frac{v}{c}\cdot\cos\alpha\right) = \frac{\nu'}{\nu}$.

Thus we oberve desired result $\boxed{\frac{\nu'}{\nu} = \frac{\Delta t'}{\Delta t}}$; q.e.d.

7

Problem 13 A source of light is fixed in S' and in that frame it emits light uniformly in all directions. Show that for large v, the light in S is mostly concentrated in a narrow forward cone; in particular, half the photons are emitted into a cone whose semi-angle is given by $\cos\theta = \frac{v}{c}$. This is called the headlight effect. Is the situation essentially different in the classical theory?

Let's consider ray going out of the source at angle α' (angle between ray direction and direction of x'-axis) in S'-frame. As light is emitted uniformly $\alpha' \in (0, \pi]$ (this corresponds to polar angle in spherical coordinates with center of sphere placed in the light source position and "North" of sphere placed in positive direction of x'-axis) $\alpha'=0$ corresponds to emition of light in direction of motion and $\alpha'=\pi$ – in opposite one $\alpha'=\frac{\pi}{2}$ – emition of light in the direction of "equator" of the sphere (half of light is emmited in the angle $0<\alpha'<\frac{\pi}{2}$) (see pic 5)



pic 5

Now we can see at what angle is photon emmited in S-frame. For this we will use formula, derived in the begining of lecture (or formula (18.3) from [Rindler]) Note formula is valid for incoming ray,

(8) While in the problem we have out going ray. So we need to substitute "$-c$" instead of "c"

$\tan\frac{\alpha}{2} = \sqrt{\frac{c-v}{c+v}}\tan\frac{\alpha'}{2}$ we see that if $v\to c$ $\tan\frac{\alpha}{2}\to 0$

and thus $\boxed{\alpha\to 0}$ which corresponds to light emmited in positive x-axis direction (i.e. in direction of movement)

Now let's understand where half of light is emmited.
In S' half of photons are emmited in North hemisphere $0<\alpha'<\frac{\pi}{2}$. Now if we substitute $\alpha'=\frac{\pi}{2}$ to the formula written above we get:

$\tan\frac{\alpha}{2} = \sqrt{\frac{c-v}{c+v}}\tan\frac{\pi}{4} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$. Now if we use following trigonometrical identity: $\tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$ we

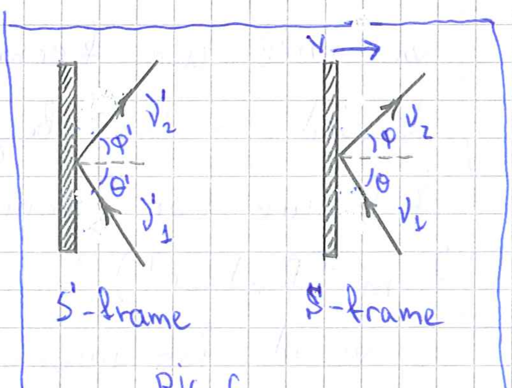
get: $\boxed{\cos\alpha = \frac{v}{c}}$ which is desired answer.

Problem 18 A plane mirror moves in the direction of its normal with uniform velocity v in a frame S. A ray of light of frequency ν_1 strikes the mirror at an angle of incidence θ, and is reflected with frequency ν_2 at an angle of reflection ϕ. Prove that

$$\frac{\tan\frac{1}{2}\theta}{\tan\frac{1}{2}\phi} = \frac{c+v}{c-v}, \quad \text{and}$$

$$\frac{\nu_2}{\nu_1} = \frac{\sin\theta}{\sin\phi} = \frac{c\cdot\cos\theta + v}{c\cdot\cos\phi - v} = \frac{c+v\cdot\cos\theta}{c-v\cdot\cos\phi};$$

In S' - proper frame of mirror everything is simple as mirror stays on the place $\phi=\theta$ and $\nu_1=\nu_2$. Now we can use already known formulas for aberration and Doppler shift to go to the S frame, in which mirror is moving with velocity "v"



pic. 6

(9) For incoming ray of light we have $\tan\frac{1}{2}\theta' = \sqrt{\frac{c-v}{c+v}}\tan\frac{1}{2}\theta$ and for out going ray:

$\tan\frac{\varphi}{2} = \sqrt{\frac{c-v}{c+v}}\tan\frac{1}{2}\varphi' = \sqrt{\frac{c-v}{c+v}}\tan\frac{1}{2}\theta' = \frac{c-v}{c+v}\tan\frac{1}{2}\theta$. So we have got desired relation

$$\boxed{\frac{\tan\frac{\theta}{2}}{\tan\frac{\varphi}{2}} = \frac{c+v}{c-v}}$$

Now let's go for frequencies. For incident light

$\frac{\nu_1'}{\nu_1} = \frac{1+\frac{v}{c}\cdot\cos\theta}{\sqrt{1-\frac{v^2}{c^2}}}$; $\frac{\nu_2'}{\nu_2} = \frac{1-\frac{v}{c}\cos\phi}{\sqrt{1-\frac{v^2}{c^2}}}$ as $\nu_1' = \nu_2'$ we immedeately get by deviding one equation with another:

$\frac{\nu_1}{\nu_2} = \frac{1-\frac{v}{c}\cos\varphi}{1+\frac{v}{c}\cos\theta}$ $\Rightarrow$ and we now obtain

$$\boxed{\frac{\nu_2}{\nu_1} = \frac{c+v\cdot\cos\theta}{c-v\cdot\cos\varphi};}$$ q.e.d.

① Problem session N4,5 (Tensors)

Theory

Formal definition

Assume we are making some coordinate transformation from coordinates $\{x^i\} = \{x^1, x^2, x^3, \dots, x^N\}$ to the coordinates $\{x^{i'}\} = \{x^{1'}, x^{2'}, x^{3'}, \dots, x^{N'}\}$;

* An object having components $A^{ij\dots n}$ in the x^i system of coordinates and $A^{i'j'\dots n'}$ in $x^{i'}$ system of coordinates is said to behave as a contravariant tensor under the transformation $\{x^i\} \to \{x^{i'}\}$ if:

$A^{i'j'\dots n'} = A^{ij\dots n} p^{i'}_i p^{j'}_j \dots p^{n'}_n$, here $p^{i'}_i$ is defined in the following way $p^{i'}_i = \frac{\partial x^{i'}}{\partial x^i}$;

* Similarly, $A_{ij\dots n}$ is said to behave as a covariant tensor under $\{x^i\} \to \{x^{i'}\}$ if

$A_{i'j'\dots n'} = A_{ij\dots n} p^i_{i'} p^j_{j'} \dots p^n_{n'}$;

* Lastly, $A^{i\dots k}_{l\dots n}$ is said to behave as a mixed tensor (contravariant in $i\dots k$ and covariant in $l\dots n$) under $\{x^i\} \to \{x^{i'}\}$ if

$A^{i'\dots k'}_{l'\dots n'} = A^{i\dots k}_{l\dots n} p^{i'}_i \dots p^{k'}_k p^l_{l'} \dots p^n_{n'}$;

So tensor are just defined as objects transforming in some particular way under coordinate transformations. The particular coordinate transformations we are usually interested in this course are Lorentz transformations.

For Lorentz transfomation we usually write $\Lambda^{\mu'}_{\mu}$ instead of $p^{i'}_i$, where general form of $\Lambda^{\mu'}_{\mu}$ is

$$\Lambda^{\mu'}_{\mu} = \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_z\beta_x}{\beta^2} \\ -\gamma\beta_y & (\gamma-1)\frac{\beta_y\beta_x}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & (\gamma-1)\frac{\beta_z\beta_x}{\beta^2} & (\gamma-1)\frac{\beta_z\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{bmatrix}$$

② here as usually $\gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$; $\beta_i = \frac{v_i}{c}$; $\beta = \frac{v}{c}$;

All expressions in Joe's notes and in Rindler book are special cases of the formulae above

For example in case of boost along x-axis $\beta_x = \beta$, $\beta_y = \beta_z = 0$. and we get:

$$\Lambda^{\mu'}_{x\,\mu} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

Usefull property of Lorentz transformation is that Lorentz transformations form $SO(3,1)$ group

3 → space rotations

1 → boost.

That means that for rotations $\Lambda^T = \Lambda^{-1}$ and for boosts $\Lambda^T = \Lambda$; or in other words $\boxed{\Lambda_{\nu\mu'}\Lambda^{\nu}{}_{\nu'} = \eta_{\mu'\nu'}}$

here $\eta_{\mu\nu}$ is metric. Minkowski flat metric is

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

As you know from lectures in notatio $\Lambda_{\nu\mu'}\Lambda^{\nu}{}_{\nu'}$ we summ over "ν" index (repeating index). This summation indicies are called "dummy". If we have some kind of product of different rank tensors, then rank of resulting tensor equals number of not dummy indicies.

For example $A_\mu B^\mu$ is scalar product and is Lorentz scalar.

Examples of 4-vectors from relativistic mechanics

4-vectors are rank 1 tensor in 4-dimensional Minkowski space

covariant vector contravariant tensor

$A_\mu = (A^0; -\bar{A})$ $A^\mu = (A^0; \bar{A})$ here $\bar{A}$ is just 3-vector in euclidian space.

③

Problem 2 An inertial observer bounces a radar signal off an arbitrary event $\mathcal{P}$. If the signal is emitted and received by him at times τ_1 and τ_2, respectively, as indicated by his standard clock, prove that the squared interval Δs^2 between his origin-event $\tau=0$ and $\mathcal{P}$ is $c^2\tau_1\tau_2$. This, in fact, constitutes a uniform method for assigning $\Delta s'^2$ to any pair of events.

Time for the signal to reach $\mathcal{P}$ is half of time difference between emition and receiving signal, i.e. $(\tau_2-\tau_1)\cdot\frac{1}{2}$. Thus distance to $\mathcal{P}$ is $\Delta x = \frac{\tau_2-\tau_1}{2}\cdot c$. Time interval between origin event $\tau=0$ and receiving signal by $\mathcal{P}$ is $\Delta\tau = \tau_1 + \frac{\tau_2-\tau_1}{2} = \frac{1}{2}(\tau_1+\tau_2)$; and thus, finally interval between $\mathcal{P}$ and observer origin-event is

$\Delta s^2 = c^2\Delta\tau^2 - \Delta x^2 = \frac{c^2}{4}(\tau_1+\tau_2)^2 - \frac{c^2}{4}(\tau_2-\tau_1)^2 = c^2\tau_1\tau_2$; and thus

$\boxed{\Delta s^2 = c^2\tau_1\tau_2;\ \text{q.e.d.}}$

Problem 7 An antisymmetric tensor $F^{\mu\nu}$ has the following components in a frame S:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix}$$

(i) Find the values of all components $F^{\mu'\nu'}$ in the usual second frame S'

S' is as usually boosted along x-axis.

Thus $F^{\mu\nu}$ is transformed under Lorentz transformations in the following way

$F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\mu}\Lambda^{\nu'}{}_{\nu}F^{\mu\nu}$ where $\Lambda^{\mu'}{}_{\mu} = \begin{bmatrix} \gamma(v) & -\frac{v}{c}\gamma(v) & 0 & 0 \\ -\frac{v}{c}\gamma(v) & \gamma(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

④ First of all to simplify our task let's note that electromagnetic tensor $F^{\mu\nu}$ is antisymmetric in all frames. Indeed:

$$F^{\mu'\nu'} = \Lambda^{\mu'}_{\ \mu}\Lambda^{\nu'}_{\ \nu}F^{\mu\nu} \ ; \quad F^{\nu'\mu'} = \Lambda^{\mu'}_{\ \mu}\Lambda^{\nu'}_{\ \nu}F^{\nu\mu} = -\Lambda^{\mu'}_{\ \mu}\Lambda^{\nu'}_{\ \nu}F^{\mu\nu} = -F^{\mu'\nu'}$$

So we don't need to find all components of $F^{\mu\nu}$, but only it's upper triangle

$$F^{0'1'} = \Lambda^{0'}_{\ \mu}\Lambda^{1'}_{\ \nu}F^{\mu\nu} = \Lambda^{0'}_{\ 0}\Lambda^{1'}_{\ 1}F^{01} + \Lambda^{0'}_{\ 1}\Lambda^{1'}_{\ 0}F^{10} = -\gamma^2 E_1 + \frac{v^2}{c^2}\gamma^2 E_1 =$$
$$= -\gamma^2\left(1-\frac{v^2}{c^2}\right)E_1 = -E_1;$$

$$F^{0'3'} = \Lambda^{0'}_{\ \mu}\Lambda^{3'}_{\ \nu}F^{\mu\nu} = \Lambda^{0'}_{\ 0}\Lambda^{3'}_{\ 3}F^{03} + \Lambda^{0'}_{\ 1}\Lambda^{3'}_{\ 3}F^{13} = \gamma(-E_3) - \frac{v}{c}\gamma \cdot B_2 =$$
$$= -\gamma(v)\left(E_3 + \frac{v}{c}B_2\right);$$ similarly we find $F^{0'2'} = -E_2' = -\gamma\left(E_2 - \frac{v}{c}B_3\right)$;

$$F^{1'2'} = \Lambda^{1'}_{\ \mu}\Lambda^{2'}_{\ \nu}F^{\mu\nu} = \Lambda^{1'}_{\ 0}\Lambda^{2'}_{\ 2}F^{02} + \Lambda^{1'}_{\ 1}\Lambda^{2'}_{\ 2}F^{12} = -\frac{v}{c}\gamma(-E_2) + \gamma(-B_3) =$$
$$= \gamma\left(\frac{v}{c}E_2 - B_3\right);$$

$$F^{1'3'} = \Lambda^{1'}_{\ 0}\Lambda^{3'}_{\ 3}F^{03} + \Lambda^{1'}_{\ 1}\Lambda^{3'}_{\ 3}F^{13} = -\frac{v}{c}\gamma(-E_3) + \gamma B_2 = \gamma\left(B_2 + \frac{v}{c}E_3\right)$$

$F^{2'3'} = \Lambda^{2'}_{\ 2}\Lambda^{3'}_{\ 3}F^{23} = -B_1$ thus we finally get:

$$F^{\mu'\nu'} = \begin{bmatrix} 0 & -E_1 & \gamma\left(\frac{v}{c}B_3 - E_2\right) & -\gamma\left(E_3 + \frac{v}{c}B_2\right) \\ E_1 & 0 & \gamma\left(\frac{v}{c}E_2 - B_3\right) & \gamma\left(B_2 + \frac{v}{c}E_3\right) \\ -\gamma\left(\frac{v}{c}B_3 - E_2\right) & -\gamma\left(\frac{v}{c}E_2 - B_3\right) & 0 & -B_1 \\ +\gamma\left(E_3 + \frac{v}{c}B_2\right) & -\gamma\left(B_2 + \frac{v}{c}E_3\right) & B_1 & 0 \end{bmatrix}$$

The thing here is that E^i and B^i don't really transform as 3-components of 4-vector (indeed E_1 and B_1 components don't even change). Really they are components of tensor $F^{\mu\nu}$ and transform in a way derived above

(ii) Verify directly that $\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \bar{B}^2 - \bar{E}^2$ is an invariant.

First of all

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix} ; \quad F_{\mu\nu} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix}$$

(5) From this 2 expressions we can easily check that indeed

$\frac{1}{2} F_{\mu\nu} F^{\mu\nu} = |\bar{B}|^2 - |\bar{E}|^2$;

Now let's show that this is lorentz invariant

As we have already mentioned as there are no not dummy indicies expression should be scalar. Let's check this explicitely on this particular example:

$$F^{\mu'\nu'} F_{\mu'\nu'} = \eta_{\mu'\rho'} \eta_{\nu'\lambda'} F^{\rho'\lambda'} F^{\mu'\nu'} = \eta_{\mu'\rho'} \eta_{\nu'\lambda} \Lambda^{\rho'}{}_{\rho} \Lambda^{\lambda'}{}_{\lambda} \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} F^{\rho\lambda} F^{\mu\nu} =$$

$$= \Lambda_{\mu'\rho} \Lambda_{\nu'\lambda} \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} F^{\rho\lambda} F^{\mu\nu}; \quad \Lambda_{\mu'\rho} \Lambda^{\mu'}{}_{\mu} = \eta_{\rho\mu};$$

thus $F_{\mu'\nu'} F^{\mu'\nu'} = \eta_{\rho\mu} \eta_{\lambda\nu} F^{\rho\lambda} F^{\mu\nu} = F_{\mu\nu} F^{\mu\nu}$; so we have shown that $\boxed{F_{\mu'\nu'} F^{\mu'\nu'} = F_{\mu\nu} F^{\mu\nu}}$, and thus $F_{\mu\nu} F^{\mu\nu}$ is indeed Lorentz invariant, q.e.d.

(iii) Show that $\overset{*}{F}_{\mu\nu} \overset{*}{F}{}^{\mu\nu} = - F_{\mu\nu} F^{\mu\nu}$ where $\overset{*}{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$

$\overset{*}{F}_{\mu\nu} \overset{*}{F}{}^{\mu\nu} = \frac{1}{4} \varepsilon_{\mu\nu\alpha\beta} \varepsilon^{\mu\nu\gamma\delta} F^{\alpha\beta} F_{\gamma\delta}$, $\varepsilon_{\mu\nu\alpha\beta}$ is absolutely antisymmetric tensor:

$$\varepsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & \text{if } \mu\nu\alpha\beta \text{ is even permut. of } 0123 \\ -1 & \text{if } \mu\nu\alpha\beta \text{ is odd permut. of } 0123 \\ 0 & \text{if some of indicies are the same} \end{cases}$$

For it we can check that (see exercise 19 from appendix)

$\varepsilon_{\mu\nu\alpha\beta} \varepsilon^{\mu\nu\gamma\delta} = 2e(\delta^{\gamma}{}_{\alpha} \delta^{\delta}{}_{\beta} - \delta^{\gamma}{}_{\beta} \delta^{\delta}{}_{\alpha})$ where $e = \eta = \det \eta_{\alpha\beta}$ is determinant of the metric. In our case $e = -1$;

thus $\overset{*}{F}_{\mu\nu} \overset{*}{F}{}^{\mu\nu} = \frac{1}{2} e (\delta^{\gamma}{}_{\alpha} \delta^{\delta}{}_{\beta} - \delta^{\gamma}{}_{\beta} \delta^{\delta}{}_{\alpha}) F^{\alpha\beta} F_{\gamma\delta} =$

$$= \frac{1}{2} e F^{\alpha\beta} F_{\alpha\beta} - \frac{1}{2} e F^{\alpha\beta} F_{\beta\alpha} = e \cdot F_{\alpha\beta} F^{\alpha\beta} = - F_{\mu\nu} F^{\mu\nu}$$

$\boxed{\overset{*}{F}_{\mu\nu} \overset{*}{F}{}^{\mu\nu} = - F_{\mu\nu} F^{\mu\nu}}$ q.e.d.

Problem 8 Consider rotation-translation transformations (i.e. rotations and translations of euclidian 3-space)

(i) Prove that if RT transformations are applied to a 4-vector $A^{\mu} = (A^0; \bar{a})$, A^0 transforms as scalar and

⑥ $\bar{a}$ as a three-vector

* Translations are acting on 3-vector only so that $A^\mu \to A^\mu + c^\mu$, where general form of c^μ looking like $c^\mu = (0, \bar{c})$ thus we see that translation transformations act in the following way on 4-vector components:

$A^0 \to A^0$: scalar-like transformation

$\bar{a} \to \bar{a} + \bar{c}$: vector-like transformation

* Rotation

general form of rotation transformation:

$$A^{\mu'} = R^{\mu'}{}_{\nu} A^\nu \quad \text{where} \quad R^{\mu'}{}_{\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & R^1{}_1 & R^1{}_2 & R^1{}_3 \\ 0 & R^2{}_1 & R^2{}_2 & R^2{}_3 \\ 0 & R^3{}_1 & R^3{}_2 & R^3{}_3 \end{bmatrix}$$

Particular example is rotation by angle θ in positive direction about z-axis.

$$R_z(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

thus if we consider transformation of A^μ components $A^{0'} = R^{0'}{}_0 A^0 = A^0$

$A^{0'} = A^0$: scalar-like transformation

$a^{i'} = R^{i'}{}_i a^i$ - 3-vector transformation.

(ii) of an RT transformation is applied to a four-tensor $T^{\mu\nu}$, then T^{00} transforms as scalar, T^{0i} and T^{i0} as three-vectors and T^{ij} as three-tensor

Let's see how all this tensor components are transformed

$T^{0'0'} = R^{0'}{}_0 R^{0'}{}_0 T^{00} = T^{00}$ - transforms as scalar

$T^{0'i'} = R^{0'}{}_0 R^{i'}{}_i T^{0i} = R^{i'}{}_i T^{0i}$ - transforms as 3-vector

$T^{i'0'} = R^{0'}{}_0 R^{i'}{}_i T^{i0} = R^{i'}{}_i T^{i0}$ - transforms as 3-vector

$T^{i'j'} = R^{i'}{}_i R^{j'}{}_j T^{ij}$ - transforms as 3-tensor

the same we can say about translation transformations

$T^{0'0'} = T^{00}$ — scalar transformation

$T^{0'i'} = T^{0i} + c_1^i$ — 3-vector transformation

$T^{i'0'} = T^{i0} + c_2^i$ — 3-vector transformation

$T^{i'j'} = T^{ij} + c_1^i \otimes c_2^j$ — tensor transformation

Problem 9 An inertial observer O has 4-velocity U_0 and a particle P has 4-acceleration A. If $U_0 \cdot A = 0$, what can you conclude about speed of P in O's rest frame.

Theory Here we meet examples of 4-vectors with physical meaning - 4-velocity and 4-acceleration.

If we parametrise trajectory of particle with the proper time τ we can introduce 4-velocity $U^\mu = \frac{dx^\mu}{d\tau}$ and 4-acceleration $A^\mu = \frac{dU^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2}$;

To relate 4-velocity U^μ with 3-velocity $\bar{u}$ remember that $\frac{dt}{d\tau} = \gamma(u)$ due to dilation of time. Thus

$U^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt} = \gamma(u)(c, \bar{u})$. Now 4-acceleration is given by $A^\mu = \frac{dU^\mu}{d\tau} = \gamma \frac{dU^\mu}{dt} = \gamma \frac{d}{dt}\gamma(u)\cdot(c, \bar{u})$;

Let's look what are values of Lorentz invariants $U^\mu U_\mu$; $A^\mu \cdot A_\mu$ and $U_\mu \cdot A^\mu$. As this all are Lorentz invariants we can obtain their values in some particular reference frame. Convinient one is instantoneous rest frame. There we get $\bar{u} = 0$ thus $\gamma(u) = 1$ and $U^\mu = (c, \bar{0})$, $A^\mu = (0, \bar{a})$, $\bar{a}$ being proper acceleration. So in instantoneous rest frame we easily get:

$$\boxed{U^\mu \cdot U_\mu = c^2;\ A^\mu \cdot A_\mu = -\bar{a}^2;\ U^\mu \cdot A_\mu = 0;}$$

and this equations are valid for all reference frames.

(8) Now let's return to the problem we are solving. We have just obtained that $U_0 \cdot A$ is Lorentz invariant so that it's equal zero in all reference frames. In O rest frame $U_0' = (c, 0)$. Now if $\bar{u}'$ is speed of P in O rest frame we get

$A' = \gamma(u') \frac{d}{dt}\{\gamma(u')(c, \bar{u}')\}$ thus as $U_0' \cdot A' = 0$:

$c \cdot \gamma(u') \frac{d}{dt} \cdot c \cdot \gamma(u') = 0$; as $\frac{d\gamma}{dt} \sim \dot{u} \cdot u = 0 \Rightarrow u = 0$ i.e. P has momentarely zero velocity in O rest frame.

Problem 15 In a given inertial frame S_0 moving with 4-velocity U_0^μ a tensor $T^{\mu\nu}$ has but a single non-vanishing component: $T^{00} = c^2$. Find the components of this tensor in the general frame S, relative to which $U_0^\mu = \gamma(u)(c, \bar{u})$

There are 2 ways of solving this problem. First one is straightforward: making Lorentz transformation

$T^{\mu'\nu'} = \Lambda^{\mu'}{}_\mu \Lambda^{\nu'}{}_\nu T^{\mu\nu} = \Lambda^{\mu'}{}_0 \Lambda^{\nu'}{}_0 c^2$ (only nonzero component is $T^{00} = c^2$) of general form

$$\Lambda^{\mu'}{}_\mu = \begin{bmatrix} \gamma & +\gamma\beta_x & +\gamma\beta_y & +\gamma\beta_z \\ +\gamma\beta_x & \cdots & \cdots & \cdots \\ +\gamma\beta_y & \cdots & \cdots & \cdots \\ +\gamma\beta_z & \cdots & \cdots & \cdots \end{bmatrix}$$ written in theoretical introduction

we easily obtain:

$T^{0'0'} = \gamma^2 c^2$; $T^{0'i'} = \gamma^2 c \cdot u^{i'}$; $T^{i'j'} = \gamma^2 u^{i'} u^{j'}$

This components in more compact form can be written as $\boxed{T^{\mu\nu} = U_0^\mu U_0^\nu}$

Second way of solving this problem is just guessing the only tensor structure we can construct from information we have is $T^{\mu\nu} = a \cdot U_0^\mu U_0^\nu$ where "a" is some scalar. In S_0 reference frame we get:

$T^{00} = c^2$ so $a = 1$ and we get the same result

$$\boxed{T^{\mu\nu} = U_0^\mu U_0^\nu;}$$

(9)

Problem 16 A particle perfoms a helical motion described by $x = R\cos\omega t$; $y = R\sin\omega t$; $z = ut$ (R, ω, u being constants)
Find it's proper acceleration.

4-velocity of particle is $u^\mu = \gamma(v)(c, -\omega R\sin\omega t; \omega R\cos\omega t; u)$
v here is absolute velocity $v = \sqrt{u^2 + \omega^2 R^2} = \text{const}$;
Then 4-acceleration is given by

$$A^\mu = \frac{du^\mu}{d\tau} = \gamma\frac{du^\mu}{dt} = -\gamma^2(0; \omega^2 R\cos\omega t; \omega^2 R\sin\omega t; 0)$$

the proper acceleration can be easely found by
$\alpha^2 = -A^\mu A_\mu = +\gamma^4\omega^4 R^2$, thus finally $\boxed{\alpha = \gamma^2\omega^2 R;}$

Problem 17 A particle moves rectilinearly with constant proper acceleration α. U and A it's 4-velocity and 4-acceleration, τ its proper time, and units are chosen to make $c=1$, prove $\frac{dA^\mu}{d\tau} = \alpha^2 U^\mu$

In this part of problem we are asked to use known form of rectilinear motion. along x-axis

$$V = c\tanh\left(\frac{\tau\alpha}{c}\right); \quad \gamma(u) = \cosh\left(\frac{\alpha\tau}{c}\right); \quad \frac{\alpha t}{c} = \sinh\left(\frac{\alpha\tau}{c}\right); \quad \frac{\alpha x}{c^2} = \cosh\left(\frac{\alpha\tau}{c}\right);$$

as $A^1 = \frac{d^2x}{d\tau^2} = \alpha\cosh\left(\frac{\alpha\tau}{c}\right)$; $U^1 = \frac{dx}{d\tau} = c\cdot\sinh\left(\frac{\alpha\tau}{c}\right)$;

we see that indeed $\frac{dA}{d\tau} = \frac{\alpha^2}{c}\sinh\left(\frac{\alpha\tau}{c}\right) = \frac{\alpha^2}{c^2}U$

if we take $c=1$; $\frac{dA}{d\tau} = \alpha^2 U$, indeed.

Prove, conversly, that this equation, without the information that α is the proper acceleration, or constant, implies both these facts.

Let's assume that equation $\frac{dA}{d\tau} = \alpha^2 U$ is valid
Now let's interpret α using equation $A\cdot U = 0 \Rightarrow$
$\frac{dA}{d\tau}\cdot U + \frac{dU}{d\tau}\cdot A = 0 \Rightarrow \frac{dA}{d\tau}\cdot U = -A\cdot A = \alpha^2$ where α is proper acceleration. Now let's check that α is constant: $\alpha^2 = -A_\mu\cdot A^\mu \Rightarrow \alpha\frac{d\alpha}{d\tau} = -A_\mu\frac{dA^\mu}{d\tau} = -\alpha^2 A_\mu U^\mu = 0$; so

(10) So we conclude that $\alpha = const.$, q.e.d.

Finally show, by integration, that the equation implies rectilinear motion in a suitable inertial frame, and thus, in fact, hyperbolic motion. Consequently $\frac{dA}{d\tau} = \alpha^2 U$ is the tensor equation characteristic of hyperbolic motion.

Let's solve $\frac{d^2 U^\mu}{d\tau^2} = \alpha^2 U^\mu$, as α is constant

$U^\mu(\tau) = U^\mu_1 e^{\alpha\tau} + U^\mu_2 e^{-\alpha\tau}$. If $U^\mu_{1,2} = 0$ for $\mu = 2,3$ (motion along x-axis) we get

$\frac{dx}{d\tau} = a e^{\alpha\tau} + b e^{-\alpha\tau}$, where a and b are constants.

we know else that $(U^0)^2 - (U^1)^2 = 1$. So we can use the following parametrisation $U^0 = \cosh x$ $U^1 = \sinh x$ if we look on expression for $\frac{dx}{d\tau}$ we can conclude that $\frac{dx}{d\tau} = \sinh \alpha\tau \Rightarrow \boxed{x = \frac{1}{\alpha}\cosh \alpha\tau + C_1}$ where C_1 is integration constant. So we see that we have indeed obtained formulae for rectlinear motion.

Appendix problems.

Problem 1 (i) A vector A^i has components $\dot{x}, \dot{y}$ in rectangular Cartesian coordinates; what are its components in polar coordinates.

$A^i = (\dot{x}; \dot{y})$ in Cartesian coordinates. Polar coordinate system basis vectors in cartesian coordinates are written in the following form: $\bar{e}_r = (\cos\theta, \sin\theta)$; $\bar{e}_\theta = (-\sin\theta; \cos\theta)$

$\dot{\bar{e}}_r = \dot{\theta}(-\sin\theta; \cos\theta)$; $\dot{\bar{e}}_\theta = -\dot{\theta}(\cos\theta; \sin\theta) = -\dot{\theta}\bar{e}_r$, now we can find $\bar{A} = \dot{\bar{r}} = (r\dot{\bar{e}}_r) = \dot{r}\bar{e}_r + r\dot{\bar{e}}_r = \dot{r}\bar{e}_r + r\dot{\theta}\bar{e}_\theta$;

Thus in polar coordinates $\boxed{\bar{A} = (\dot{r}; r\dot{\theta});}$

(ii) The same for vector $\bar{B} = \ddot{\bar{r}}$

$\ddot{\bar{r}} = \ddot{r}\bar{e}_r + \dot{r}\dot{\bar{e}}_r + \dot{r}\dot{\theta}\bar{e}_\theta + r\ddot{\theta}\bar{e}_\theta + r\dot{\theta}\dot{\bar{e}}_\theta = \ddot{r}\bar{e}_r + 2\dot{r}\dot{\theta}\bar{e}_\theta + r\ddot{\theta}\bar{e}_\theta -$
$- r\dot{\theta}^2\bar{e}_r$; thus $\bar{B} = (\ddot{r} - r\dot{\theta}^2)\bar{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\bar{e}_\theta$;

$\boxed{B = (\ddot{r} - r\dot{\theta}^2; 2\dot{r}\dot{\theta} + r\ddot{\theta});}$

Problem 3 Use the quotient rule to show that $\delta^i{}_j$ is a tensor?

Quotient rule says that if a set of components when combined by a given type of multiplication with arbitrary tensor of a given valence yields a tensor, then the set constitutes a tensor. For example if A^i and B^i are vectors $A^i = C^i{}_j B^j$ and as we know for $\delta^i{}_j$: $A^i = \delta^i{}_j A^j$ so $\delta^i{}_j$ is indeed tensor

Problem 5 (i) If $C_{ij}A^iA^j$ is a scalar for an arbitrary vector A^i, prove that $(C_{ij} + C_{ji})$ is a tensor

Let's look how C_{ij} transforms. As $C_{ij}A^iA^j$ is scalar
$C_{i'j'}A^{i'}A^{j'} = C_{ij}A^iA^j$ and as $A^{i'} = A^i p^{i'}_i$ we get

$(C_{i'j'}p^{i'}_i p^{j'}_j - C_{ij})A^iA^j = 0$ and as A^i is arbitrary vector and we can take first vector with one nonzero component thus eliminating summation, then vector with 2 nonzero components and so on we get:

$C_{ij} = C_{i'j'}p^{i'}_i p^{j'}_j$ which is rule of tensor transformation. thus C_{ij} is tensor. Now we can make standart procedure of symmetrisation: $C_{ij}A^iA^j = \frac{1}{2}(C_{ij} + C_{ji})A^iA^j$

Thus object $(C_{ij} + C_{ji})$ can be treated on equal footing with C_{ij} and thus it is tensor too, q.e.d.

(ii) If $C_{ij}A^iB^j$ is a scalar for two arbitrary vectors A^i, B^i, prove that C_{ij} is a tensor.

⑫ The line of proof is just the same:

$$C_{ij} A^i B^j = C_{i'j'} A^{i'} B^{j'} = C_{i'j'} p^{i'}{}_i p^{j'}{}_j A^i B^j$$ and thus

$$(C_{ij} - C_{i'j'} p^{i'}{}_i p^{j'}{}_j) A^i B^j = 0$$ and as in the previous part of problem we conclude

$$C_{ij} = C_{i'j'} p^{i'}{}_i p^{j'}{}_j$$ and we conclude that C_{ij} is tensor.

Problem 9 If $g_{ij} = 0$ for $i \neq j$, prove that $g^{ij} = 0$ for $i \neq j$, and $g^{ii} = \frac{1}{g_{ii}}$

By definition $g^{ij} g_{jk} = \delta^i{}_k$ as $g_{ij} = 0$ for $i \neq j$

we write $g^{ij} g_{jj} = \delta^i{}_j$ (no summation over j here)

Thus $g^{ij} g_{jj} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$ so if $i \neq j$ $g^{ij} = 0$ and if $i = j$ $g^{ii} = \frac{1}{g^{ii}}$, q.e.d.

Problem 17 For any antisymmetric tensor F_{ij} we define a dual tensor $\overset{*}{F}_{ij} = \frac{1}{2} \varepsilon_{ijkl} F^{kl}$. Prove that the dual of the dual is plus or minus the original tensor

$$\frac{1}{2} \varepsilon_{ijkl} \overset{*}{F}{}^{kl} = e F_{ij}$$

Let's go streightforward $\frac{1}{2} \varepsilon_{ijkl} \overset{*}{F}{}^{kl} = \frac{1}{4} \varepsilon_{ijkl} \varepsilon^{klmn} F_{mn} =$

$= \frac{1}{4} \varepsilon_{ijkl} \varepsilon^{mnkl} F_{mn} = \frac{1}{2} e (\delta^m{}_i \delta^n{}_j - \delta^m{}_j \delta^n{}_i) F_{mn} = \frac{1}{2} e (F_{ij} - F_{ji}) =$

$= \frac{1}{2} e F_{ij} \cdot 2 = e F_{ij}$. here we have used equation (ii) from exercise 16: $\varepsilon_{ijkl} \varepsilon^{ijmn} = 2e(\delta^m{}_k \delta^n{}_l - \delta^m{}_l \delta^n{}_k)$ and antisymmetric property of F_{ij}: $F_{ij} = -F_{ji}$;

so we indeed get $\boxed{\frac{1}{2} \varepsilon_{ijkl} \overset{*}{F}{}^{kl} = e F_{ij}}$, q.e.d.

① Seminar 6 (relativistic particle mechanics)

Theory In previous lecture we have introduced some 4-vectors that will be usefull in seminars 6 and 7. This is mainly 4-momentum vector which is introduced in the way analogeous to the usual 3-momentum $p^i = m_0 u^i$ and $p^\mu = m_0 u^\mu$

$\downarrow$ nonrelativistic, $\downarrow$ relativistic.

m_0 is calle the rest mass. If we write it in the components form: $p^\mu = m_0 \gamma(u)(c, \bar{u})$. There are 2 conclusions can be drawn from this expression

* first of all if we look on spatial components we get following expression for 3-momentum:

$\bar{p} = m_0 \gamma(u) \cdot \bar{u} = m\bar{u}$ the last form looks very much like usual nonrelativistic expression, in which we have introduced velocity-dependent relativistic mass of particle $\boxed{m(u) = m_0 \gamma(u)}$

* second we should interpret somehow p^0. Right interpretation coming from nonrelativistic limit is that $p^0 = \frac{E}{c}$, i.e this is energy of particle. Difference with nonrelativistic mechanics is that particle posses energy even being at rest ($\boxed{E_0 = m_0 c^2}$) Kinetic energy is then given by $\boxed{T = E - E_0 = m_0 c^2 (\gamma - 1)}$;

Important and usefull law we will widely use is 4-momentum conservation

$\sum_{\text{before react}} p_\mu = \sum_{\text{after react}} p_\mu$. Using just this single relation a lot of problems can be solved.

Another thing that will be usefull to us is relation for $p_\mu p^\mu$: $p_\mu p^\mu = m_0^2 c^2$ and invariance of scalar product.

②

Problem 1 How fast must particle move before its kinetic energy equals it's rest energy?

Let m_0 be the rest mass of particle. Then as we have already written energy is given by $E = m_0 c^2 \gamma$, rest energy is $E_0 = m_0 c^2$ and kinetic is $T = m_0 c^2 (\gamma - 1)$. So if we want kinetic energy to be equal to rest energy we should obtain:

$T = E_0 \Rightarrow \gamma - 1 = 1 \Rightarrow \boxed{\gamma = 2}$

Problem 2 How fast must a 1kg cannon ball move to have the same kinetic energy as a cosmic-ray proton moving with γ-factor 10^{11}?

To solve this we should know proton mass:

$m_p = 938 \frac{MeV}{c^2}$ or in SI units we get: $m_p = 1{,}6 \cdot 10^{-27}$ kg.

Thus kinetic energy is $T_p = m_p c^2 (\gamma - 1) \approx m_p c^2 \gamma$ as $\gamma \gg 1$

canon ball mass is 27 orders bigger then the proton's one, so cannon ball will be nonrelativistic for sure. and we can approximate it's kinetic energy by standart non-relativistic formulae. $T_b = \frac{1}{2} m_b v^2$ where $m_b = 1$kg is canon ball rest mass. thus we get

$\frac{1}{2} m_b v^2 = \gamma m_p c^2 \Rightarrow v = \sqrt{\frac{2\gamma m_p c^2}{m_b}} = \sqrt{\frac{2 \cdot 10^{11} \cdot 1{,}6 \cdot 10^{-27}}{1}}\, c =$

$= 1{,}79 \cdot 10^{-8} \cdot 3 \cdot 10^{8} \frac{m}{s} = 5{,}36 \frac{m}{s}$; thus $\boxed{v = \sqrt{\frac{2\gamma m_p c^2}{m_b}} = 5{,}36 \frac{m}{s}}$

Problem 3 What is the γ-factor of a proton accelerated to an energy of 20 TeV?

The rest mass of the proton is $m_p = 938 \frac{MeV}{c^2}$

Energy of the proton is $\mathcal{E}_p = m_p \cdot \gamma \cdot c^2$ thus $\gamma = \frac{\mathcal{E}_p}{m_p c^2}$

$\gamma = \frac{20 \cdot 10^{12}}{938 \cdot 10^{6}} \simeq 21300$ $\boxed{\gamma = \frac{\mathcal{E}_p}{m_p c^2} \approx 21300}$

③

Problem 4 The mass of hydrogen atom is 1.00814 a.m.u., that of a neutron is 1.00898 a.m.u., and that of a helium atom (2 hydrogen atoms and 2 neutrons) is 4.00388 a.m.u. Find the binding energy as a fraction of the total energy of a helium atom.

$m_H = 1,00814$ a.m.u. ; $m_n = 1,00898$ a.m.u. ; $m_{He} = 4,00388$ a.m.u. ;

mass excess equals $\Delta m = 2m_p + 2m_n - m_{He} = 2 \cdot 1,00814$ a.m.u. $+$

$+ 2 \cdot 1,00898$ a.m.u. $- 4,00388$ a.m.u. $= 0,03036$ a.m.u.

mass excess is related to the bonding energy directly: $E_{Bond} = \Delta m \cdot c^2$; thus

$$\frac{E_{Bond}}{E_{He}} = \frac{\Delta m \cdot c^2}{m_{He} c^2} = \frac{\Delta m}{m_{He}} = \frac{0,03036}{4,00388} = 0,00758 \ (0,76\%)$$

$$\boxed{\frac{E_{Bond}}{E_{He}} = 2\,\frac{m_p + m_n}{m_{He}} - 1 = 0,76\%}$$

Problem 6 A rocket propels itself rectilinearly by giving portions of its mass a constant (backward) velocity U relative to its instantaneous rest frame. It continious to do so until it attains a velocity V relative to its initial rest frame. Prove that the ratio of the initial to the final rest mass of the rocket is

$$\frac{M_i}{M_f} = \left(\frac{c+V}{c-V}\right)^{\frac{c}{2U}}$$

Note that the least expenditure of mass needed to attain a given velocity occurs when $U=c$, i.e. when the rocket propels itself with a jet of photons

Let S be the initial rest frame and S' - instantaneous rest frame at some time t measured in S and corresponding to the proper time τ (proper time of ship)

Let's consider system rocket + ejected gas in instentoneous rest frame between moments of time

④ τ and $\tau + d\tau$. During this time ship ejects mass $(-dM)$ fuel which gives it momentum $-dM \cdot U = M\,du'$ where du' is velocity in instanteneous frame.

In initial rest frame in corresponding moment of time we get due to velocity transformation formulae:

$$V + dV = \frac{V + du'}{1 + \frac{V du'}{c^2}} \Rightarrow V + dV + \frac{V}{c^2} dV du' + \frac{V^2}{c^2} du' = V + du'$$ thus

we finally get: $dV = du'\left(1 - \frac{V^2}{c^2}\right) = -\frac{dM}{M} u \left(1 - \frac{V^2}{c^2}\right)$

on the second step we have used relation obtained from momentum conservation law. Thus we get:

$$-\frac{dM}{M} \cdot u = \frac{dV}{1 - \frac{V^2}{c^2}} = \frac{1}{2} \frac{dV}{1 + \frac{V}{c}} + \frac{1}{2} \frac{dV}{1 - \frac{V}{c}}$$

This equation can be easily integrated:

$-u \cdot \ln \frac{M_f}{M_i} = \frac{1}{2} c \log \frac{c + V}{c - V}$ and thus we finally get desired answer: $\boxed{\frac{M_i}{M_f} = \left(\frac{c + V}{c - V}\right)^{\frac{c}{2u}}, \text{ q.e.d.}}$

(ii) By reference to exercise II (15), prove that, if the rocket moves with constant proper acceleration α for a proper time interval τ, then $\frac{M_i}{M_f} = \exp\left(\frac{\alpha \tau}{u}\right)$. If $u = c$, $\alpha = g$ and $\tau = n$ yr, prove $\frac{M_i}{M_f} \approx e^n$;

By definition proper acceleration is $\alpha = \frac{du'}{d\tau} = -\frac{u}{M} \frac{dM}{d\tau}$;

Integration of this equation gives us

$\tau \alpha = u \log \frac{M_i}{M_f}$; and thus $\frac{M_i}{M_f} = \exp \frac{\alpha \tau}{u}$, substituting here $u = c \approx 1$, $\tau = n$ years; $\alpha = g \approx 1 \frac{\text{l.yr.}}{\text{yr.}}$ we get:

$$\boxed{\frac{M_i}{M_f} = e^n, \text{ q.e.d.}}$$

(5)

Problem 7 Two particles with rest masses m_1 and m_2 move collinearly in some inertial frame, with uniform velocities u_1 and u_2 respectively. They collide and form a single particle with rest mass m moving at velocity u. Prove that

$$m^2 = m_1^2 + m_2^2 + 2m_1 m_2 \gamma(u_1)\gamma(u_2)\left(1 - \frac{u_1 u_2}{c^2}\right)$$

and also find u.

Here we will see how convinient 4-momentum conservation is. Here it takes form

$P^\mu = P_1^\mu + P_2^\mu$; Let's square both sides. We get:

$P^2 = (P_1 + P_2)^2$ (as usually by p^2 we mean $p_\mu p^\mu$)

Here in particular $P^\mu = \gamma(u)\, m\,(c, \bar{u})$; $P_1^\mu = \gamma(u_1) m_1 (c; \bar{u}_1)$ and $P_2^\mu = \gamma(u_2) m_2 (c; \bar{u}_2)$ as initial movement of particles 1 and 2 is collinear we conclude that $\bar{u}_1 \parallel \bar{u}_2 \parallel \bar{u}$ (collinearity of $\bar{u}$ with inital velocities follows from conservation of 3-momentum. Transverse momentum can't appear after collision due to conservation). So we get

$P^2 = P_1^2 + P_2^2 + 2P_1 \cdot P_2$ $\quad P_i^2 = m_i^2 c^2$ $\quad$ and $P_1 \cdot P_2 = \gamma(u_1)\gamma(u_2) m_1 m_2 \times c^2\left(1 - \frac{u_1 u_2}{c^2}\right)$

thus

$$\boxed{m^2 = m_1^2 + m_2^2 + 2\gamma(u_1)\gamma(u_2) m_1 m_2 \left(1 - \frac{u_1 u_2}{c^2}\right);} \quad (*)$$

to find u let's consider conservation of energy

$P^0 = P_1^0 + P_2^0; \Rightarrow \gamma(u) = \frac{m_1}{m}\gamma(u_1) + \frac{m_2}{m}\gamma(u_2)$; thus

$$\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{m_1\gamma(u_1) + m_2\gamma(u_2)}{m} \Rightarrow \frac{u^2}{c^2} = 1 - \frac{m^2}{(m_1\gamma_1 + m_2\gamma_2)^2};$$

now substituting m^2 from $(*)$ into this equation as $(1 - \gamma_i^2) = -\frac{u_i^2}{c^2}\gamma_i^2$

$$\frac{u^2}{c^2} = \frac{m_1^2(\gamma_1^2 - 1) + m_2^2(\gamma_2^2 - 1) + 2\gamma_1\gamma_2 m_1 m_2 \frac{u_1 u_2}{c^2}}{(m_1\gamma_1 + m_2\gamma_2)^2}$$

$$\boxed{u = \frac{|m_1\gamma_1 u_1 + m_2\gamma_2 u_2|}{m_1\gamma_1 + m_2\gamma_2}}$$

Problem 8 Consider a head-on elastic collision of a bullet of rest mass M with a stationary target of rest mass m. Prove that the post-collision γ-factor of the bullet cannot exceed $\frac{(m^2+M^2)}{2mM}$. This means that for large bullet energies (with γ factors much larger than this critical value), the relative transfer of energy from bullet to target is almost total. The situation is radically different in Newtonian mechanics, where the pre- and post-collision velocities of the bullet are related by $\frac{u}{u'} = \frac{M+m}{M-m}$. Prove this

* relativistic case

Let P and Q be 4 momentum of bullet and target correspondingly, before the collision. P' and Q' - corresponding 4-momenta after collision.

We will use time-like nature of 4-momentum. In fact we can consider $(P'-Q)^2$ and go to the center of momentum system where only nonzero component of $(P'-Q)$ is $(P'^0 - Q^0)$ so that we can say that in this system $(P'-Q)^2 = (P'^0 - Q^0)^2 > 0$ and thus as $(P'-Q)^2$ is Lorentz invariant it will be positive in all frames

in the rest frame of target we have

$P'^\mu = \gamma(u')(c;\bar{u}')m$; $Q^\mu = (cM;0)$;

$(P'-Q)^2 = M^2c^2 + m^2c^2 - 2P'_\mu Q^\mu$; substituting explicit form of 4-mometa into this expression we get:

$M^2c^2 + m^2c^2 - 2c^2Mm\gamma(u') \geq 0$ thus $\boxed{\gamma(u') \leq \frac{M^2+m^2}{2Mm}}$, q.e.d

(7)

* nonrelativistinc case

Here we just write momentum and energy conservation

$Mu = Mu' + mv'$

$\frac{1}{2}Mu^2 = \frac{1}{2}Mu'^2 + \frac{1}{2}mv'^2$

from the first equation we get:

$v' = \frac{M}{m}(u-u')$; substituting this into second equation we get: $\frac{1}{2}Mu^2 = \frac{1}{2}Mu'^2 + \frac{1}{2}\frac{M^2}{m}(u-u')^2$ thus

$mu^2 = mu'^2 + Mu^2 + Mu'^2 - 2Mu'u$ If we now denote $x = \frac{u}{u'}$; $(m-M)x^2 + 2Mx - (M+m) = 0$;

$D = 4M^2 + 4m^2 - 4M^2 = 4m^2$; $x = \frac{-M \pm m}{m-M}$ "+" sign give us trivial result. when collision doesn't effect the motion of bullet $x = \frac{m+M}{M-m}$ is solution we are interested in. so $\boxed{\frac{u}{u'} = \frac{M+m}{M-m}}$; difference with the relativistic case is that there we have obtained the limit for velocity of bullet after collision. Here in classical case there is no such limit and velocity after collision linearly depends on velocity before the collision.

Problem 11 Show that a photon cannot spontaneously disintegrate into an electron-positron pair. But in the presence of a stationary nucleus (acting as a kind of catalyst) it can. If the rest mass of nucleus is N, and that of the electron is m, what is the threshold frequency of photon? Verify that for large N the efficiency is ~ 100 percent so that the nucleus then comes close to being a pure catalyst.

4-momentum conservation give us $P^\mu = P_1^\mu + P_2^\mu$; thus $(P_1 + P_2)^2 = 0$;

(8) $2m_e^2c^2 + 2P_1 \cdot P_2 = 0$; in electron-positron COM

$P_1^\mu + P_2^\mu = (P_1^0 + P_2^0; \bar{0})$ thus in this system for $(P_1+P_2)^2=0$ we should obtain $P_1^0 = -P_2^0$ which can be satisfied only if $P_1^0 = P_2^0 = 0$ which in turn can't be satisfied as $P_{1,2}^0 \geq m_e c$; Anothe way to show the same fact is write down $2m_e^2c^2 + 2P_1 \cdot P_2 = 2m_e^2c^2 + 2m_e^2c^2\gamma(u_1)\gamma(u_2)\left(1 - \frac{u_1 u_2}{c^2}\right) = 0$ and as velocities of electron and positron are smaller the the speed of light we get $1 - \frac{u_1u_2}{c^2} \geq 0$ and thus $2m_e^2c^2 + 2m_e^2c^2\gamma(u_1)\gamma(u_2)\left(1 - \frac{u_1u_2}{c^2}\right) > 0$ always. We can say that decay is forbidden because we can't go to the photons rest frame.

Now assume we have one more body in the story, namely, nuclei

$$P^\mu + N^\mu = Q_1^\mu + Q_2^\mu + N'^\mu;$$

threshold energy occurs when in the COM frame of resulting particles all they have zero momentum, taking square of both sides we get:

$(P_\mu + N_\mu)^2 = c^2N^2 + 2Nc\,h\nu = (Q_1^\mu + Q_2^\mu + N'^\mu)^2 = (2m_e^2 + N^2)c^2 + 4m_eNc^2 + 2m_e^2 \cdot c^2$ wher we have used following 4-momentas

$$\left.\begin{aligned} P^\mu &= \tfrac{1}{c}h\nu(1, \bar{n}); \\ N^\mu &= N \cdot (c, \bar{0}); \end{aligned}\right\} \text{lab. frame}$$

$$\left.\begin{aligned} N'^\mu &= N \cdot (c, \bar{0}); \\ Q_{1,2}^\mu &= m_e \cdot (c, \bar{0}); \end{aligned}\right\} \text{threshold in COM;}$$

$\frac{1}{c}h\nu \cdot 2Nc = 4m_e^2c^2 + 4m_eNc^2$; $\boxed{\nu = \frac{2m_e^2c^2 + 2m_eNc^2}{hN}}$

(9) A fast electron of rest mass m decelerates in a

Problem 13 collision with a heavy nucleus and emits a Bremsstrahlung photon. Prove that the energy of the photon can range all the way up to $(\gamma-1)mc^2$, the kinetic energy of electron

What we have is process:

$N + e^- \to N + e^- + \gamma$, 4-momentum conservation gives us

$$N^\mu + p^\mu = N'^\mu + p'^\mu + q^\mu$$

energy of photon is maximal when energy of system electron + nuclei is minimal and this happens when in COM frame of electron and nuclei both particles are in rest

---

Note: To understand this fact conside $(p_1+p_2)^\mu = \left(\frac{\mathcal{E}_1+\mathcal{E}_2}{c}; \bar{p}_1+\bar{p}_2\right)$

$(p_1+p_2)^2 = \frac{1}{c^2}(\mathcal{E}_1+\mathcal{E}_2)^2 + (\bar{p}_1+\bar{p}_2)^2 = m_1^2c^2 + m_2^2c^2 + 2p_1\cdot p_2$;

r.h.s. of this equation is minimal when $p_1 p_2$ is minimal.

as $p_1 p_2$ is Lorentz invariant we can go into COM frame where we have $\bar{p}_1 = -\bar{p}_2 = \bar{p}$ and thus $p_1 p_2 = \frac{1}{c^2}\mathcal{E}_1\mathcal{E}_2 + |\bar{p}|^2$ which is minimal, in turn, when $\bar{p}=0$, i.e. both particles are in rest.

Now we write 4-momentum consevation in the following form: $N^\mu + p^\mu - q^\mu = N'^\mu + p'^\mu$ and square it

$(N+p-q)^2 = (N'+p')^2$. It is reasonable to consider l.h.s in lab. frame and r.h.s. in COM-frame of nuclei and electron. Then

$$\left.\begin{aligned} N &= (m_N c, \bar{0}) \\ p &= m_e\gamma(u)(c; \bar{u}) \\ q &= \frac{h\nu}{c}(1, \bar{n}) \end{aligned}\right\} \text{lab. fram. } \bar{n} \text{ is direction of photon velocity}$$

$$\left.\begin{aligned} N' &= (m_N c; \bar{0}) \\ p' &= (m_e c; \bar{0}) \end{aligned}\right\} \text{N-e COM-frame.}$$

Here m_N and m_e are masses of nuclei and electron

(10) respectively and ν is frequency of photon. Thus we get:

$$m_N^2 \cdot c^2 + m_e^2 c^2 + 2 m_N m_e c^2 \gamma(u) - 2 \frac{h\nu}{c} \cdot m_N c - 2 \frac{h\nu}{c} \gamma(u) m_e (c - u \cos\theta) =$$
$$= c^2 (m_e + m_N)^2 \text{ ; thus we get}$$

$$h\nu = \frac{m_e m_N c^2 (\gamma(u) - 1)}{m_N + m_e \gamma(u)\left(1 - \frac{u}{c} \cdot \cos\theta\right)} \leqslant \frac{m_e m_N c^2 (\gamma(u) - 1)}{m_N + m_e \gamma(u)\left(1 - \frac{u}{c}\right)}\text{, maximum}$$

is obtained when $\cos\theta = 1$. Then maximal energy of photon is given by $\varepsilon_{max} = \dfrac{m_e m_N c^2 (\gamma(u) - 1)}{m_N + m_e \sqrt{\dfrac{1 - \frac{u}{c}}{1 + \frac{u}{c}}}}$; as

$\sqrt{\dfrac{1 - \frac{u}{c}}{1 + \frac{u}{c}}} \leqslant 1$ and $m_e \ll m_N$ we get $\boxed{\varepsilon_{max} = m_e c^2 (\gamma - 1) \text{, q.e.d.}}$

①

Seminar 7 (relativistic particle mechanics II)

Theory

Some facts from previous class

The main kind of problems we will be solving on this class are collision problems and main tool is 4-momentum conservation

$$\sum_{\substack{\text{before}\\\text{collision}}} p^{\mu}_i = \sum_{\substack{\text{after}\\\text{collision}}} p^{\mu}_j$$ (latin indicies correspond to different particles)

Main trick here is to consider different 4-momentum conservation squared. As $p_{\mu} p^{\mu}$ is Lorentz invariant quantity we can consider it in convinient frame. For example, useful frame is centre of momentum (CM) frame, thus the frame in which $P^{\mu} = \sum p^{\mu}$ has no spatial components, thus $\sum \bar{p} = 0$. This are main things we need. For better understanding we should consider some concrete examples.

Problem 14 A particle of rest mass m decays from rest into a particle of rest mass m' and photon. Find the separate energies of this end products.

Let's write down 4-momentum conservation:

$p^{\mu} = p'^{\mu} + q^{\mu}$ here we have introduced following notations:

$p^{\mu} = (mc; \bar{0})$ - 4-momentum of decaying particle

$p'^{\mu} = m'\gamma(u')(c, \bar{u}')$ - 4-momentum of massive product of decay

$q^{\mu} = h\nu(1, \bar{n})$ - 4-momentum of photon, $\bar{n}$ is photon's direction of movement, $\bar{u}'$ - massive particle velocity.

* first let's consider massive particle

Let's exclude photon by writing:

$q^{\mu} = p^{\mu} - p'^{\mu}$ Squaring this we obtain $(p-p')^2 = 0$ as

② $q^2 = 0$ for photon (Note: remember that for massive particle $p^2 = m^2c^2$ and for massless particles $q^2 = 0$) thus

$(m^2 + m'^2)c^2 - 2m\mathcal{E}' = 0$ and finally we get

$$\boxed{\mathcal{E}' = \frac{m^2 + m'^2}{2m} \cdot c^2;}$$

* now let's find energy of photon

$p^\mu - q^\mu = p'^\mu$ → in the same way as we excluded photon dynamics in previous part of the problem, in this part we exclude massive product of decay. Now we get: $(p-q)^2 = m^2c^2 - 2\mathcal{E}_\gamma m$; $p'^2 = m'^2c^2$; thus

$m^2c^2 - 2\mathcal{E}_\gamma m = m'^2c^2$; and finally we get:

$$\boxed{\mathcal{E}_\gamma = \frac{(m^2 - m'^2)c^2}{2m};}$$

Problem 15 An excited atom, of total mass m, is at rest in a given frame. It emits a photon and thereby loses internal (i.e. rest) energy ΔE. Calculate the exact frequency of the photon, making due allowance for the recoil of the atom.

In fact in this problem we have just the same process as in the previous problem, i.e. 4-momentum conservation. $p^\mu = p'^\mu + q^\mu$ where p^μ is 4-momentum of atom of mass m, p'^μ – 4-momentum of atom of mass m' and q^μ – 4-momentum of photon. In our case

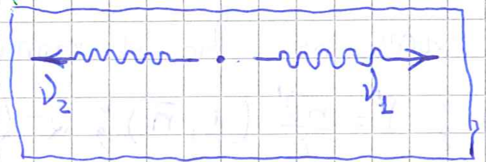
$m' = m - \frac{\Delta E}{c^2}$; So we can directly apply formulae obtained in previous problem:

$$\mathcal{E}_\gamma = h\nu = \frac{(m^2 - m'^2)c^2}{2m} = \frac{c^2}{2m}\left(m^2 - \left(m - \frac{\Delta E}{c^2}\right)^2\right) = \frac{c^2}{2m}\left(\frac{2\Delta E}{c^2}\cancel{m} - \frac{(\Delta E)^2}{c^4}\right) =$$

$= \Delta E\left(1 - \frac{\Delta E}{2mc^2}\right)$, and thus $\boxed{\nu = \frac{\Delta E}{h}\left(1 - \frac{\Delta E}{2mc^2}\right)}$ q.e.d.

(3)

Problem 18 In an inertial frame S, 2 photons of frequencies ν_1 and ν_2 travel in the positive and negative x-directions respectively. Find the velocity of the CM frame of these photons.



; $p_1^\mu = \frac{h\nu_1}{c}(1;\hat{x})$

$p_2^\mu = \frac{h\nu_2}{c}(1;-\hat{x})$

here we have written down 4-momentum of photons, $\hat{x}$ is positive direction of x-axis. As we have written befor CM-frame is one were spatial components of 4-momentum equals zero. Total 4-momentum

$$P^\mu = p_1^\mu + p_2^\mu = \frac{h}{c}(\nu_1+\nu_2;\ \nu_1-\nu_2;0;0)$$

Now let's transform this 4-momentum with Lorentz transformation. $P^{\mu'} = \Lambda^{\mu'}{}_\nu P^\nu$, where $\Lambda^{\mu'}{}_\nu$ is standart lorentz transformation matrix

$$\Lambda^{\mu'}{}_\nu = \begin{bmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

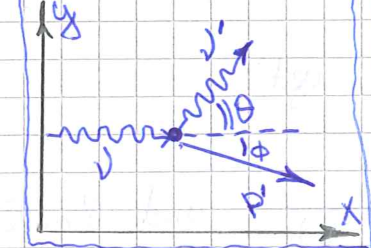
In particular:

$$P^{1'} = \Lambda^{1'}{}_\mu P^\mu = -\frac{v}{c}\gamma P^0 + \gamma P^1 = \frac{h}{c}\gamma(\nu_1-\nu_2 - \frac{v}{c}\nu_1 - \frac{v}{c}\nu_2)$$

$P^{2'} = \Lambda^{2'}{}_\mu P^\mu = P^2 = 0$; $P^{3'} = \Lambda^{3'}{}_\mu P^\mu = P^3 = 0$; So as we can see if we want momentum to be zero we need

$$P^{1'} = \frac{h}{c}\gamma(\nu_1-\nu_2-\frac{v}{c}\nu_1-\frac{v}{c}\nu_2) = 0$$ so that $\boxed{\frac{v}{c} = \frac{\nu_1-\nu_2}{\nu_1+\nu_2}}$, q.e.d.

Problem 19 For the Compton collision discussed in [Rindler] section 33 prove the relation $\tan\varphi = \left(1+\frac{h\nu}{mc^2}\right)^{-1}\cot\frac{1}{2}\theta$;



Let P, P' are pre- and post-collision four-momenta of the photon and Q, Q' those of electrons (this are notations of Rindler). Let first recall the main results of Compton scattering derived in book. Then 4-momentum conservation is:

(4) $P^\mu + Q^\mu = P'^\mu + Q'^\mu$; To find frequency ν' after collission we should read off electron after collision

$P^\mu + Q^\mu - P'^\mu = Q'^\mu$ squaring this we get:

$(P+Q-P')^2 = Q'^2$ thus $P^2+Q^2+P'^2+2P\cdot Q - 2P\cdot P' - 2P'\cdot Q = Q'^2$

Now we use the following expressions for 4-momentum in lab. frame: $P = \frac{h\nu}{c}(1;\hat{x})$; $P' = \frac{h\nu'}{c}(1;\bar{n})$; $Q = (mc;\bar{0})$;

$Q' = m\gamma(u)(c;\bar{u})$; where $\bar{n}$ is vector pointing in direction of outgoing photon momentum; $P^2 = P'^2 = 0$; $Q^2 = Q'^2 = m^2c^2$;

Then we get:

$h(\nu - \nu')m - \frac{1}{c^2}h^2\nu\nu'(1-\cos\theta) = 0$ where θ is the angle between $\bar{n}$ and $\hat{x}$, using trigonometric identity

$1-\cos\theta = 2\sin^2\frac{\theta}{2}$; $\boxed{\sin^2\frac{\theta}{2} = \frac{mc^2}{2h}\left(\frac{1}{\nu'} - \frac{1}{\nu}\right);}$ q.e.d. (*)

Now to find angle between electron's direction of motion after collision let's consider separately conservations of all components of momentum, assuming that reaction plane coinsides with x-y plane

x: $P^1 + Q^1 = P'^1 + Q'^1 \Rightarrow \frac{h\nu}{c} = p'\cos\varphi + \frac{h\nu'}{c}\cos\theta$;

y: $P^2 + Q^2 = P'^2 + Q'^2 \Rightarrow p'\sin\varphi = \frac{h\nu'}{c}\cdot\sin\theta$;

so we get $p'\sin\varphi = \frac{h\nu'}{c}\sin\theta$; $p'\cos\varphi = \frac{h\nu}{c} - \frac{h\nu'}{c}\cos\theta$;

Deviding one equation with another we get:

$$\tan\varphi = \frac{\nu'\sin\theta}{\nu - \nu'\cos\theta} = \frac{\sin\theta}{\frac{\nu}{\nu'} - \cos\theta};$$ now from (*) we

know that $\frac{\nu}{\nu'} = 1 + \frac{2h\nu}{mc^2}\sin^2\frac{\theta}{2}$; substituting this

expression into equation above we get:

$$\tan\varphi = \frac{\sin\theta}{1-\cos\theta + \frac{2h\nu}{mc^2}\sin^2\frac{\theta}{2}} = \frac{\sin\theta}{2\sin^2\frac{\theta}{2}}\,\frac{1}{1+\frac{h\nu}{mc^2}}$$ and finally

$$\boxed{\tan\varphi = \cot\frac{\theta}{2}\left(1 + \frac{h\nu}{mc^2}\right)^{-1}, \quad \text{q.e.d.}}$$

(5)

Problem 21 Taking $h = 6,63 \cdot 10^{-27}$ ergs and $c = 3 \cdot 10^5 \frac{km}{s}$; calculate how many photons of wavelength $5 \cdot 10^{-5}$cm must fall per second on a blackened plate to produce a force of one dyne.

Force is just momentum absorbed per unit time: $F = \frac{\Delta p}{\Delta t}$. Let N denote total number of photons absorbed. Then total absorbed momentum is $\frac{Nh\nu}{c} = \frac{Nh}{\lambda}$ thus we get that

$$\frac{N}{\Delta t} = \frac{F\lambda}{h} = \frac{5 \cdot 10^{-5} cm \cdot 1 dyne}{6,63 \cdot 10^{-27} erg} = 7,5 \cdot 10^{21} s^{-1};$$

$$\boxed{\frac{N}{\Delta t} = \frac{F\lambda}{h} = 7,5 \cdot 10^{21} s^{-1};}$$

Problem 23 A point source of light moves at constant velocity directly towards (or away from) an observer O. In it's rest frame it radiates isotropically with total luminosity (power) L. Prove that, if it was at distance r when it emitted the light whereby it is now seen, the energy flux due to it at O is $\frac{L}{4\pi r^2} D^4$, where $D = \frac{(\text{frequency of reception})}{(\text{frequency of emition})}$ for a typical spectral line.

It is instructive to work the problem in two ways: once in the rest frame of the source, and once in the rest frame of the observer.

We will consider only source moving away from an observer O. First of all let's find out how luminosity changes when we go from one frame to another. Luminosity is proportional to the energy emmited per second by the light source. Let's consider this process from corpuscular point of

⑥ view. Let's say that source emits 1 photon of frequency ν' in $\Delta t'$ time in O'-frame moving with source.
In O we get due to Dopler effect $\nu = \nu' \sqrt{\frac{c-v}{c+v}}$ and due to time dilation and movement of source (remember here the way of Doppler effect derivation)
$\Delta t = \gamma(\Delta t' + \Delta t' \cdot \frac{v}{c}) = \Delta t' \cdot \gamma(1+\frac{v}{c}) = \Delta t' \cdot \sqrt{\frac{c+v}{c-v}}$. So when we go from one frame to another luminosity changes in the following way:

$$\frac{L}{L'} = \frac{\nu}{\nu'}\frac{\Delta t'}{\Delta t} = \frac{c-v}{c+v} = \left(\frac{\nu}{\nu'}\right)^2; \quad \text{So} \quad \boxed{L = \frac{c-v}{c+v}L'; \; \nu = \sqrt{\frac{c-v}{c+v}}\,\nu';}$$

Now we go further and try to find change of flux while going from one frame to another. This can be done in two frames:

Ⓘ O'-frame In this frame it is quite easy to do. As in O' radiation is isotropic we get in O' flux equal to

$F = \frac{L}{4\pi x^2}$ where $L = \frac{c-v}{c+v}L'$ - luminosity that observer O observes, and x is distance to observer in O'-frame
$x = \gamma(r + v\Delta t)$ wher Δt is time in O-frame essential for photon to reach observer O: $\Delta t = \frac{r}{c}$. So we get $x = r \cdot \gamma \cdot (1+\frac{v}{c}) = r \cdot \sqrt{\frac{c+v}{c-v}}$; Then we get

$F = \frac{L'}{4\pi r^2}\left(\frac{c-v}{c+v}\right)^2$; as $\nu = \nu' \cdot \sqrt{\frac{c-v}{c+v}}$ thus $D = \sqrt{\frac{c-v}{c+v}}$ and indeed $\boxed{F = \frac{L'}{4\pi r^2} D^4 \text{, q.e.d.}}$

Note: First of all notations are a bit messy. What we call L' here is L in notations of Rindler Book. Second - all reasonings remain the same for source of light moving towards observer (we should just make change $v \to -v$) and conclusions remain the same.

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II Now let's find flux F' considering problem in O frame. In this frame source doesn't emmit light isotropic due to "headlight" effect (see problem 13 from chapter III). So let's find the ratio $\frac{d\Omega}{d\Omega'}$ where $d\Omega$ is element of solid angle in O and $d\Omega'$ - solid angle in O'. In problem 13 (chapter III) we have derived relation between polar angles $\tan\frac{\theta}{2} = \sqrt{\frac{c-v}{c+v}}\tan\frac{\theta'}{2}$; If source is moving away from observer, this corresponds to angles close to π: $\theta' = \pi - \alpha'$ where α' is small. Then we get by taking derivative:

$$\frac{d\theta}{\cos^2\frac{\theta}{2}} = \sqrt{\frac{c-v}{c+v}}\frac{d\theta'}{\cos^2\frac{\theta'}{2}}$$

, now using $\cos^2\theta = \frac{1}{1+\tan^2\theta}$ we get:

$$d\theta = d\theta' \cdot \sqrt{\frac{c-v}{c+v}} \frac{1+\tan^2\frac{\theta'}{2}}{1+\tan^2\frac{\theta}{2}} = d\theta'\sqrt{\frac{c-v}{c+v}}\cdot\frac{1+\tan^2\frac{\theta'}{2}}{1+\frac{c-v}{c+v}\cdot\tan^2\frac{\theta'}{2}},$$

now as $\theta \to \pi$ we get: $\boxed{d\theta = d\theta' \cdot \sqrt{\frac{c+v}{c-v}}};$

Solid angle is given by $d\Omega = \sin\theta \, d\theta \, d\varphi$

$d\varphi$ doesn't change after Lorentz transformations, because it corresponds to rotations transversal to boost direction (boost is made along x-axis) so we need else to find relation between "$\sin\theta$" and "$\sin\theta'$"

$\tan\frac{\theta'}{2} = \tan\left(\frac{\pi}{2} - \frac{\alpha'}{2}\right) \approx \frac{2}{\alpha'}$; then we get

$\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{c-v}{c+v}\left(\frac{2}{\alpha'}\right)^2 = A \to \infty$ so that $A - A\cos\theta = 1 + \cos\theta$

and $\cos\theta = \frac{1-A}{1+A}$ and $\sin^2\theta = \frac{4A}{(1+A)^2} \approx \frac{4}{A} = \frac{c+v}{c-v}\alpha'^2 = \frac{c+v}{c-v}\cdot\sin^2\alpha' =$

$= \frac{c+v}{c-v}\sin^2\theta'$ and finally $\sin\theta = \sqrt{\frac{c+v}{c-v}}\sin\theta'$. Thus

$\frac{d\Omega'}{d\Omega} = \frac{\sin\theta' \cdot d\theta'}{\sin\theta \, d\theta} = \frac{c-v}{c+v}$, and for flux we get

$\frac{F}{F'} = \frac{d\Omega'}{d\Omega} = \frac{\sin\theta' \cdot d\theta'}{\sin\theta \, d\theta} = \frac{c-v}{c+v} = \mathcal{D}^2$ here F' is flux without taking into account "headlight" effect

8 i.e. $F' = \frac{L}{4\pi r^2}$ so we get $\boxed{F = \frac{L'}{4\pi r^2} D^4 \quad , q.e.d.}$

Problem 27 A particle moves rectilinearly under a rest-mass preserving force in some inertial frame. Show that the product of its rest mass and its instanteneous proper acceleration equals the magnitude of the relativistic 3-force acting on the particle in that frame. Show also that this is not necessarily true when the motion is not rectilinear.

First let's recall some theory from lectures

By definition 4-force is $F^\mu \equiv \frac{dp^\mu}{d\tau}$; $F^\mu = \gamma(v)\left(\frac{1}{c}\frac{dE}{dt}; \bar{f}\right)$, and $\bar{f} = \frac{d\bar{p}}{dt}$. Mass-preserving or pure force is one that doesn't change the rest energy. Let's show some properties of pure force $U_\mu F^\mu = \gamma^2\left(\frac{dE}{dt} - \bar{f}\cdot\bar{u}\right)$, from the other point of view $U_\mu F^\mu$ is Lorentz invariant and thus can be considered for example in particle rest frame where we get $U_\mu F^\mu = \gamma c^2 \frac{dm_0}{dt}$ thus for pure force $U_\mu F^\mu = 0$ and $\frac{dE}{dt} = \bar{f}\cdot\bar{u}$ and finally $F^\mu = m_0 A^\mu = \gamma(u)\left(\frac{1}{c}\bar{f}\cdot\bar{u}; \bar{f}\right)$

thus $A^\mu = \frac{\gamma(u)}{m_0}\left(\frac{1}{c}\bar{f}\cdot\bar{u}; \bar{f}\right)$, finally

$\alpha^2 = -A_\mu A^\mu = -\frac{\gamma^2(u)}{m_0^2}\left(\frac{1}{c^2}(\bar{f}\cdot\bar{u})^2 - |\bar{f}|^2\right)$ then we use

$\bar{f}\cdot\bar{u} = |\bar{f}|\,|\bar{u}|\cdot\cos\theta$ where θ is angle between $\bar{f}$ and $\bar{u}$ 3-vectors. then we finally get:

$\alpha^2 m_0^2 = \gamma^2(u)\cdot|\bar{f}|^2\left(1 - \frac{u^2}{c^2}\cdot\cos^2\theta\right)$. If motion is rectilinear we get $\bar{f} \parallel \bar{u}$, thus $\theta = 0, \cos\theta = 1$ and we get $\alpha^2 m_0^2 =$ $= \gamma^2(u)\left(1 - \frac{u^2}{c^2}\right)|\bar{f}|^2 = |\bar{f}|^2$ where we have used $\gamma^2(u) = \left(1 - \frac{u^2}{c^2}\right)^{-1}$ if motion is not rectilinear $\cos^2\theta < 1$ $\left(1 - \frac{u^2}{c^2}\cos^2\theta\right) > \left(1 - \frac{u^2}{c^2}\right)$ and thus $\alpha^2 m_0^2 > |\bar{f}|^2$; so we get

$\boxed{\text{rectilinear } \alpha^2 m_0^2 = |\bar{f}|^2 \; ; \; \text{not rectilinear } \alpha^2 m_0^2 > |\bar{f}|^2}$ q.e.d.

1

Seminar 8 (electrodynamics I)

General note: We here use more convinient notations not the one in Rindler's book.

First of all we introduce antisymmetric tensor of electromagnetic field $F^{\mu\nu}$. It's defined by equation $\partial_\mu F^{\mu\nu} = \varkappa j^\nu$ where j^ν is current density $j^\nu = \rho_0 \gamma (c, \bar{v})$ where ρ_0 is charge density in the rest frame of charge flow, and v is velocity of flow; Actually $\partial_\mu F^{\mu\nu} = \varkappa j^\nu$ can be shawn to be just well known to us if $\varkappa = \frac{1}{c\varepsilon_0}$ and

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -cB_3 & cB_2 \\ E_2 & cB_3 & 0 & -cB_1 \\ E_3 & -cB_2 & cB_1 & 0 \end{bmatrix}$$

So we see that E^i and B^i are actualy components of tensor rather then vector. We else need Lorentz equations that can be written in 2 ways

* "relativistic" form $\boxed{\partial_\mu F^{\mu\nu} = \frac{1}{c\varepsilon_0} j^\nu ; \ \partial_\mu \tilde{F}^{\mu\nu} = 0;}$ where we have introduced dual tensor $\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$; where $\varepsilon^{\mu\nu\rho\sigma}$ is antisymmetric tensor.

* "usual" form

$$\boxed{\bar{\nabla}\bar{E} = \frac{1}{\varepsilon_0} \rho \ ; \quad -\frac{1}{c^2}\frac{\partial \bar{E}}{\partial t} + \bar{\nabla} \times \bar{B} = \mu_0 \bar{j}} ; \quad \varepsilon_0 \mu_0 = \frac{1}{c^2}$$

– this 2 equations are called "first pair of Maxwell equations" and correspond to equation $\partial_\mu F^{\mu\nu} = \frac{1}{c\varepsilon_0} j^\nu$

$$\boxed{\bar{\nabla} \cdot \bar{B} = 0 \ ; \quad \frac{\partial \bar{B}}{\partial t} + \bar{\nabla} \times \bar{E} = 0 ;}$$

second pair of Maxwell equations corresponding to equation $\partial_\mu \tilde{F}^{\mu\nu} = 0$;

Now from tensor properties of $F^{\mu\nu}$ we can derive how do electric and magnetic fields $\bar{E}$ and $\bar{B}$ transform under Lorentz transformation:

$E_1' = E_1$; $E_2' = \gamma(E_2 - vB_3)$; $E_3' = \gamma(E_3 + vB_2)$;

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$B'_1 = B_1$; $B'_2 = \gamma(B_2 + \frac{v}{c^2}E_3)$; $B'_3 = \gamma(B_3 - \frac{v}{c^2}E_2)$;

- There are 2 Lorentz invariants:
  * $F_{\mu\nu}F^{\mu\nu}$ (electromagnetic field action);
  * $F_{\mu\nu}\tilde{F}^{\mu\nu}$;

The very last thing we will need is Lorentz force, i.e. force acting on the charged particle placed in external electromagnetic field.

$$\boxed{\bar{f} = q(\bar{E} + \bar{v} \times \bar{B});}$$

Problem 1 (i) A particle of rest mass m and charge q is injected at velocity $\bar{u}$ into a constant pure magnetic field $\bar{B}$ at right angles to the field lines. Use the Lorentz force law to establish that the particle will trace out a circle of radius $\frac{mu\gamma(u)}{qB}$ with period $\frac{2\pi m\gamma(u)}{qB}$.

In the case of pure magnetic field Lorentz force acting on the charged particle is given by: $\bar{f} = q\bar{u}\times\bar{B}$ where $\bar{u}$ is the velocity of the particle. We can now write "relativistic Newton law"

$\bar{f} = \gamma m\bar{a} + \frac{\bar{u}}{c^2}(\bar{f}\cdot\bar{u})$ but as $\bar{f}\cdot\bar{u} = \bar{u}\cdot(\bar{u}\times\bar{B})\cdot q = 0$

we get simply: $\gamma(u)m\frac{du}{dt} = \bar{f} = q(\bar{u}\times\bar{B})$

As we are free to make any convinient choice of axes we can direct $\bar{B}$ in positive $\hat{z}$ direction:

$\bar{B} = B\cdot\hat{z}$, so that: $\bar{u}\times\bar{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_1 & u_2 & u_3 \\ 0 & 0 & B \end{vmatrix} = \hat{x}u_2B - \hat{y}u_1B =$

$= \begin{bmatrix} Bu_2 \\ -Bu_1 \\ 0 \end{bmatrix}$, so we can now write down equation of motion in components:

③ $$\begin{cases} m\gamma(u)\dfrac{du_1}{dt} = qBu_2 ; \\ m\gamma(u)\cdot\dfrac{du_2}{dt} = -qBu_1 ; \\ m\gamma(u)\dfrac{du_3}{dt} = 0 ; \end{cases}$$

We know that magnetic field doesn't do any work so energy $\mathcal{E} = mc^2\gamma(u)$ is preserved. We can show it directly by considering time derivative of energy:

$\dfrac{d\mathcal{E}}{dt} = mc^2 \dfrac{\partial\gamma}{\partial u}\dfrac{du}{dt}$, at the same time

$\dfrac{d(u^2)}{dt} = 2\bar{u}\dfrac{d\bar{u}}{dt} = 2\bar{u}\cdot\dfrac{q}{m\gamma(u)}(\bar{u}\times\bar{B}) = \dfrac{2q}{m\gamma(u)}\bar{B}(\bar{u}\times\bar{u}) = 0$. So

we get $\dfrac{d\mathcal{E}}{dt} = 0$ indeed and $u = const$. Taking one more derivative with respect to "t" we get:

$\ddot{u}_x = \dfrac{qB}{m\gamma}\cdot\dot{u}_y = -\left(\dfrac{qB}{m\gamma}\right)^2 u_x = -\omega^2 u_x$ where we have introduced

$\boxed{\omega = \dfrac{qB}{m\gamma}}$; In the same way $\ddot{u}_y = -\dfrac{qB}{m\gamma}\dot{u}_x = -\left(\dfrac{qB}{m\gamma}\right)^2 u_y = -\omega^2 u_y$

So we get
$$\ddot{u}_x = -\omega^2 u_x \Rightarrow u_x = A_1 e^{i\omega t} + B_1 e^{-i\omega t} = A_1' \cos\omega t + B_1' \sin\omega t;$$
$$\ddot{u}_y = -\omega^2 u_y \Rightarrow u_y = A_2 e^{i\omega t} + B_2 e^{-i\omega t} = A_2' \cos\omega t + B_2' \sin\omega t;$$
$$\dot{u}_z = 0 \Rightarrow u_z = C = const$$

Here A_1, A_2, B_1, B_2 and C are integration constants to be determined by initial conditions.

In part (i) particle is injected at right angle to the field lines of magnetic field $\bar{B}$, i.e. $u_x(0) = u$; $u_y(0) = 0$; $u_z(0) = 0$

as $u_z(0) = 0$ we immideately get $u_z(t) = 0$

as $u_y(0) = 0 \Rightarrow u_y(t) = B_2' \sin\omega t$ (i.e. $A_2' = 0$)

as $u_x(0) = u$ and $u_x^2 + u_y^2 = u^2 = const$ we get $A_1' = u$

$$u^2\cos^2\omega t + B_1' u\cdot\sin 2\omega t + B_1'^2 \sin^2\omega t + B_2' \sin^2\omega t = u^2$$

$$u^2 + B_1' u\cdot\sin 2\omega t + (B_1'^2 + B_2'^2 - u^2)\sin^2\omega t = u^2$$

the easiest way to satisfy this equation is to take $B_1' = 0$ and $B_2' = u$ so that $\boxed{u_x(t) = u\cos\omega t;\ u_y(t) = u\sin\omega t;\ u_z(t) = 0;}$ integrating once more

$$\boxed{x(t) = \frac{u}{\omega}\sin\omega t + x_0;\ y(t) = -\frac{u}{\omega}\cos\omega t + y_0;\ z(t) = z_0;}$$

④ This is circular motion with period $\boxed{T = \frac{2\pi}{\omega} = \frac{2\pi m \gamma(u)}{qB};}$

and the radius of the circle is $\boxed{R = \frac{u}{\omega} = \frac{u m \gamma(u)}{qB};}$ q.e.d.

(ii) If the particle is injected into the field with the same velocity but at an angle $\theta \neq \frac{\pi}{2}$ to the field lines, prove that the path is a helix, of smaller radius, but that the period for one complete cycle is the same as before.

In this case initial conditions take form $u_x(0) = u_r$; $u_y(0) = 0$; $u_z(0) = u_z$ and $u_r^2 + u_z^2 = u^2$. The difference in solution is that now velocity in the plane transverse to magnetic field lines is $u_r < u$ and velocity along z-axis is constant

$$\boxed{\begin{array}{l} u_x(t) = u_r \cos\omega t; \; u_y(t) = u_r \sin\omega t; \; u_z(t) = u_z; \\ x(t) = \frac{u_r}{\omega}\sin\omega t + x_0; \; y(t) = -\frac{u_r}{\omega}\cos\omega t + y_0; \; z(t) = u_z t; \end{array}}$$

and integration gives this is indeed helix motion with period $T = \frac{2\pi}{\omega}$ but now radius of helix is now $\boxed{r_H = \frac{u_r}{\omega} < r}$ as $u_r < u$;

Problem 2 (i) A particle of rest mass m and charge q is released from rest in a frame S' in which there are constant and orthogonal $\bar{E}$ and $\bar{B}$ fields say $\mathbf{E} = (0; E_0; 0)$ and $\bar{B} = (0; 0; B_0)$ such that $0 < E_0 < cB_0$. How much time elapses before the particle is momentarely at rest again

First of all remember that we have 2 Lorentz invariants for electromagnetic field:

$\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = c^2B^2 - E^2 > 0$ and $\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = c(\bar{B}\cdot\bar{E})$. In particular as $E_0 < cB_0$ we can alway go to the frame where we obtain only magnetic field. Using Lorentz transformation formulas we can find the velocity of this frame:

(5) $E_1' = E_1 = 0$; $E_2' = \gamma(E_2 - B_3 V)$; $E_3' = \gamma(E_3 + VB_2) = 0$

if we want $\bar{E}' = 0$: $V = \frac{E_2}{B_3} = \frac{E_0}{B_0}$, i.e. $\boxed{V = \frac{E_0}{B_0};}$

Magnetic field in this frame is

$B_1' = B_1 = 0$; $B_2' = \gamma(B_2 + \frac{V}{c^2}E_3) = 0$; $B_3' = \gamma(B_3 - \frac{V}{c^2}E_2) = \gamma(B_0 - \frac{E_0^2}{c^2 B_0})$

$\gamma = \left(1 - \frac{E_0^2}{c^2 B_0^2}\right)^{-\frac{1}{2}} \Rightarrow B_3' = B_0\left(1 - \frac{E_0^2}{c^2 B_0^2}\right)^{\frac{1}{2}} = \frac{B_0}{\gamma}$

So in S' frame we have particle that in initial moment has velocity $u' = (-V; 0; 0)$. As we have seen this corresponds to circular movement in S'-frame, which is periodic and period is given by $T' = \frac{2\pi m \gamma(V)}{qB'} = \frac{2\pi m \gamma^2}{qB}$, due to time dilation $T = \gamma T'$ so we finally get $T = \frac{2\pi m \gamma^3}{qB_0} \Rightarrow \boxed{T = \frac{2\pi m c^3 B_0^2}{q(c^2B_0^2 - E_0^2)^{3/2}}; \quad \text{q.e.d.}}$

(ii) Show also that if $0 < cB_0 < E_0$, the particle ultimately moves with constant proper acceleration $\frac{q}{m}(E_0^2 - c^2B_0^2)^{\frac{1}{2}}$ in a direction making an angle $\cos^{-1}\left(\frac{cB_0}{E_0}\right)$ with the x-axis.

Now as $0 < cB_0 < E_0$ we can go to the frame such that $\bar{B}' = 0$. Using Lorentz transformation formulas $B_1' = B_1 = 0$; $B_2' = \gamma(B_2 + \frac{V}{c^2}E_3)$; $B_3' = \gamma(B_3 - \frac{V}{c^2}E_2)$; this is boos along x-axis. To make all components zero we should assume $B_3 = \frac{V}{c^2}E_2 \Rightarrow \boxed{V = c^2\frac{B_0}{E_0};}$

Electric field in this frame is given by

$E_1' = E_1 = 0$; $E_2' = \gamma(E_2 - VB_3) = \gamma E_0\left(1 - \frac{c^2B_0^2}{E_0^2}\right) = \frac{E_0}{\gamma}$; $E_3' = \gamma(E_3 + VB_2) = 0$;

So that $\bar{E}' = (0, \frac{E_0}{\gamma}, 0)^T$ And 3-force acting on charged particle is $\bar{f}' = qE'$; and equation of motion is $\dot{p}_x' = 0$ and $\dot{p}_y' = qE'$. Initial conditions in this frame

⑥ (let's call it S'-frame) correspond to the particle moving in negative direction of x-axis. So solution to this equations is

$\boxed{p'_x = p'_0 ;\ p'_y = qE't ;}$ the energy of particle is given by $\mathcal{E}' = \sqrt{(p'c)^2 + (m'c^2)^2}$ where $m' = m\gamma(v)$ is the "relativistic mass" of particle in S'-frame

Substituting obtained solutions into this expression we get $\mathcal{E}' = \sqrt{\mathcal{E}_0'^2 + (qE'c\,t')^2}$ Where $\mathcal{E}_0'^2 = (p'_0 c)^2 + (m'c^2)^2$

If t' is big enough (we need to describe ultimate movement) we get: $\mathcal{E}' = qE't\left(1 + \frac{\mathcal{E}_0'^2}{2(qE't)^2}\right)$; Another thing we need velocities of particle. As $\bar{p} = \gamma(v) m\bar{v}$; $\mathcal{E} = \gamma(v) mc^2$; we can conclude that $\bar{v} = \frac{c^2\bar{p}}{\mathcal{E}}$ In our case we get

$$u'_x = \frac{p'_0 c^2}{\mathcal{E}'} ;\ u'_y = \frac{qE't'c^2}{\mathcal{E}'} ;\ \mathcal{E}' = \sqrt{\mathcal{E}_0'^2 + (qE'ct)^2} ;\ \mathcal{E}_0'^2 = (p'_0 c)^2 + (m'c^2)^2$$

then $\frac{1}{\gamma^2(u)} = 1 - \frac{u_x'^2 + u_y'^2}{c^2} = 1 - \frac{(p'_0 c)^2 + (qE't'c)^2}{\mathcal{E}'^2} = \frac{(m'c^2)^2}{\mathcal{E}'^2}$

$\boxed{\gamma(u) = \frac{\mathcal{E}'}{m'c^2}}$ here $m' = \gamma(v)\,m$ - is mass of particle in S'-frame

4-force is given by:

$$F^{\mu'} = \gamma(u)\left(\frac{1}{c}\frac{d\mathcal{E}'}{dt'} ; \bar{f}'\right) \quad \text{and} \quad \frac{d\mathcal{E}'}{dt'} = \frac{(qE'c)^2 t'}{\mathcal{E}'}$$

now we can find proper acceleration using formula

$$\alpha^2 = -A_\mu A^\mu = -\frac{1}{m^2} F_{\mu'} F^{\mu'} = -\frac{\gamma^2(u)}{m^2}\left(\frac{1}{c^2}\left(\frac{d\mathcal{E}'}{dt'}\right)^2 - (\bar{f}')^2\right) =$$

$$= -\frac{\gamma^2(u)}{m^2}(qE')^2\left(\frac{(qE'c\,t')^2}{\mathcal{E}'^2} - 1\right) = \left(\frac{\mathcal{E}'}{m'c^2}\right)^2 \frac{(qE')^2}{m^2}\frac{\mathcal{E}_0'^2}{\mathcal{E}'^2}$$ now substituting

$E' = \frac{E_0}{\gamma(v)}$; $m' = \gamma(v) m$; $\mathcal{E}'_0 = m\gamma(v)c^2$, thus

$\alpha^2 = \frac{(qE_0)^2}{m^2\gamma^2(v)} = \left(\frac{q}{m}\right)^2 (E_0^2 - c^2B_0^2)$ (we have used here

$\gamma(v) = \left(1 - \left(\frac{cB_0}{E_0}\right)^2\right)^{\frac{1}{2}}$, thus, finally: $\boxed{\alpha = \frac{q}{m}\sqrt{E_0^2 - c^2B_0^2};\ \text{q.e.d.}}$

To determine direction of particles movement we should find velocity direction in S-frame

(7) We know that

$$u'_x = \frac{p'_0 c^2}{\mathcal{E}'} ; \quad u'_y = \frac{qE_0 t' c^2}{\gamma \mathcal{E}'}$$ Using formulas of

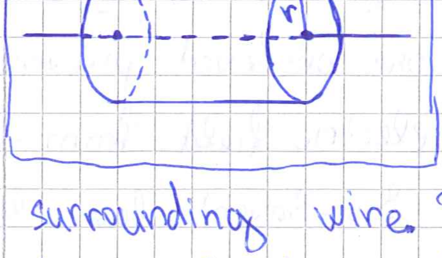
Lorentz transformation of velocities we get:

$$V_x = \frac{V'_x + V}{1 + \frac{u'_x V}{c^2}} ; \quad V_y = \frac{V'_y}{\gamma(V)\left(1 + \frac{u'_x V}{c^2}\right)} ;$$

thus direction of motion makes φ angle with x-axis such that: $\tan\varphi = \frac{V_y}{V_x} = \frac{V'_y}{\gamma(V)(V'_x + V)}$ When time is large enough we can take limit $V'_x \to 0$; $V'_y \to c$ and thus $\tan\varphi = \frac{c}{V\gamma(V)}$ and $\cos^2\varphi = \frac{1}{1+\tan^2\varphi} \Rightarrow$

$\Rightarrow 1 + \tan^2\varphi = 1 + \frac{c^2}{V^2}\left(1 - \frac{V^2}{c^2}\right) = \frac{c^2}{V^2}$ and $\cos\varphi = \frac{V}{c} = \frac{cB_0}{E_0}$ so we finally get $\boxed{\varphi = \arccos \frac{cB_0}{E_0}, \text{q.e.d.}}$

Problem 4 Prove, by any method, than the electric field $\bar{E}$ at a point P due to an infinite straight-line distribution of static charge, λ per unit length, is given by $\bar{E} = \frac{\lambda \bar{r}}{2\pi\varepsilon_0 r^2}$; where $\bar{r}$ is the perpendicular vector-distance of P from the line. Deduce, by transforming to a frame in which this lines moves, that the magnetic field $\bar{B}$ at P due to an infinitely long straight current $\bar{I}$ is given by $\bar{B} = \frac{\bar{I} \times \bar{r}}{2\pi\varepsilon_0 c^2 r^2}$;



To find electric field we should use For this purpose we consider flux through cylinder of radius r, surrounding wire. Due to the symmetry of problem electric field sholdn't have any components along wire. As $\bar{\nabla}\cdot\bar{E} = \rho$ integration over the volume of cylinder $\int dV \bar{\nabla}\cdot\bar{E} = \oint_{\partial V} \bar{E}\cdot d\bar{S} = \int dV \rho$ so that:

(7) $2\pi r l E_r = \lambda l$ where l is length of cylinder and E_r is electric field strength pointing out of cylinder as shown on picture. So we get $E_r = \frac{\lambda}{2\pi r \varepsilon_0}$ or

$$\boxed{\bar{E} = \frac{\lambda \bar{r}}{2\pi r^2 \varepsilon_0}, \text{ q.e.d.}}$$

Now let's go to the S' frame moving with velocity v along x-axis. In this frame we have magnetic field that can be found using Lorentz transformation.

$$B_1' = B_1 = 0 \,; \quad B_2' = \gamma(v)\left(B_2 + \frac{v}{c^2}E_3\right) = \frac{v\gamma(v)}{c^2}\,\frac{\lambda x_3}{2\pi\varepsilon_0 r^2}\,;$$

$$B_3' = \gamma(v)\left(B_3 - \frac{v}{c^2}E_2\right) = -\frac{v\gamma(v)}{c^2}\,\frac{\lambda x_2}{2\pi\varepsilon_0 r^2}\,;$$

Now in S'-frame we get instead of static charge disturbed along the wire, current flowing along this wire. This current is given by $I^{\mu'} = \lambda\gamma(v)(c; -\bar{v})$ (see formula (6.5) in Joe's lectures). Now we can find magnetic field in S'-frame, using Bio-Savart law

$$\bar{B} = \frac{\bar{I} \times \bar{r}}{2\pi\varepsilon_0 c^2 r^2} \quad \text{as} \quad \bar{I}' = \lambda\gamma(v)(-v; 0; 0). \text{ Thus:}$$

$$\bar{I}' \times \bar{r}' = \begin{Vmatrix} \hat{x}' & \hat{y}' & \hat{z}' \\ -\lambda\gamma v & 0 & 0 \\ x' & y' & z' \end{Vmatrix} = \lambda\gamma(v)\cdot v \begin{bmatrix} 0 \\ z' \\ -y' \end{bmatrix};$$

Thus $\boxed{\bar{B}' = \frac{\lambda\cdot\gamma(v)\cdot v}{2\pi\varepsilon_0 c^2 r^2}\begin{bmatrix} 0 \\ x_3' \\ -x_2' \end{bmatrix}}$ note that x_3 and x_2 are directions transverse to the boost direction. so that $x_3' = x_3$ and $x_2' = x_2$ and our answer coinsides with the one observed previously by Lorentz transformation of electric field from S-frame to S'-frame. Let's now check Bio-Savart law by direct substitution into Maxwell equations.

First of all in Maxwell equations we have current density instead of current and they are related in the following way : $\bar{j} = \frac{\bar{I}}{\pi r^2}$ so that

$\bar{B} = \frac{\bar{j} \times \bar{r}}{2\varepsilon_0 c^2}$ now $\bar{\nabla} \times \bar{B} = \frac{1}{2\varepsilon_0 c^2} \bar{\nabla} \times \bar{j} \times \bar{r} = \frac{1}{2\varepsilon_0 c^2} \varepsilon^{ijk} \varepsilon^{klm} \partial_j \partial_e r_m =$

$= (\delta^{il}\delta^{jm} - \delta^{im}\delta^{ej}) \frac{1}{2\varepsilon_0 c^2} \partial_j (j_e r_m) = \frac{1}{2\varepsilon_0 c^2} (\bar{j}(\bar{\nabla} \cdot \bar{r}) - \bar{j}) = 2\bar{j} \frac{1}{2\varepsilon_0 c^2} =$

$= \frac{\bar{j}}{\varepsilon_0 c^2}$ thus we have obtained $\bar{\nabla} \times \bar{B} = \frac{\bar{j}}{\varepsilon_0 c^2}$ which is just Maxwell equation, q.e.d.

Problem 8 Obtain the Lienard - Wiechert potentials $\left(\varphi = \frac{Q}{r(1+\frac{u_r}{c})} ; \bar{A} = \frac{Q}{c^2}\left[\frac{\bar{u}}{r(1+\frac{u_r}{c})}\right], Q = \frac{q}{4\pi\varepsilon_0} ;\right)$ of an arbitrary moving charge q by the following alternative method: Assume, first, that the charge moves uniformely and that in it's rest frame the potential is given by $\varphi = \frac{Q}{r}$; $\bar{A} = \bar{0}$. Then transform this to the general frame, using the 4-vector property of A^μ

Note here we use more common notations of 4-potential $A^0 = \varphi$; $A^i = cw^i$ → this is correspondence between our notations and notations of Rindler book.

In rest frame of charged particle (let's call this frame S') we get

$A^{0'} = \frac{Q}{r'}$; $\bar{A} = \bar{0}$; A^μ transforms as 4-vector under Lorentz transformation: $A^\mu = \Lambda^\mu{}_{\nu'} A^{\nu'}$ where

$\Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & \frac{v}{c}\gamma & 0 & 0 \\ \frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ - we assume particle to be moving along x-direction, and here we represent boost back to S-frame from S'-frame. In our case Lorentz trasformation is easy as only nonzero component of A^μ in S'-frame is $A^{0'}$ so that:

(9) $A^0 = \Lambda^0{}_{0'} A^{0'} = \frac{\gamma Q}{r'}$; $A^1 = \Lambda^1{}_{0'} A^{0'} = \frac{\gamma Q}{r'}\frac{v}{c}$; $A^3 = \Lambda^3{}_{0'} A^{0'} = 0$;
$A^2 = \Lambda^2{}_{0'} A^{0'} = 0$. So we have got

$A^\mu = \gamma \frac{Q}{r'} \left(1; \frac{v}{c}; 0; 0\right)^T$. Now we should relate r and r'.
Distance from charge to observer is given by $r = ct$ and $r' = ct'$. Using Lorentz transformation for time it takes light to travel

$t' = \gamma\left(t - \frac{\bar{u}\cdot\bar{r}}{c^2}\right) = \gamma t\left(1 + \frac{u_r}{c}\right)$ here we have used general form of Lorentz transformation. and
$\bar{u}\cdot\bar{r} = -u_r \cdot r$ where u_r is radial velocity away from the observer. So what we finally get:

$$\boxed{\phi = A^0 = \frac{Q}{r(1+\frac{u_r}{c})} \quad \text{and} \quad \bar{w} = \frac{\bar{A}}{c} = \frac{Q\bar{u}}{c^2 r(1+\frac{u_r}{c})}, \text{ q.e.d.}}$$

Note that here we have written more general $\bar{A} \sim \bar{u}$ rather then just one component of 4-potential.

Problem 9 Obtain field $\bar{E} = \frac{Q\bar{r}_0}{\gamma^2 r_0^3\left[1 - \frac{u^2}{c^2}\sin^2\theta\right]^{3/2}}$; and $\bar{B} = \frac{1}{c^2}(\bar{u}\times\bar{E})$;
of a uniformly moving charge q by the following alternative method: assume that the field in the rest frame S' of the charge is given by:
$\bar{E}' = \frac{Q}{r'^3}(x', y', z')$; $\bar{B}' = 0$, $r'^2 = x'^2 + y'^2 + z'^2$;
then transform this field to the usual second frame S at $t = 0$.

So in the rest frame of particle we have

$\bar{E}' = \frac{Q}{r'^3}(x', y', z')$; $\bar{B}' = 0$

Now without loss of generality let's assume that particle is moving along x-axis in observers rest frame S. So we can find electromagnetic field in S-frame simply doing boost in negative x direction

(10) $E_1 = E_1' = \frac{Q}{r'^3}x'$; $E_2 = (E_2' + vB_3')\gamma = \frac{Q}{r'^3}y'\gamma(v)$;

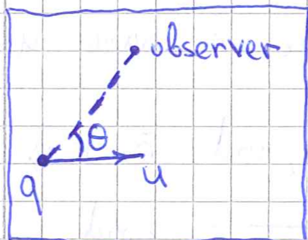
$E_3 = \gamma(E_3' - vB_2') = \frac{Q}{r'^3}z'\gamma(v)$;

$B_1 = B_1' = 0$; $B_2 = \gamma(v)\left(B_2' - \frac{v}{c^2}E_3'\right) = -\frac{v\gamma(v)}{c^2}E_3'$;

$B_3 = \gamma(v)\left(B_3' + \frac{v}{c^2}E_2'\right) = \frac{v\gamma(v)}{c^2}E_2'$

also we should transform coordinates in the following way:

$x' = \gamma x$; $y' = y$; $z' = z$; So that $r'^2 = x'^2 + y'^2 + z'^2 = \gamma^2 x^2 + y^2 + z^2 =$
$= \gamma^2(x^2+y^2+z^2) - (\gamma^2-1)(y^2+z^2) = \gamma^2 r^2\left(1 - \frac{u^2}{c^2}\frac{y^2+z^2}{r^2}\right) =$
$= \gamma^2 r^2\left[1 - \frac{u^2}{c^2}\sin^2\theta\right]$, θ here is angle between $\bar{r}$ and $\bar{u}$, which in our case is directed along x-axis.



So we now conclude the following

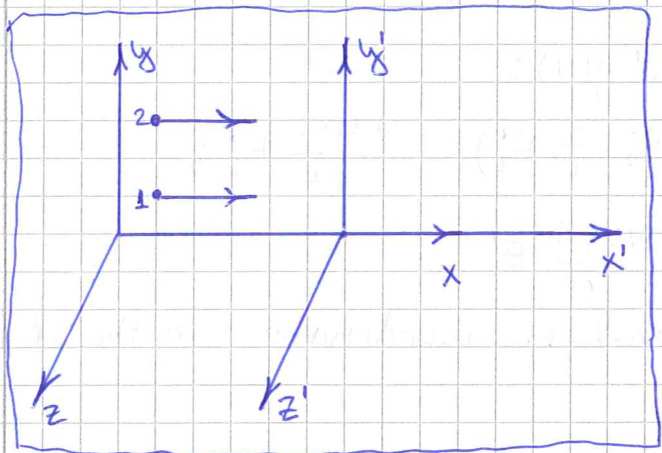
$$E_1 = \frac{Q\gamma x}{\gamma^3 r^3\left[1-\frac{u^2}{c^2}\sin^2\theta\right]^{3/2}}; \quad E_2 = \frac{Qy}{\gamma^2 r^3\left[1-\frac{u^2}{c^2}\sin^2\theta\right]^{3/2}};$$

$$E_3 = \frac{Qz}{\gamma^2 r^3\left[1-\frac{u^2}{c^2}\sin^2\theta\right]^{3/2}};$$ or written in vector form:

$$\boxed{\bar{E} = \frac{Q\bar{r}}{\gamma^2 r^3\left[1-\frac{u^2}{c^2}\sin^2\theta\right]^{3/2}};}$$ q.e.d.

Now $\bar{B} = \frac{v\gamma(v)}{c^2}\begin{bmatrix}0\\-E_3'\\E_2'\end{bmatrix}$; $\bar{E} = \begin{bmatrix}E_1'\\\gamma E_2'\\\gamma E_3'\end{bmatrix}$ now we can find that $\frac{1}{c^2}\bar{v}\times E = \frac{1}{c^2}\begin{Vmatrix}\hat{x} & \hat{y} & \hat{z}\\ v & 0 & 0\\ E_1' & \gamma E_2' & \gamma E_3'\end{Vmatrix} = \frac{1}{c^2}\begin{bmatrix}0\\-E_3'\\E_2'\end{bmatrix}\cdot\frac{v\gamma(v)}{c^2} = \bar{B}$, so we have directly shown that indeed $\boxed{\bar{B} = \frac{1}{c^2}\bar{v}\times E}$, q.e.d.

Problem 10 In a fram S, two identical particles with electric charge q move abreast along lines parallel to the x-axis, a distance r apart and with velocity V. Determine the force, in S, that each exerts on the other.



There are two ways to determine force:

(i) Let's calculate electromagnetic fields fields created by charge 1 at the position of charge 2. For this we use formulas (41.5), (41.6), and (38.16) from [Rindler]. Force acting on charge is given by Lorenz force $\bar F = q(\bar E + \bar u \times \bar B)$ and fields created by charge q are:

$\bar E = \frac{Q\bar r}{\gamma^2 r^3 \left[1-\frac{u^2}{c^2}\right]^{3/2}}$; $\bar B = \frac{1}{c^2}\bar u \times \bar E$ here we have used $\sin\theta = 1$ as particles move abreast, and $\bar r = r \cdot \hat y$ and we get

$$\boxed{\begin{aligned}\bar E &= \frac{Q\gamma(u)}{r^2}\hat y; \\ \bar B &= \frac{Q\gamma(u)\cdot u}{r^2 c^2}\hat z;\end{aligned}}$$

$\bar B = \frac{1}{c^2} u \frac{Q\hat x \times \hat y}{\gamma^2 r^2 \left[1-\frac{u^2}{c^2}\right]^{3/2}}$; and

And now we are able to find force acting on second particle: $F = q(\bar E + \bar v \times \bar B) = q\left(\bar E - \frac{u^2}{c^2}\bar E\right) = q\bar E \frac{1}{\gamma^2} = \frac{qQ}{\gamma r^2}\hat y$ we have used here following formula for triple vector product $\bar u \times \bar u \times \bar E = \bar u \cdot \underbrace{(\bar E \cdot \bar u)}_{0} - \bar E \cdot u^2$, so the force equals to $\boxed{\bar F = \frac{q^2}{4\pi\varepsilon_0 r^2 \gamma(u)}\hat y;}$

(ii) Second way to derive same result is to consider problem in the rest frame of charges, i.e. frame S' moving with velocity v in positive $\hat x$-direction.
In S' particles are at rest and we need just Coulomb force $\bar f' = \frac{q^2 \bar r'}{4\pi\varepsilon_0 r'^3} = \frac{q^2 \hat y}{4\pi\varepsilon_0 r^2}$ we have taken here $r' = r$ as there are no changes in distancies in

(12) As for zero-component $F^{0'}$ it is vanishing $F^{0'}=0$ as Coulomb force is mass preserving.
So 4-force we get is $F^{\mu'}=(0;\bar{f}')$. It transforms as 4-vector under Lorentz transformations, so we get

$$f^1=\frac{1}{\gamma}F^1=\frac{1}{\gamma}(\Lambda^1{}_{0'}F^{0'}+\Lambda^1{}_{1'}F^{1'})=\gamma^{-1}\gamma F^{1'}=f^{1'}=0;$$

$$f^2=\frac{1}{\gamma}F^2=\frac{1}{\gamma}F^{2'}=\frac{q^2}{4\pi\varepsilon_0\gamma r^2};\quad f^3=\frac{1}{\gamma}F^3=0;$$ thus we get

$$\boxed{\bar{f}=\frac{q^2}{4\pi\varepsilon_0\gamma r^2}\hat{y}\text{, q.e.d.}}$$

Problem 12 If $\bar{E}\cdot\bar{B}\neq 0$, prove there are infinitely many frames in which $\bar{E}\parallel\bar{B}$ precisely one of this moves in the direction $\bar{E}\times\bar{B}$, it's velocity being given by the smaller root of the quadratic $\beta^2-R\beta+1=0$, where $\beta=\frac{v}{c}$ and $R=\frac{E^2+c^2B^2}{|\bar{E}\times c\bar{B}|}$. For the reality of β $R>2$ inequality should be satisfied.

The frame in which $\bar{E}'\parallel\bar{B}'$ is one where $\bar{E}'\times c\bar{B}'=0$.
Let's first assume that $\bar{E}'\times c\bar{B}'\neq 0$. We always can choose axes in such way that $\bar{E}'\times c\bar{B}'\parallel\hat{x}$, so that
$(\bar{E}'\times c\bar{B}')^{2'}=(\bar{E}'\times c\bar{B}')^{3'}=0$ and

$(\bar{E}'\times c\bar{B}')^1=cE_2'B_3'-cE_3'B_2'=\{$using Lorentz transformation formulas$\}=\gamma^2((E_2-vB_3)(cB_3-\frac{v}{c}E_2)-(E_3+vB_2)(cB_2+\frac{v}{c}E_3))=$

$=\gamma^2(cE_2B_3-cE_3B_2+\frac{v^2}{c^2}(cE_2B_3-cB_2E_3)-\frac{v}{c}(|cB|^2+E^2))=$

$=\gamma^2(1+\frac{v^2}{c^2})(\bar{E}\times c\bar{B})^1-\gamma^2\frac{v}{c}(|E|^2+c^2|B|^2)$

Here we have made boost of velocity v along x-axis direction. We want this expression to be zero. So that
$(1+\frac{v^2}{c^2})(\bar{E}\times c\bar{B})^1-\frac{v}{c}(|E|^2+c^2|B|^2)=0$ we have chosen axes in such way that $(\bar{E}\times c\bar{B})^1=|\bar{E}\times c\bar{B}|$ and we get:

$\boxed{\beta^2-R\beta+1=0;\quad R=\frac{|E|^2+c^2|B|^2}{|\bar{E}\times c\bar{B}|}}$; the solution of this

(13) equation will give us the value of velocity v of boost along $(\bar{E} \times c\bar{B})$ direction that will make $\bar{E} \parallel \bar{B}$. Starting from this frame we can do any boosts in $\bar{E}$ directon and $\bar{E}$ will stay parallel to $\bar{B}$ after all this boosts, q.e.d!!!

For the equation we have derived to have real solutions we want

$$D = R^2 - 4 \geq 0 \quad \text{so} \quad R \geq 2 \Rightarrow |\bar{E}|^2 + c^2|\bar{B}|^2 \geq 2|\bar{E} \times c\bar{B}| = 2c|\bar{E}| \cdot |\bar{B}| |\sin\theta| <$$

$$< 2c|\bar{E}| \cdot |\bar{B}| \quad \text{So we get} \quad |\bar{E}|^2 + c^2|\bar{B}|^2 \geq 2c|\bar{E}| \cdot |\bar{B}| \Rightarrow$$

$$\Rightarrow (|\bar{E}| - c|\bar{B}|)^2 \geq 0 \text{ which is always satisfied q.e.d!!!}$$

① Seminar 9 (Combining analytical mechanics and special relativity)

Theory

First of all as usually in analytical mechanics we introduce action, which in case of relativistic massive particle is given by

$$S = -\int mc^2 d\tau = -\int mc\sqrt{dx_\mu \cdot dx^\mu} = -\int mc\sqrt{\frac{dx^\mu}{d\lambda}\cdot\frac{dx_\mu}{d\lambda}}\,d\lambda$$

The most used form of action is the last one

$$\boxed{S = -\int mc\sqrt{\frac{dx^\mu}{d\lambda}\cdot\frac{dx_\mu}{d\lambda}}\,d\lambda;}$$ This form of action possess reparamtrisation invariance, i.e.

we can go from one parametrisaton to another: $\lambda \to \lambda'(\lambda)$ and action remains the same. Lagrangian of relativistic particle is $L = -mc\sqrt{\frac{\partial x^\mu}{\partial \lambda}\frac{\partial x_\mu}{\partial \lambda}}$, equations of motion $\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0$ gives $-\frac{d}{d\lambda}\left(mc\frac{\dot{x}_\mu}{\sqrt{\dot{x}_\mu \cdot \dot{x}^\mu}}\right) = 0$ if we now take $\lambda = \tau$ we get $-\frac{d}{d\tau}(\dot{x}_\mu m) = 0$ or $\frac{dp^\mu}{d\tau} = 0$ which

is just free relativistic particle equation of motion.

But all formulas above are aplicable only for massive particle. To include massless particles we should introduce more general action:

Problem 1

Another formulation of relativistic action

ⓐ Show that the action

$$S = -\frac{1}{2}\int\left(e^{-1}\frac{\partial x^\mu}{\partial \lambda}\frac{\partial x_\mu}{\partial \lambda} + c^2 e m^2\right)d\lambda$$

is equivalent to the action above. Here, e is a Lagrange multiplier.

To reduce this action to the previous one we should read off Lagrange multiplier e using it's equations of motion: $\frac{\partial L}{\partial e} = 0$ which leads to

(2) $-\frac{1}{e^2}\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda}+m^2c^2=0 \Rightarrow \boxed{e=\frac{1}{mc}\sqrt{\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda}};}$

now substituting this value back to action we get:

$$S=-\frac{1}{2}\int\left(mc\sqrt{\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda}}+mc\sqrt{\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda}}\right)d\lambda \Rightarrow$$

$\Rightarrow \quad S=-mc\int\sqrt{\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda}}\,d\lambda;$ q.e.d. So, we have reduced initial action to the usual one. Why we need this new action at all is because it is usefull in the case of $m=0$, i.e. massless particle while initial action doesn't make any sence in $m=0$ limit.

(b) Find how e must transform under the reparametrisation $\lambda\to\lambda'(\lambda)$ such that action is invariant under it.

If $\lambda\to\lambda'(\lambda)$; $\frac{\partial x^\mu}{\partial \lambda'}=\frac{\partial x^\mu}{\partial \lambda}\frac{d\lambda}{d\lambda'}$; $d\lambda'=\frac{d\lambda'}{d\lambda}d\lambda$; So under reparametrisation we get:

$$S=-\frac{1}{2}\int\left(\frac{1}{e}\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda}+em^2c^2\right)d\lambda \to -\frac{1}{2}\int\left(\frac{1}{e'(\lambda')}\frac{\partial x^\mu}{\partial \lambda'}\frac{\partial x_\mu}{\partial \lambda'}+e'(\lambda')m^2c^2\right)d\lambda'=$$

$$=-\frac{1}{2}\int\left(\frac{1}{e'(\lambda')}\left(\frac{d\lambda}{d\lambda'}\right)^2\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda}+e'(\lambda')m^2c^2\right)\frac{d\lambda'}{d\lambda}d\lambda=$$

$$=-\frac{1}{2}\int\left(\frac{1}{e'(\lambda')}\frac{d\lambda}{d\lambda'}\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda}+e'(\lambda')\frac{d\lambda'}{d\lambda}m^2c^2\right)d\lambda=$$

$=-\frac{1}{2}\int\left(\frac{1}{e(\lambda)}\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda}+e(\lambda)m^2c^2\right)d\lambda$ where in the last equation we supposed $e(\lambda)=e'(\lambda')\frac{d\lambda'}{d\lambda}\Rightarrow \boxed{e(\lambda)d\lambda=e(\lambda')d\lambda';}$

this is how Lagrange multiplier should be changed.

(c) Now let $m=0$ and "choose the gauge" $e=1$. Show that this gives the equations of motion for a massless particle.

If we take $m=0$ and $e=1$ we are left with the Lagrangian $L=\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda}$ Note here that we don't have reparametrisation invariance now as we fixed

③ parametrisation by choosing some particular "gauge" $e=1$. Equation of motion gives

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}_1}{\partial \dot{x}^\mu}\right)=0 \Rightarrow \boxed{\frac{d^2x^\mu}{d\lambda^2}=0;}$$ this is

indeed equation of motion for massless particle. (If we write down wave vector as $k^\mu = \frac{dx^\mu}{d\lambda}$ we get $\frac{d^2x^\mu}{d\lambda^2}=0$ as $dk^\mu=0$ - wave vector doesn't change along beam of light, or, in other words, particle trajectory)

Theory

Now comes the question how can we couple charged particles to electromagnetic fields, so that we can describe their motion in external fields. There are usually two conditions that should be satisfied:

(I) Lorentz invariance

(II) gauge invariance.

Appropriate term is $S_{coup} = -\frac{1}{c}\int q A_\mu dx^\mu$; this term is obviously Lorentz invariant. As for gauge invariance, under gauge transformation we get:

$$\int q A_\mu dx^\mu \mapsto \int q A_\mu dx^\mu + \int q\, \partial_\mu \Phi\, dx^\mu$$

this last term is total derivative and doesn't contribute to e.o.m. and physics.

So action for particle of mass m and charge q in external gauge field looks like:

$$\boxed{S = -\int mc^2 d\tau - \frac{1}{c}\int q A_\mu dx^\mu;}$$

④ The action and the magnetic flux.

Problem 3 Suppose we have a particle of charge q in the presence of a magnetic field. Show that for motion around a loop, the contribution to the action from the magnetic field is $q\Phi$, where Φ is the magnetic flux through the loop.

Let's take only coupling term of action:

$S_c = -\frac{1}{c}\int q A_\mu dx^\mu$. As we have only magnetic field we can suppose $A_\mu = (0; -\bar{A})$, so that:

$S_c = \frac{1}{c}\int q \bar{A} d\bar{x}$. Trajectory is closed so that line integral goes to contour integral:

$$S_c = \frac{1}{c}\oint_{\partial S} q \bar{A} d\bar{x} = \frac{1}{c}\int_S q(\bar{\nabla}\times\bar{A})d\bar{S} = q\int \bar{B} d\bar{S} = q\Phi$$

Here we have used Stokes formula when going from line to contour integral and definition of magnetic field:

$\bar{B} = \frac{1}{c}\bar{\nabla}\times\bar{A}$. So we have shown that:

$\boxed{S_c = q\Phi,\ \text{q.e.d.}}$

Problem 4 Consider the Pauli-Lubanski vector

$W_\mu \equiv \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma} p^\nu \Theta^{\lambda\sigma}$, where $\Theta^{\lambda\sigma} = -\Theta^{\sigma\lambda}$ - is generalized angular momentum. Assume $\Theta^{\lambda\sigma} = L^{\lambda\sigma} + S^{\lambda\sigma}$ where $L^{\lambda\sigma}$ is the orbital contribution:

$$L^{\lambda\sigma} = x^\lambda p^\sigma - x^\sigma p^\lambda;$$

and $S^{\lambda\sigma}$ is the contribution from an internal spin.

ⓐ Show that $p^\mu W_\mu = 0$

$p_\mu W^\mu = p^\mu W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma} p^\mu p^\nu \Theta^{\lambda\sigma}$, product of symmetric and antisymmetric tensors is always zero. Indeed

$\varepsilon_{\mu\nu\lambda\sigma} p^\mu p^\nu = \varepsilon_{\nu\mu\lambda\sigma} p^\nu p^\mu = -\varepsilon_{\mu\nu\lambda\sigma} p^\mu p^\nu = 0$ thus $\varepsilon_{\mu\nu\lambda\sigma} p^\mu p^\nu = 0$

here on the first step we have just renamed dummy indicies and on the second permuted them

(5) So we have shown, that indeed $\boxed{p_\mu W^\mu = 0}$, q.e.d.

(b) Show that contribution of the orbital part to W_μ is zero.

Let's consider orbital momentum contribution

$$W^4_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma} p^\nu L_4^{\lambda\sigma} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma} p^\nu x^\lambda p^\sigma - \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma} p^\nu x^\sigma p^\lambda$$

Considering 2 terms separately we get:

$*\ \varepsilon_{\mu\nu\lambda\sigma} p^\nu p^\sigma = \varepsilon_{\mu\sigma\lambda\nu} p^\nu p^\sigma = \{3 \text{ permutations } \sigma\leftrightarrow\lambda;\ \sigma\leftrightarrow\nu;\ \lambda\leftrightarrow\nu\} =$
$= -\varepsilon_{\mu\nu\lambda\sigma} p^\nu p^\sigma = 0$

$*\ \varepsilon_{\mu\nu\lambda\sigma} p^\nu p^\lambda = \varepsilon_{\mu\lambda\nu\sigma} p^\nu p^\lambda = -\varepsilon_{\mu\nu\lambda\sigma} p^\nu p^\lambda = 0$, so that:

$\boxed{W^4_\mu = 0}$, q.e.d.

(c) In the particles rest frame, $S_{0i} = 0$; $S_{ij} = \varepsilon_{ijk} S_k$; Assuming that $\bar{S}\cdot\bar{S} = \hbar^2 s(s+1)$, find $W_\mu W^\mu$ in terms of the particle's rest mass m and spin s.

In the particle rest frame $p^\mu = (mc; \bar{0})$, so only nonzero component is p^0

$W_\mu = \frac{1}{2}\varepsilon_{\mu 0\lambda\sigma} mc \cdot S^{\lambda\sigma}$, if 2 indicies of $\varepsilon_{\mu\nu\alpha\beta}$ coincide it turns to zero so $\mu\lambda\sigma$ are all spatial. Thus

$W_i = \frac{1}{2}\varepsilon_{0ijk}\varepsilon^{jkp} S_p mc = -\frac{1}{2}\varepsilon_{jk}\varepsilon^{jkp}\cdot S_p mc$, using $\varepsilon_{jki}\varepsilon^{jkp} = -2\delta_{pi}$, so that $W_i = S_i mc$ and $W_0 = 0$

then $W_\mu W^\mu = (W^0)^2 - |\bar{W}|^2 = -m^2c^2\,\bar{S}\cdot\bar{S} = -(mc\hbar)^2 s(s+1)$

So we get $\boxed{W_\mu W^\mu = -(mc\hbar)^2 s(s+1)}$

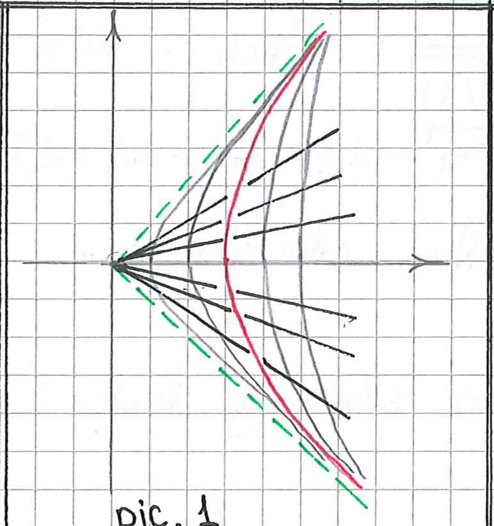
(d) Show that $hP_\mu = W_\mu$.

For massless particle $W_\mu W^\mu = 0$ so that $W^0 = |\bar{W}|$ and the same for p^μ: $P^0 = |\bar{P}|$ then if $P_\mu W^\mu = 0$, as we have shown in (a) then $P^0 W^0 - P^i W^i = 0 \Rightarrow$
$\Rightarrow |\bar{P}|\cdot|\bar{W}| = (\bar{P}\cdot\bar{W}) \Rightarrow \bar{P} \parallel \bar{W}$ and so they are collinear and $W^\mu = P^\mu \cdot h$, where h is constant called helicity

1

Rindler coordinates

* We have already considered observer moving with uniform acceleration (i.e. constant proper acceleration). As we know this observer moves along hyperbolas $\boxed{x^2 - t^2 = \frac{1}{\alpha_0^2};}$ (see pic. 1) (α_0 is proper acceleration)



pic. 1

— light cone
— world line of acc. observer O
— $\bar{x}$=const lines
— $\bar{t}$=const lines

If we calculate proper time that this accelerated observer measures we get:

$$\boxed{\tau = \frac{1}{\alpha_0}\,\mathrm{arctanh}\left(\frac{t}{x}\right);}$$

* natural choise of coordinates is to choose

• lines of constant coordinate $\bar{x}$ is chosen to be other hyperbolas. You can think of this as set of uniformly accelerating observers moving with proper acceleration $\alpha = \frac{1}{\bar{x}_0}$ where $\bar{x}_0$ is distanse to observer O at the moment of time $t=0$.

• lines of constant $\bar{t}$ are just streight lines (see pic. 1) This corresponds to choise $\bar{t} = \tau$, i.e. we choose proper time of O-observer as $\bar{t}$-coordinate.

• Summarizing:

$$\boxed{\begin{array}{ll} \bar{t} = \frac{1}{\alpha_0}\,\mathrm{arctanh}\left(\frac{t}{x}\right); & \bar{y} = y; \\ \bar{x} = \mathrm{sign}(x)\sqrt{x^2 - t^2}; & \bar{z} = z; \end{array}}$$

this are Rindler coordinates

• Let's consider light emmited at $\bar{t}=0$ from $\bar{x}_1 < \frac{1}{\alpha_0}$

In usual ref frame world line of beam is

② described by $x = t + \bar{x}_1$ thus it will intersect world-line of accelerated observer at the point where $x^2 = t^2 + \frac{1}{\alpha_0^2} = (t+\bar{x}_1)^2$;

At this moment of time accelerated observer sees red-shifted light as it is moving with the velocity $V = \frac{dx}{dt} = \frac{t}{x} = \frac{\alpha_0 t}{\sqrt{(\alpha_0 t)^2+1}}$; which in turn gives red shift: $\frac{\lambda_0}{\lambda} = \sqrt{\frac{1-V}{1+V}} = \sqrt{(\alpha_0 t)^2+1} - \alpha_0 t = \alpha_0 x - \alpha_0 t = \alpha_0 \bar{x}_1$

so that $\boxed{\frac{\lambda_0}{\lambda} = \alpha_0 \bar{x}_1}$ red shift. So that observer O will think that clocks at $\bar{x}_1$ are slow by factor $\alpha_0 \bar{x}_1$

Problem I

Derive metric of Rindler coordinates.

Let's start with the definition of Rindler coordinates

$$\bar{t} = \frac{1}{\alpha_0} \operatorname{arctanh}\left(\frac{t}{x}\right); \quad \bar{y} = y; \qquad \Rightarrow \qquad \frac{t}{x} = \tanh(\alpha_0 \bar{t})$$

$$\bar{x} = \operatorname{sign}(x)\sqrt{x^2 - t^2}; \quad \bar{z} = z; \qquad x^2 - t^2 = \bar{x}^2 \Rightarrow$$

$$x^2\left(1 - \tanh^2(\alpha_0 \bar{t})\right) = \frac{x^2}{\cosh^2(\alpha_0 \bar{t})} = \bar{x}^2 \Rightarrow$$

$$\Rightarrow \quad x = \pm \bar{x} \cdot \cosh(\alpha_0 \bar{t}); \quad y = \bar{y};$$
$$t = \pm \bar{x} \cdot \sinh(\alpha_0 \bar{t}); \quad z = \bar{z};$$

$$dx = \pm\left(d\bar{x} \cdot \cosh(\alpha_0 \bar{t}) + d\bar{t}\,\alpha_0 \bar{x} \cdot \sinh(\alpha_0 \bar{t})\right); \quad dy = d\bar{y};$$
$$dt = \pm\left(d\bar{x} \cdot \sinh(\alpha_0 \bar{t}) + d\bar{t}\,\alpha_0 \bar{x} \cdot \cosh(\alpha_0 \bar{t})\right); \quad dz = d\bar{z};$$

So Minkowski metric transforms in the following way:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = -d\bar{x}^2 \cdot \sinh^2(\alpha_0 \bar{t}) - \alpha_0^2 \bar{x}^2 \cdot \cosh^2(\alpha_0 \bar{t}) d\bar{t}^2$$
$$- \cancel{2\alpha_0 \bar{x} \cdot d\bar{x} \cdot d\bar{t} \cdot \sinh(\alpha_0 \bar{t}) \cdot \cosh(\alpha_0 \bar{t})} + d\bar{x}^2 \cdot \cosh^2(\alpha_0 \bar{t}) + \alpha_0^2 \bar{x}^2 d\bar{t}^2 \cdot \sinh^2(\alpha_0 \bar{t})$$
$$+ \cancel{2\alpha_0 \bar{x}\, d\bar{x}\, d\bar{t} \cdot \sinh(\alpha_0 \bar{t}) \cdot \cosh(\alpha_0 \bar{t})} + d\bar{y}^2 + d\bar{z}^2 =$$
$$= -\alpha_0^2 \bar{x}^2 \cdot d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2; \quad \text{So that:}$$

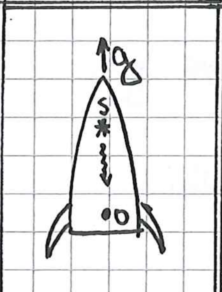
3

$$\boxed{ds^2 = -\alpha_0^2 \bar{x}^2 d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2;}$$

Notice that time separation between two spatially separated events is given by $\Delta t' = \alpha_0 \bar{x} \Delta t$ which is consistent with our previous results on the red shift.

Problem II

Consider rocket moving with acceleration $g = 10 \frac{m}{s^2}$. Let's say at some point one astronaut sitting in front end of the ship sends signal with the wavelength $\lambda_0 = 5000\,\text{Å}$. Another astrounaut in the back of the ship which is $L = 100\,m$ away get's this signal. What wavelength λ does he observe.



Pic. 2

Let's use red-shift formula we have derived

$\frac{\lambda_0}{\lambda} = g \cdot \bar{x}$ (notice that we have derived this formula using system which has $c = 1$, to return speed of light into play we should use dimension analysis)

$[g] = \frac{m}{s^2}$; $[\bar{x}] = m$; $[g \cdot \bar{x}] = \frac{m^2}{s^2}$; so that $\frac{g\bar{x}}{c^2}$ is dimensionless. So that $\underline{\frac{\lambda_0}{\lambda} = \frac{1}{c^2} g \cdot \bar{x}}$

Now we should understand what is $\bar{x}$?

Position of Observer O at moment of radiation is $\frac{1}{g}$ and then coordinate of source is $\bar{x} = \frac{1}{g} + L$, so that $\frac{\lambda_0}{\lambda} = \frac{g}{c^2}\left(\frac{c^2}{g} + L\right) = 1 + \frac{gL}{c^2}$ so that

$\boxed{\lambda = \lambda_0 \left(1 + \frac{gL}{c^2}\right)^{-1};}$ we see that $\lambda < \lambda_0$, i.e. light is blue-shifted

(4) • Another way to obtain the same result is to use weak equivalence principle, which states that accelrated frame is equivalent to fixed frame in gravitation field with $g = \alpha_0$.
Now let's consider source in gravitation field which sends signal to observer who is down by 100m.
Due to weak equivalence principle this problem is identical to previous one.
Now imagine that photon "falling" L meters down get additional potential energy as usual body for which we have:

* $E_0 = mc^2$ – energy before falling down
* $E = E_0 + mgL = mc^2\left(1 + \frac{gL}{c^2}\right) = E_0\left(1 + \frac{gL}{c^2}\right)$ – energy of body after the fall.

Assuming that the same holds for photon we get:

$h\nu = h\nu_0\left(1 + \frac{gL}{c^2}\right)$ so that $\lambda = \lambda_0\left(1 + \frac{gL}{c^2}\right)^{-1}$ which coinsides with the previous result.
Now if we do calculation:

$\lambda = 5 \cdot 10^3 \text{ Å} \cdot \left(1 + \frac{10 \cdot 100}{9 \cdot 10^{16}}\right)^{-1} \approx 5 \cdot 10^3 (1 - 10^{-14}) \text{ Å}$ as we see correction is really small but was measured

Problem 3

$ * excited atom emits photon at height h of frequency ν_0

o * another atom can adsorb photon of frequency ν_0 only and become excited.

Due to gravitational blue shift atom can't adsorb photon. But if we will move atom at some velocity V in order to compensate blue shift

(5) What should the velocity of the second atom be in order for photon to be adsorbed?

As we have figured out in previous problem light is blue-shifted so that $\lambda' = \lambda_0 \left(1 + \frac{gh}{c^2}\right)^{-1}$

Now if second is moving with velocity v down we get redshift $\lambda = \lambda' \cdot \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = \lambda_0 \left(1 + \frac{gh}{c^2}\right)^{-1} \sqrt{\frac{1+v/c}{1-v/c}}$

so this 2 shifts will compensate each other if

$\left(1 + \frac{gh}{c^2}\right)^{-1} \sqrt{\frac{c+v}{c-v}} = 1$ in order for this equation to be satisfied $v \ll c$ velocity should be small then

$\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = \frac{1 + \frac{1}{2}\frac{v}{c}}{1 - \frac{v}{2c}} = 1 + \frac{v}{c}$ so that $1 + \frac{v}{c} = 1 + \frac{gh}{c^2}$

and thus speed of atom should be equal to

$\boxed{v = \frac{gh}{c};}$ in Pound-Rebka experiment that was set in 1959 yr. h was equal to 22,5m

so that $v = \frac{22{,}5 \cdot 10}{3 \cdot 10^8} \approx 7{,}5 \cdot 10^{-7} \frac{m}{s}$.

①

Seminar 10 (exam 2011)

Problem 3 Consider a particle of mass m and charge e moving in the presence of electric and magnetic fields $\bar{E}=-\bar{\nabla}\Phi$ and $\bar{B}=\bar{\nabla}\times\bar{A}$. The Lagrangian of this particle is

$L=\frac{1}{2}m\dot{\bar{r}}^2-e\Phi+\frac{e}{c}\dot{\bar{r}}\bar{A}$ where $\bar{r}$ is the position of the particle.

ⓐ Derive the Euler-Lagrange equations of motion

Euler-Lagrange equations are $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_i}-\frac{\partial L}{\partial r_i}=0$

in index notation $L=\frac{1}{2}m\dot{r}_i^2-e\Phi(r_i)+\frac{e}{c}\dot{r}_iA_i$;

$\frac{\partial L}{\partial \dot{r}_i}=m\dot{r}_i+\frac{e}{c}A_i$; $\frac{\partial L}{\partial r_i}=-e\frac{\partial \Phi}{\partial r_i}+\frac{e}{c}\dot{r}_j\partial_jA_i$ (here in all expressions we assume summation over repeating index)

So that $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_i}=m\ddot{r}_i+\frac{e}{c}\dot{A}_i+\frac{e}{c}\dot{r}_j\partial_jA_i+e\frac{\partial\Phi}{\partial r_i}-\frac{e}{c}\dot{r}_j\partial_iA_j=0$

So that equations of motion we finally get:

$m\ddot{r}_i=-e\left(\frac{1}{c}\dot{A}_i+\frac{\partial\Phi}{\partial r_i}\right)+\frac{e}{c}\dot{r}_j(\partial_iA_j-\partial_jA_i)$

Now we use definitions of electric and magnetic field. $E_i=-\frac{1}{c}\dot{A}_i-\frac{\partial\Phi}{\partial r_i}$; $B_i=\varepsilon_{ijk}\partial_jA_k$ so that

$\partial_iA_j=\frac{1}{2}\varepsilon_{ijk}B_k \Rightarrow \dot{r}_j\cdot(\partial_iA_j-\partial_jA_i)=\frac{1}{2}(\varepsilon_{ijk}\dot{r}_jB_k-\varepsilon_{ijk}\dot{r}_kB_j)=$

$=\varepsilon_{ijk}\dot{r}_jB_k=\bar{v}\times\bar{B}$. So, finally, we get:

$\boxed{m\ddot{\bar{r}}=e(\bar{E}+\frac{1}{c}(\bar{v}\times\bar{B}))}$ So the force acting on charged particle is $\boxed{\bar{F}=e(\bar{E}+\frac{1}{c}(\bar{v}\times\bar{B}))}$ which is just Lorentz force and that's the answer for part ⓔ of the problem

ⓑ Compute the momentum $\bar{p}$ conjugate to $\bar{r}$

By definition $p_i=\frac{\partial L}{\partial \dot{r}_i}=m\dot{r}_i+\frac{e}{c}A_i$; $\boxed{\bar{p}=m\dot{\bar{r}}+\frac{e}{c}\bar{A};}$ so that $\dot{\bar{r}}=\frac{1}{m}\bar{p}-\frac{e}{mc}\bar{A}$;

ⓒ Compute the Hamiltonian, H as a function of $\bar{p}$ and $\bar{r}$.

② $H = \bar{p}\cdot\dot{\bar{r}} - L = \frac{\bar{P}^2}{2m} - \frac{e}{mc}\bar{p}\bar{A} + e\Phi - \frac{e}{mc}\bar{p}\bar{A} + \frac{e^2}{mc^2}\bar{A}^2;$

so that $\boxed{H = \frac{1}{2m}(\bar{p} - \frac{e}{c}\bar{A})^2 + e\Phi;}$

(d) Check that the Hamiltonian equations of motion gives the same force law as in (a);

Canonical equations are

$\dot{\bar{p}} = -\frac{\partial H}{\partial q} \Rightarrow \quad \dot{p}_i = (p_j - \frac{e}{c}A_j)\partial_i A_j \frac{e}{mc} - e\partial_i\Phi$

$\dot{r}_i = \frac{\partial H}{\partial p^i} = \frac{1}{m}(p_i - \frac{e}{c}A_i)$ then

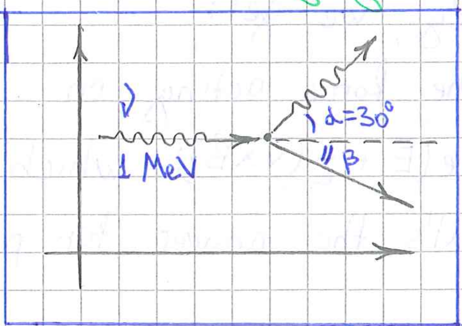
$m\ddot{r}_i = \dot{p}_i - \frac{e}{c}\dot{A}_i - \frac{e}{c}\dot{r}_j\partial_j A_i = \frac{e}{mc}(p_j - \frac{e}{c}A_j)\partial_i A_j - e\partial_i\Phi - \frac{e}{c}\dot{A}_i -$

$- \frac{e}{c}\dot{r}_j\partial_j A_i = \underbrace{\frac{e}{c}\dot{r}_j(\partial_i A_j - \partial_j A_i)}_{\frac{e}{c}(\bar{v}\times\bar{B})} - e\underbrace{(\frac{1}{c}\dot{A}_i + \partial_i\Phi)}_{-\bar{E}},$

$\boxed{m\ddot{\bar{r}} = e\{\bar{E} + \frac{1}{c}(\bar{v}\times\bar{B})\};}$ which coinside with result obtained in part (a).

Problem 4 A neutrino with energy 1 MeV scatters off an electron at rest ($m_e c^2 = 0{,}511$). The neutrino emerges from the interaction at a thirty degree angle away from its original trajectory. Assume that the neutrino has zero mass. You may also assume that the neutrino is not superluminal.

(a) Find the outgoing energy of the neutrino.



Let P and P′ are momentum of neutrino before and after scattering Q and Q′ momentum of electro before and after collision. As we take neutrino mass to be zero, this is just Compton scattering. Four-momentum conservation is:

$P_\mu + Q_\mu = P'_\mu + Q'_\mu \Rightarrow Q'_\mu = P_\mu + Q_\mu - P'_\mu;$ taking square we get:

$Q'^2 = P^2 + Q^2 + P'^2 + 2P\cdot Q - 2P'Q - 2P'P$; $Q^2 = Q'^2 = m^2c^2$; $P'^2 = P^2 = 0$;

So that we get $PQ - P'Q - P'P = 0$;

(3) Writing down all 4-momenta

$P = \frac{h\nu}{c}(1, \hat{x})$; $P' = \frac{h\nu'}{c}(1, \bar{n})$; $Q = (mc, 0)$; $Q' = (\frac{\mathcal{E}'}{c}, p')$;

where $\hat{x}$ is identity vector in positive x-direction, and $\bar{n}$ is vector along outgoing neutrino. So conservation equation becomes:

$mh(\nu - \nu') - \frac{h^2}{c^2}\nu'\nu(1 - \bar{n}\hat{x}) = 0$, $\bar{n}\hat{x} = \cos\alpha = \frac{\sqrt{3}}{2}$ so that

$\frac{mc^2}{h}\left(\frac{1}{\nu'} - \frac{1}{\nu}\right) = \frac{2-\sqrt{3}}{2} \Rightarrow \frac{1}{\mathcal{E}'} = \frac{1}{\mathcal{E}} + \frac{2-\sqrt{3}}{2mc^2}$, so finally neutrino energy is given by $\mathcal{E}' = \frac{2\mathcal{E}mc^2}{2mc^2 + (2-\sqrt{3})\mathcal{E}}$ if we substitute $\mathcal{E} = 1$ MeV and $mc^2 = 0{,}5$ MeV we get:

$\mathcal{E}' \approx 0{,}79$ MeV ;

$$\boxed{\mathcal{E}' = \frac{2\mathcal{E}mc^2}{2mc^2 + (2-\sqrt{3})\mathcal{E}} \approx 0{,}79 \text{ MeV}}; \quad \mathcal{E}' = \frac{\mathcal{E}mc^2}{mc^2 + (1-\cos\alpha)\mathcal{E}};$$

(b) Find the outgoing angle of the electron.

To find angle of outgoing electron let's write down momentum conservation in X and Y axis

X-axis $\quad \frac{h\nu}{c} = \frac{h\nu'}{c}\cos\alpha + p'\cos\beta$

Y-axis $\quad \frac{h\nu'}{c}\sin\alpha = p'\sin\beta$; $\Rightarrow p' = \frac{h\nu'}{c}\frac{\sin\alpha}{\sin\beta}$;

substitution to firs equation gives us:

$h\nu = h\nu'(\cos\alpha + \sin\alpha \cdot \mathrm{ctg}\,\beta)$ so we get

$\mathrm{ctg}\,\beta = \frac{1}{\sin\alpha}\left(\frac{\mathcal{E}}{\mathcal{E}'} - \cos\alpha\right)$; $\frac{\mathcal{E}}{\mathcal{E}'} = 1 + \frac{\mathcal{E}}{mc^2}\left(1 - \frac{\sqrt{3}}{2}\right)$

$\mathrm{ctg}\,\beta = (2-\sqrt{3})\left(1 + \frac{\mathcal{E}}{mc^2}\right) = 6 - 3\sqrt{3}$

$$\boxed{\beta = \mathrm{arccot}(6 - 3\sqrt{3})}$$

(c) finally energy of an electron:

We can find from energy conservation law.

$\mathcal{E} - \mathcal{E}' = -mc^2 + \mathcal{E}_e \Rightarrow \mathcal{E}_e = mc^2 \quad \mathcal{E}\left(\frac{mc^2}{mc^2 + (1-\cos\alpha)\mathcal{E}} - 1\right)$

$$\boxed{\mathcal{E}_e = \frac{(1-\cos\alpha)\mathcal{E}^2}{mc^2 + (1-\cos\alpha)\mathcal{E}} + mc^2 = 0{,}71 \text{ MeV}}$$

④ Problems

Using Cartesian coordinates in the plane, the Lagrangian for the motion of the Foucault pendulum may be written

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}m\Omega^2(x^2+y^2) + \omega_z(x\dot{y} - y\dot{x}),$$

where $\Omega = \sqrt{\frac{g}{e}}$ is the usual pendulum frequency and ω_z represents the Coriolis force. In the (x,y) coordinates, the motion is inseparable. Show that by going to polar coordinates in the plane, (ρ,φ) the Hamiltonian becomes separable due to the cyclicity of φ. Solve the HJE, i.e. find Hamiltons principal function for this problem up to elementary integrals.

Let's go to polar coordinates (ρ, φ)

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases} \quad \dot{x}^2 + \dot{y}^2 = \dot{\rho}^2 + \rho^2\dot{\varphi}^2 \;;\; x^2+y^2 = \rho^2;$$

$$x\dot{y} - y\dot{x} = \rho\dot{\rho}\cos\varphi\cdot\sin\varphi + \rho^2\dot{\varphi}\cos^2\varphi - \rho\dot{\rho}\sin\varphi\cdot\cos\varphi + \rho^2\dot{\varphi}\sin^2\varphi = r^2\dot{\varphi}$$

$$\begin{cases} \dot{x} = \dot{\rho}\cos\varphi - \rho\dot{\varphi}\sin\varphi \\ \dot{y} = \dot{\rho}\sin\varphi + \rho\dot{\varphi}\cos\varphi \end{cases}$$

So Lagrangian Becomes

$$\boxed{L = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\varphi}^2 - \frac{1}{2}m\Omega^2\rho^2 + \omega_z\rho^2\dot{\varphi};}$$

*canonical momentum $p_\rho = \frac{\partial L}{\partial\dot{\rho}} = m\dot{\rho}$; $p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = m\rho^2\dot{\varphi} + \omega_z\rho^2$

*inverting this expressions we get $\dot{\rho} = \frac{p_\rho}{m}$; $\dot{\varphi} = \frac{p_\varphi}{m\rho^2} - \frac{\omega_z}{m}$;

* making Legendre transformation we get:

$$H = p_\rho\dot{\rho} + p_\varphi\dot{\varphi} - L = \frac{p_\rho^2}{m} + \frac{p_\varphi^2}{m\rho^2} - \frac{p_\varphi\omega_z}{m} - \frac{p_\rho^2}{2m} - \frac{p_\varphi^2}{2m\rho^2} + \frac{p_\varphi\omega_z}{m} - $$
$$- \frac{\rho^2\omega_z^2}{2m} + \frac{1}{2}m\Omega^2\rho^2 - \frac{\omega_z p_\varphi}{m} + \frac{\omega_z^2\rho^2}{m};$$

$$\boxed{H = \frac{p_\rho^2}{2m} + \frac{1}{2m\rho^2}(p_\varphi^2 - 2\omega_z\rho^2 p_\varphi) + \frac{\rho^2\omega_z^2}{2m} + \frac{1}{2}m\Omega^2\rho^2}$$

Note that φ-coordinate is not presented in Hamiltonian so it is cyclic variable and we can put $p_\varphi = J = \text{const}$. and we get effective Lagrangian:

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$$H = \frac{1}{2m} p_\rho^2 + \frac{1}{2m\rho^2} J^2 - \frac{\omega_z J}{m} + \frac{\rho^2 \omega_z^2}{2m} + \frac{1}{2} m \Omega^2 \rho^2;$$

Now we can introduce Hamilton's principal function S and write down HJE

$$\frac{1}{2m}\left(\frac{\partial S}{\partial \rho}\right)^2 + \frac{1}{2m\rho^2} J^2 - \frac{\omega_z J}{m} + \frac{\rho^2 \omega_z^2}{2m} + \frac{1}{2} m \Omega^2 \rho^2 + \frac{\partial S}{\partial t} = 0$$

as Hamiltonian is time-independent we make ansatz: $S(\rho, t) = W(\rho) - \alpha_1 t$ and we get equation for characteristic equation

$$\frac{1}{2m}(W_\rho')^2 + \frac{1}{2m\rho^2} J^2 - \frac{\omega_z J}{m} + \frac{\rho^2 \omega_z^2}{2m} + \frac{1}{2} m \Omega^2 \rho^2 - \alpha_1 = 0$$

So that $$W_\rho' = \left[2m\alpha_1 + 2\omega_z J - m^2 \Omega^2 \rho^2 - \omega_z^2 \rho^2 - \frac{J^2}{\rho^2}\right]^{\frac{1}{2}}$$

So we finally get Hamilton's principal function:

$$\boxed{S(\rho, t) = \int d\rho \left[2\alpha_1 m + 2\omega_z J - m^2 \Omega^2 \rho^2 - \omega_z^2 \rho^2 - \frac{J^2}{\rho^2}\right]^{\frac{1}{2}} - \alpha_1 t;}$$

Problem 6 The LHC is a collider between opposite beams of protons. The protons in a beam are not actually equally spaced but come in "bunches". At present, the spacing between the bunches is about 50 ns ($5 \cdot 10^{-8}$ s) You may assume that each beam has 3,5 TeV protons and that the rest mass of the proton is 1 GeV.

(a) Find the distance between the bunches as measured by a physicist at the LHC

In lab frame distance is simply equal to the path passed by first bunch in 50 ns:

$$L_0 = vt = ct \cdot \sqrt{1 - \frac{1}{\gamma^2}} \approx ct = 5 \cdot 10^{-8} \cdot 3 \cdot 10^8 = 15\,m$$

Here we have used the fact that γ-factor equals to $\gamma = \frac{\mathcal{E}}{mc^2} = 3,5 \cdot 10^3$ and velocity $v = c\sqrt{1 - \frac{1}{\gamma^2}} \approx c$

So $\boxed{L_0 = 15\,m}$

⑥ (b) Find the distance between the bunches as measured in the rest frame of the bunches.

* Let's consider coordinates of particles separated by L_0 distance. Now

$x_1 = ut_1$ $\quad$ $x_2 = ut_2 + L_0$

Let's go to some other frame moving with velocity v. In this frame velocity of bunches is given by $u' = \frac{u-v}{1-\frac{uv}{c^2}}$; Now times are conserved as

$$t_1' = \gamma\left(t_1 - \frac{v}{c^2}x_1\right) = \gamma t_1\left(1 - \frac{uv}{c^2}\right); \quad t_2' = \gamma t_2\left(1-\frac{uv}{c^2}\right) - \gamma L_0 \frac{v}{c^2}$$

inverting this expressions we get

$$t_1 = \frac{t_1'}{\gamma\left(1-\frac{uv}{c^2}\right)}; \quad t_2 = \frac{t_2'}{\gamma\left(1-\frac{uv}{c^2}\right)} + \frac{v}{c^2}\frac{L_0}{1-\frac{uv}{c^2}};$$

now we can write down coordinates of particles

$$x_1' = \gamma(x_1 - vt_1) = \gamma(u-v)t_1 = \frac{u-v}{1-\frac{uv}{c^2}}t_1' = u't_1';$$

$$x_2' = \gamma(x_2 - vt_2) = \gamma(u-v)t_2 + \gamma L_0 = u't_2' + \gamma\frac{u'v}{c^2}L_0 + \gamma L_0$$

So we finally get

$$x_1' = u't_1'; \quad x_2' = u't_2' + \gamma L_0\left(1+\frac{u'v}{c^2}\right) = u't_2' + L' \Rightarrow \boxed{L' = \gamma L_0\left(1+\frac{u'v}{c^2}\right)}$$

Now if we substitute $u'=0$ (rest frame of bunches) we get $\boxed{L = \gamma L_0 \approx 52{,}5k}$. This is just formula given by Lorentz contraction, here proper length is L and L_0 is length in lab frame.

(c) Find the distance between the bunches as measured in the rest frame of the bunches in the other beam.

Now we take frame moving with velocity "$-u$" using formula derived we get $L'' = \gamma L_0\left(1+\frac{u'v}{c^2}\right)$ in this case $u' = \frac{2u}{1+\frac{u^2}{c^2}}$ so we get

$$L'' = \gamma L_0\left(1 - \frac{2u^2}{c^2}\frac{1}{1+\frac{u^2}{c^2}}\right) = \frac{L_0\left(1-\frac{u^2}{c^2}\right)}{\sqrt{1-\frac{u^2}{c^2}}\left(1+\frac{u^2}{c^2}\right)};$$

So

$$\boxed{L'' = \frac{L_0}{\gamma\left(1+\frac{u^2}{c^2}\right)} \approx \frac{L_0}{\gamma(u)} = 2\cdot 10^{-3}\,m = 2\,mm}$$

(7) the same result can be obtained from Lorentz contraction formula. Second bunch moves with velocity $u' = -\frac{2u}{1+\frac{u^2}{c^2}}$ So that

$$\frac{1}{\gamma(u')} = \sqrt{1 - \frac{4u^2}{c^2\left(1+\frac{u^2}{c^2}\right)^2}} \Rightarrow \gamma(u') = \frac{1+u^2/c^2}{1-u^2/c^2} = \gamma^2(u)\left(1+\frac{u^2}{c^2}\right)$$

Now we can use Lorentz contraction

$$L'' = \frac{L}{\gamma(u')} = \frac{\gamma(u)}{\gamma(u')} L_0 = \frac{L_0}{\gamma(u)\left(1+\frac{u^2}{c^2}\right)}$$ which coinsides with the formula obtained by previous methode.