I Conformal transformations and conformal algebra (approx 2h 30m)

Plan:

1. Review of Poincare transformations:
   - def. of transform
   - Killing vector equation and generic solution.
   - Algebra.
2. Conformal transformation:
   - definition.
   - Conformal Killing eq. and solutions
   - Finite form of transform.
   - Comment on SCT: iversion, non-linearity.
   - Conformal group

1. Let's start with reviewing Poincare transformations:

- Poincare transformation is transf. $x \to \tilde{x}(x)$ that preserves form of the metric:

$$g_{\mu\nu}\, d\tilde{x}^\mu\, d\tilde{x}^\nu = g_{\alpha\beta}\, dx^\alpha\, dx^\beta \Rightarrow \boxed{g_{\mu\nu} = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}} \quad (1)$$

- In this lectures we will concentrate on the case of flat space so that

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{cases} \text{diag}(-1\ +1\ +1\ +1) & \text{Minkowski signature} \\ \text{diag}(+1\ +1\ +1\ +1) & \text{Euclidian signature} \end{cases}$$

- Infinitesemal analysis of transformation.

Let's consider transf. close to identity.

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x), \quad (2)$$

small term $|\xi^\mu(x)| \ll |x^\mu|$ (Killing vector)

$\xi = \xi^\mu \partial_\mu$ – vector field generating transf.

- Let's plug infinitesemal transform (2) into metric transformation (1);

we assume flat metric here

$$\frac{\partial \tilde{x}^\mu}{\partial x^\alpha} = \delta^\mu{}_\alpha + \partial_\alpha \xi^\mu \Rightarrow \eta_{\mu\nu}(\delta^\mu{}_\alpha + \partial_\alpha \xi^\mu)(\delta^\nu{}_\beta + \partial_\beta \xi^\nu) = \eta_{\alpha\beta} \Rightarrow$$

$$\Rightarrow \boxed{\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0} \quad (3) \text{ – Killing vector equation.}$$

taking one more derivative
+ permuting indicies:

$$\left.\begin{aligned}
&\partial_\alpha \partial_\nu \xi_\mu + \partial_\alpha \partial_\mu \xi_\nu = 0 \\
&\quad \downarrow (\alpha \leftrightarrow \nu) \qquad + \\
&\partial_\alpha \partial_\nu \xi_\mu + \partial_\nu \partial_\mu \xi_\alpha = 0 \\
&\quad \downarrow (\mu \leftrightarrow \nu) \qquad - \\
&\partial_\mu \partial_\nu \xi_\alpha + \partial_\alpha \partial_\mu \xi_\nu = 0
\end{aligned}\right\} \quad \partial_\alpha \partial_\nu \xi_\mu = 0 \Rightarrow \xi_\mu \text{ is at most linear:}$$

$$\xi_\mu = a_\mu + \omega_{\mu\nu} x^\nu \quad (4)$$

- $a_\mu \to$ translations # of param. $= d$ ↳ dimension of spacetime.
- Substituting gen. solution (4) into (3) we get:

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0 \Rightarrow \omega_{\mu\nu} \text{ is antisymmetric matrix.}$$

this corresponds to the Lorentz transf.

# of paramet. $= d(d-1)/2$

Comments/extras on Lorentz transf.

- In our infinitesemal analysis we reproduce only part of the full Lorentz group connected to identity.
- In general Lorentz transf. is written as

$\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$ with $\Lambda^\mu{}_\nu$ obeying $\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$

or in matrix form $\Lambda^T \eta \Lambda = \eta$ - elements Λ forms orthogonal group $\begin{cases} \text{Euclidian: } O(d) \\ \text{Mink: } O(d-1,1) \end{cases}$

- Infinitesemal transformation has the form: $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ which we reproduce earlier and which corresponds to $SO(d)$ in Euclid. or $SO(d-1,1)$ in Mink. spaces. This is called proper Lorentz transformation. ($\det \Lambda = 1$, $\Lambda^0{}_0 \geq 1$, ortochronous)
- from $\eta_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \Rightarrow$ $\boxed{\det\Lambda = \pm 1;\ (\Lambda^0{}_0)^2 \geq 1;}$ $(\Lambda^0{}_0)^2 - \Lambda^i{}_0 \Lambda^i{}_0 = 1$
- So there are ~~three~~ extra possibilities of discrete transform not connected to one:

① parity transf.	② time reversal.	③ PT transform.
$\mathcal{P}^\mu{}_\nu = \begin{bmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$	$\mathcal{T}^\mu{}_\nu = \begin{bmatrix} -1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & +1 \end{bmatrix}$	$(\mathcal{PT})^\mu{}_\nu = \begin{bmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$
orthochronous, improper	non-ortochronous improper	non-orthochronous proper

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- Now finally let's move to the conformal transf.

By def. conf. transform. is coordinate transform. $x \to \tilde{x}(x)$ which rescales the metric

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Lambda(x)\, g_{\mu\nu}(x) \qquad (5)$$

at the same time $\tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = g_{\alpha\beta} dx^\alpha dx^\beta$

Combining this with (5) we obtain:

$$\boxed{\Lambda^{-1}(x)\, g_{\mu\nu} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} g_{\alpha\beta}} \qquad (6)$$

- Comments: ① Obviously $\Lambda(x)=1$ correspond to the Lorentz tranform. Hence Poincare is a subgroup of conformal group.

② We can introduce matrix $\Lambda^\mu{}_\nu = \Lambda^{-\frac{1}{2}}(x) \frac{\partial x^\mu}{\partial \tilde{x}^\nu}$

this matrix corresponds as before to the Lorentz transform as can be seen from comparing (6) and (1)

- As before we consider infinitesemal transform:

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x); \quad |\xi^\mu| \ll |x^\mu|$$

$$\Lambda(x) = 1 - f(x);$$

- Substituting this into (6):

as usually we assume flat space metric.

$$(1+f(x))\,\eta_{\mu\nu} = (\delta^\alpha_\mu + \partial_\mu \xi^\alpha)(\delta^\beta_\nu + \partial_\nu \xi^\beta)\,\eta_{\alpha\beta} \Rightarrow$$

⑥

$$\Downarrow$$

$$\boxed{(2-2d)\,\partial^2 f=0} \quad (12)$$

- Comments:

① In $d=1$ eq. (11) is satisfied for any f, i.e. any transform is conformal.

② In $d=2$ situation is also specific:

Consider CKE in $d=2$ (in Euclidian)

- $\mu=\nu=1$ $\quad 2\partial_1\xi_1=\partial_1\xi_1+\partial_2\xi_2$
- $\mu=1\ \nu=2$ $\quad \partial_1\xi_2+\partial_2\xi_1=(\partial_1\xi_1+\partial_2\xi_2)\cdot 0$

$$\Downarrow\ \boxed{\begin{array}{l}\partial_1\xi_1=\partial_2\xi_2;\\ \partial_1\xi_2=-\partial_2\xi_1;\end{array}}$$

→ Cauchy-Riemann e

Solution → any holomorphic functi in the plane $(x_1+ix_2, x_1-i$

- Hence in $d=2$ any holomorphic map is conformal !!!
- More details ⇒ further in the course.

- Now let's go further concentraiting on the case $d\geq 3$. From eq (11), (12) $\partial_\mu\partial_\nu f=0$, hence f is at most linear $f(x)=A+B_\mu x^\mu$

- Then for the Killing vector:

$$2\partial_\mu\partial_\nu\xi_\rho=\eta_{\mu\rho}\cdot\partial_\nu f+\eta_{\nu\rho}\cdot\partial_\mu f-\eta_{\mu\nu}\cdot\partial_\rho f=\eta_{\mu\rho}B_\nu+\eta_{\nu\rho}B_\mu-\eta_{\mu\nu}$$

$\Rightarrow$ ξ_μ is at most quadratic $\Rightarrow$ $\boxed{\xi_\mu=a_\mu+b_{\mu\nu}x^\nu+c_{\mu\nu\rho}x^\nu x^\rho;}$

$\Rightarrow$ $\boxed{\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = f(x)\,\eta_{\mu\nu}}$ (7) – conformal Killing vector equation. (CKE)

Resolving these equations:

① take the trace of both parts:

$$2\,\partial_\alpha \xi^\alpha = f(x)\cdot d$$

② Similarly to the usual Killing vector equation: let's take one more derivative and permute indices:

$$\partial_\alpha \partial_\mu \xi_\nu + \partial_\alpha \partial_\nu \xi_\mu = \partial_\alpha f(x)\cdot \eta_{\mu\nu}$$

$\downarrow \alpha \leftrightarrow \mu$ $\quad +$

$$\partial_\alpha \partial_\mu \xi_\nu + \partial_\mu \partial_\nu \xi_\alpha = \partial_\mu f(x)\cdot \eta_{\alpha\nu}$$

$\downarrow \mu \leftrightarrow \nu$ $\quad -$

$$\partial_\alpha \partial_\nu \xi_\mu + \partial_\mu \partial_\nu \xi_\alpha = \partial_\nu f(x)\cdot \eta_{\alpha\mu}$$

$$\Downarrow$$

$$\eta^{\alpha\mu}\cdot|\; 2\partial_\alpha \partial_\mu \xi_\nu = \eta_{\mu\nu}\,\partial_\alpha f + \eta_{\alpha\nu}\,\partial_\mu f - \eta_{\alpha\mu}\,\partial_\nu f;$$

$$\Downarrow$$

$$2\partial^2 \xi_\nu = \partial_\nu f\cdot(2-d) \quad (8)$$

③ Consider following actions of diff. op's:

$$\partial_\mu\cdot(8) \Rightarrow \; 2\partial^2 \partial_\mu \xi_\nu = (2-d)\,\partial_\mu \partial_\nu f \quad (9)$$

$$\partial^2\cdot(7) \Rightarrow \; \partial^2 \partial_\mu \xi_\nu + \partial^2 \partial_\nu \xi_\mu = \eta_{\mu\nu}\cdot \partial^2 f \quad (10)$$

$\Downarrow$ (9) + (9)($\mu \leftrightarrow \nu$) – (10)×2

$$2(2-d)\,\partial_\mu \partial_\nu f = 2\eta_{\mu\nu}\cdot \partial^2 f \quad (11)$$

$\Downarrow$ contract with $\eta^{\mu\nu}$

⑦. Let's consider parameters one by one

① a_μ – no constraints, d parameters, translations

② $b_{\mu\nu}$ – substitue $\xi_\mu = b_{\mu\nu} \cdot x^\nu$ into CKE (7)

$$b_{\mu\nu} + b_{\nu\mu} = f(x)\eta_{\mu\nu} = \frac{2}{d}\partial_\mu \xi^\mu \eta_{\mu\nu} = \frac{2}{d} b^\alpha{}_\alpha \eta_{\mu\nu}$$

then the solution is $b_{\mu\nu} = \alpha\eta_{\mu\nu} + \omega_{\mu\nu}$, $1 + \frac{d(d-1)}{2}$ param

trace part scale transf. $\omega_{\mu\nu} = -\omega_{\nu\mu}$ (Lorentz)

③ $c_{\mu\nu\rho}$ – substitute $\xi_\mu = c_{\mu\nu\rho} x^\nu x^\rho$ into CKE, in particular into

$$\rightarrow f = \frac{4}{d} \cdot c^\mu{}_{\mu\alpha} x^\alpha$$

$$2\partial_\mu \partial_\nu \xi_\rho = \eta_{\mu\rho} \cdot \partial_\nu f + \eta_{\nu\rho} \cdot \partial_\mu f - \eta_{\mu\nu} \cdot \partial_\rho f$$

$$\downarrow$$

$$4 \cdot c_{\rho\mu\nu} = 4\eta_{\mu\rho} \cdot b_\nu + 4\eta_{\nu\rho} \cdot b_\mu - 4 \cdot \eta_{\mu\nu} \cdot b_\rho$$

$$\rightarrow b_\mu = \frac{1}{d} c^\alpha{}_{\alpha\mu}, \quad \underbrace{c_{\mu\nu\rho} = c_{\mu\rho\nu}}_{\text{we have also used this}}$$

• hence infinitesemal transf. can be written as:

$$\tilde{x}^\mu = x^\mu + \xi^\mu(x) = x^\mu + c^{\mu\nu\rho} x_\nu x_\rho = x^\mu + x_\nu x_\rho (\eta^{\mu\nu} b^\rho + \eta^{\mu\rho} b^\nu - \eta^{\nu\rho} b^\mu)$$

$$= x^\mu + 2x^\mu (x \cdot b) - x^2 \cdot b^\mu$$

$$\Downarrow$$

$$\tilde{x}^\mu = x^\mu + 2x^\mu (x \cdot b) - x^2 b^\mu;$$

$$\downarrow$$

Special Conformal Transformation

d parameters (components of b_μ) (SCT)

- Summary table of conformal transf:

Transformation	Infinetesemal	Finite	# of paramet.
Translation	$x^\mu \to x^\mu + a^\mu$	$x^\mu \to x^\mu + a^\mu$	$d, \Lambda(x)=1$
Lorentz	$x^\mu \to x^\mu + \omega^{\mu\nu} x_\nu$ $(\omega_{\mu\nu} = -\omega_{\nu\mu})$	$x^\mu \to \Lambda^\mu{}_\nu x^\nu$ $\eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = \eta_{\alpha\beta}$	$\frac{d(d-1)}{2}, \Lambda(x)=1$
Scale (dilatation)	$x^\mu \to (1+\alpha) x^\mu$	$x^\mu \to \lambda x^\mu$	$1, \Lambda(x)=\lambda^{-2}$
SCT	$x^\mu \to x^\mu + 2x^\mu (b \cdot x) - x^2 b^\mu$	$x^\mu \to \dfrac{x^\mu - b^\mu x^2}{1-2(b\cdot x)+b^2x^2}$	$d, \Lambda(x) = (1-2(b\cdot x)+x^2b^2)^2$
			total: $\frac{1}{2}(d+1)(d+2)$

Comments:

① First 2 lines form Poincare subgroup of conformal group.

② Eponentiation of SCT is highly nontrivial. Instead we can check that finite form given above results in correct infinitesemal limit:

$$\frac{x^\mu - b^\mu x^2}{1-2(b\cdot x)+b^2x^2} \longrightarrow (x^\mu - b^\mu x^2)(1+2(b\cdot x)+O(b^2)) =$$

$$= x^\mu + 2x^\mu (b\cdot x) - b^\mu \cdot x^2, \text{ q.e.d.}$$

③ Finite SCT has very nice interpretation:

- Consider inversion transformation: $x'^\mu = I \cdot x^\mu = \dfrac{x^\mu}{x^2}$

Properties of inversion:

⑨ ⓐ Inversion is conformal transformation:

indeed $\frac{\partial \tilde{x}^\mu}{\partial x^\alpha} = \frac{\delta^\mu{}_\alpha}{x^2} - \frac{x^\mu \cdot 2x_\alpha}{x^4} = x^{-2}\left(\delta^\mu{}_\alpha - 2\frac{x^\mu x_\alpha}{x^2}\right)$

then $\tilde{\eta}_{\mu\nu} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \cdot \frac{\partial \tilde{x}^\beta}{\partial x^\nu} \eta_{\alpha\beta} = x^{-4}\left(\delta^\alpha{}_\mu - 2\frac{x^\alpha x_\mu}{x^2}\right)\left(\delta^\beta{}_\nu - 2\frac{x^\beta x_\nu}{x^2}\right)\eta_{\alpha\beta} =$

$= x^{-4}\left(\eta_{\mu\nu} - 4\cancel{\frac{x_\mu x_\nu}{x^2}} + 4\cancel{\frac{x^2 x_\mu x_\nu}{x^4}}\right) = x^{-4}\eta_{\mu\nu}$

and hence inversion is conf. transf with $\Lambda(x) = x^{-4}$;

ⓑ Obviously inversion is discrete transform not connected to the identity. It swaps orientation of spacetime

ⓒ SCT = inversion + translations + inversions:

This fact can be shown streight forward:

$$\underset{\substack{\uparrow \\ \text{SCT transf}}}{K_b}\, x^\mu \overset{?}{=} I \underset{\substack{\uparrow \\ \text{transl.}}}{P_b} I \cdot x^\mu = I P_b \cdot \frac{x^\mu}{x^2} = I \cdot \left(\frac{x^\mu}{x^2} - b^\mu\right) = \frac{x^\mu - b^\mu x^2}{x^{-2}(x^\mu - x^2 b^\mu, x_\nu - x \ldots}$$

$$= \frac{x^\mu - b^\mu x^2}{x^{-2}(x^2 + x^4 b^2 - 2x^2(xb))} =$$

$$= \frac{x^\mu - b^\mu x^2}{1 - 2(xb) + x^2 b^2} \equiv K_b x^\mu \quad \text{q.e.d.!}$$

• Another way to see it is to notice

$$\frac{\tilde{x}^\mu}{\tilde{x}^2} = \frac{x^\mu}{x^2} - b^\mu \quad \text{for SCT}$$

Exercise: Do streight forward calculation to show equation above.

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$$\frac{\tilde{x}^\mu}{\tilde{x}^2} = \frac{(x^\mu - b^\mu x^2)(1-2(b\cdot x)+x^2 b^2)}{(x^\mu - b^\mu x^2)(x_\nu - b_\nu x^2)} = \frac{(x^\mu - b^\mu x^2)(1-2(b\cdot x)+x^2 b^2)}{(x^2 + b^2 x^4 - 2x^2(b\cdot x))} =$$

$$= \frac{x^\mu}{x^2} - b^\mu \quad , \text{ q.e.d.}$$

(d) Just as usual QFT is not necessarily P, T-inv., CFT is not required to be inv. w.r.t. inversion.

(e) Calculation of $\Lambda(x)$ factors is trivial in all cases except SCT. In this case the calculation is straightforward but lengthy.

Exercise: Show that for SCT $\Lambda(x) = (1-2(b x)+b^2 x^2)^2$;

- Generators of the transformation and conformal group.

- Consider infinitesemal transform

$$x^\mu \to \tilde{x}^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a}$$

$\uparrow$ infinitesemal parameter (ω_a).

- For conformal transformations:

Translations: $x^\mu \to \tilde{x}^\mu + a^\mu$, $\omega_\mu = a_\mu$; $\frac{\delta x^\mu}{\delta a^\nu} = \delta^\mu{}_\nu$;

Lorentz: $x^\mu \to \tilde{x}^\mu = (\delta^\mu{}_\nu + \omega^\mu{}_\nu) x^\nu$; $\frac{\delta x^\mu}{\delta \omega^{\alpha\beta}} = (\delta^\mu{}_\alpha x_\beta - \delta^\mu{}_\beta x_\alpha)$

Dilations: $x^\mu \to \tilde{x}^\mu = x^\mu + \alpha x^\mu$; $\frac{\delta x^\mu}{\delta \alpha} = x^\mu$;

SCT: $x^\mu \to \tilde{x}^\mu = x^\mu + 2x^\mu (x\cdot b) - x^2 b^\mu$; $\frac{\delta x^\mu}{\delta b^\nu} = 2x^\mu x_\nu - x^2 \delta^\mu{}_\nu$;

(11) • Assume there is scalar function $f(x)$ which satisfies $\tilde{f}(\tilde{x}) = f(x)$

Then we define generator of transformation as:

$$\delta_\omega f(x) \equiv \tilde{f}(x) - f(x) \equiv -i\omega_a G_a f(x)$$

generator.

then as $\tilde{f}(\tilde{x}) = f(x) = f\left(\tilde{x} - \omega_a \frac{\delta x}{\delta \omega_a}\right) = f(\tilde{x}) - \omega_a \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu f(\tilde{x})$

$$\Rightarrow \quad \boxed{i G_a f(x) = \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu f(x)} \quad (13)$$

• For conformal transformations:

Translations: $i\hat{P}_\mu f = \delta^\nu{}_\mu \partial_\nu f \Rightarrow \hat{P}_\mu = -i\partial_\mu;$

Lorentz: $i\hat{L}_{\mu\nu} f(x) = (\delta^\alpha_\mu x_\nu - \delta^\alpha{}_\nu x_\mu)\partial_\alpha f \Rightarrow \hat{L}_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$

Dilations: $i\hat{D} f = x^\mu \partial_\mu f \Rightarrow \hat{D} = -i x^\mu \partial_\mu;$

SCT: $i\hat{K}_\mu f = (2x^\nu x_\mu - x^2 \delta^\nu{}_\mu)\partial_\nu f \Rightarrow \hat{K}_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)$

• Summary of generators:

$$\boxed{\begin{array}{l} \hat{P}_\mu = -i\partial_\mu; \quad \hat{L}_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu); \quad \hat{D} = -i x^\mu \partial_\mu; \\ \hat{K}_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu); \end{array}}$$

• Using this representation of conformal group we can find commutation relations defining algebra:

(1) $[\hat{D}, \hat{P}_\mu] = -[x^\nu \partial_\nu, \partial_\mu] = -x^\nu \cancel{\partial_\nu \partial_\mu} + \delta^\nu_\mu \partial_\nu + x^\nu \cancel{\partial_\nu \partial_\mu} =$

$= i\hat{P}_\mu$

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② $[\hat{D},\hat{K}_\mu] = -[x^\alpha\partial_\alpha, 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu] = -2x_\mu x^\nu \cancel{x^\alpha}\partial_\nu\partial_\alpha -$

$-2\delta^\alpha_\mu x^\nu \cancel{x_\alpha}\partial_\nu - 2\delta^\alpha_\nu x_\mu \cancel{x_\alpha}\partial^\nu + x^\alpha \cancel{x^2}\partial_\alpha\partial_\mu + 2\cancel{x^\alpha x_\alpha}\partial_\mu +$

$+2x_\mu x^\nu \cancel{x^\alpha}\partial_\nu\partial_\alpha + 2x^\alpha x_\mu\delta^\nu_\alpha\partial_\nu + 2x^\alpha x^\nu\eta_{\mu\alpha}\partial_\nu - x^\alpha\cancel{x^2}\partial_\mu\partial_\alpha -$

$-2\cancel{x^\alpha x_\alpha}\partial_\mu = -2x^\nu \cancel{x_\mu}\partial_\nu - 2x_\mu\partial^2 + 2x_\mu x_\nu\partial^\nu + 2x_\mu x^\nu\partial_\nu$

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② $[\hat{D}, \hat{K}_\mu] = -[x^\alpha \partial_\alpha, 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu] =$

$= -2x^\alpha \eta_{\mu\alpha} x^\nu \partial_\nu - 2x^\alpha \delta^\nu_\alpha x_\mu \partial_\nu + 2x^\alpha x_\alpha \partial_\mu + 2x_\mu x^\nu \delta^\alpha_\nu \partial_\alpha -$

$- x^2 \delta^\alpha_\mu \partial_\alpha = -2x_\mu x^\nu \partial_\nu - \cancel{2x_\mu x^\nu \partial_\nu} + 2x^2 \partial_\mu + \cancel{2x_\mu x^\nu \partial_\nu} - x^2 \partial_\mu$

$= -(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) = -i\hat{K}_\mu$

③ ..... Exercise: Derive all further commutation relations.

- Conformal algebra:

$$\boxed{\begin{aligned} &[\hat{D}, \hat{P}_\mu] = i\hat{P}_\mu; \quad [\hat{D}, \hat{K}_\mu] = -i\hat{K}_\mu; [\hat{K}_\mu, \hat{P}_\nu] = 2i(\eta_{\mu\nu}\hat{D} - L_{\mu\nu}); \\ &[\hat{K}_\rho, \hat{L}_{\mu\nu}] = i(\eta_{\rho\mu}\hat{K}_\nu - \eta_{\rho\nu}\hat{K}_\mu); \\ &[\hat{P}_\rho, \hat{L}_{\mu\nu}] = i(\eta_{\rho\mu}\hat{P}_\nu - \eta_{\rho\nu}\hat{P}_\mu); \rightarrow \text{Poincare subalgebra.} \\ &[\hat{L}_{\mu\nu}, \hat{L}_{\rho\sigma}] = i(\eta_{\nu\rho}\hat{L}_{\mu\sigma} + \eta_{\mu\sigma}\hat{L}_{\nu\rho} - \eta_{\mu\rho}\hat{L}_{\nu\sigma} - \eta_{\nu\sigma}\hat{L}_{\mu\rho}); \end{aligned}}$$

- It is convenient to introduce new notations:

$$\boxed{\hat{J}_{\mu\nu} = \hat{L}_{\mu\nu}; \quad \hat{J}_{-1,0} = \hat{D}; \quad \hat{J}_{-1,\mu} = \tfrac{1}{2}(\hat{P}_\mu - \hat{K}_\mu); \quad \hat{J}_{0,\mu} = \tfrac{1}{2}(\hat{P}_\mu + \hat{K}_\mu);}$$

These new generators satisfy commutation relations:

- $[\hat{J}_{\mu\nu}, \hat{J}_{\alpha\beta}] = i(\eta_{\nu\alpha}\hat{J}_{\mu\beta} + \eta_{\mu\beta}\hat{J}_{\nu\alpha} - \eta_{\mu\alpha}\hat{J}_{\nu\beta} - \eta_{\nu\beta}\hat{J}_{\mu\alpha});$
- $[\hat{J}_{-1,0}, \hat{J}_{-1,\mu}] = \frac{1}{2}[\hat{D}, \hat{P}_\mu - \hat{K}_\mu] = \frac{i}{2}(\hat{P}_\mu + \hat{K}_\mu) = i \cdot \hat{J}_{0,\mu};$
- $[\hat{J}_{-1,0}, \hat{J}_{0,\mu}] = \frac{1}{2}[\hat{D}, \hat{P}_\mu + \hat{K}_\mu] = \frac{i}{2}(\hat{P}_\mu - \hat{K}_\mu) = -i\hat{J}_{-1,\mu};$

Exercise: Compute other commutators.

(13) • Summarizing these commutators:

$$\underline{[\hat{J}_{ab}, \hat{J}_{cd}] = i(g_{ad}\hat{J}_{bc} + g_{bc}\hat{J}_{ad} - g_{ac}\cdot\hat{J}_{bd} - g_{bd}\cdot\hat{J}_{ac})}$$

$$g_{ad} = \mathrm{diag}(-1, 1, \mp 1, 1, \ldots, 1), \quad a,d = -1, 0, 1, \ldots, d.$$

metric on $\mathbb{R}^{2,d}$ or $\mathbb{R}^{1,d+1}$

$$\begin{cases} \text{"}-\text{"} - \text{Mink} \\ \text{"}+\text{"} - \text{Euclid.} \end{cases}$$

Lorentz transform. for the metric g_{ab}

$$\boxed{\begin{cases} SO(2,d) & \text{Minkowski} \\ SO(1,d+1) & \text{Euclidean.} \end{cases}}$$

Comments:

(1) # of parameters of $SO(2,d)$ or $SO(1,d+1)$ is $\frac{1}{2}(d+2)(d+1)$, which coinsides with our previous counting of the conf. transform.

(2) If we add inversions I above will become $O(2,d)$ or $O(1,d+1)$

(3) Poincare group + dilations → subgroup of full conf. group. Hence theory can be scale invariant but not inv. under SCT.

(4) In principle Poincare + Inversions generate full conf. group.
• Inv. + transl = SCT
• SCT + transl = dilations (from commutator)

Ⅱ Action of conf. transform. on operators.

(approx 2h 15m)

There are two ways to define action of conf. transf on operators (fields) of CFT

① All actions of transform. can be split in two kinds:

ⓐ Geometrical / "orbital" transform: just coordinate transform. $x \to \tilde{x} = g \cdot x$ inside the field. (universal)

ⓑ Internal / "spin" transform: Transform. of operator under certain representation of the Lorentz group (depends on the choice of oper. Φ^A)

Together transform looks like

$$\Phi^A(x) \xrightarrow{g} \tilde{\Phi}^A(\tilde{x}) = L^A{}_B(g)\, \Phi^B(x)$$

$\Downarrow$ relabeling coordinates

$$\tilde{\Phi}^A(x) = \underbrace{L^A{}_B(g)}_{\text{"spin"}}\, \Phi^B(\underbrace{g^{-1}x}_{\text{"orbital"}})$$

"spin" given by matrix rep. of Lorentz group

• As $L^A{}_B$ forms matrix rep. it should satisfy consistency condition

② $$L^A{}_B(g_1)\, L^B{}_C(g_2) = L^A{}_C(g_1 \cdot g_2)$$

- Usually $L^A{}_B(g)$ is written in the following form: $L^A{}_B = \left(e^{-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}}\right)^A{}_B$

$S^{\mu\nu}$ → matrix rep. for Lorentz

- For infinitesemal transformation:

$$L^A{}_B = \delta^A{}_B - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^A{}_B;$$

examples:

$$(S^{\mu\nu})^A{}_B = \begin{cases} 0 : \text{scalar field;} \\ \frac{i}{2}[\Gamma^\mu, \Gamma^\nu] : \text{spinor;} \\ i(\delta^{\mu A}\delta^\nu{}_B - \delta^\mu{}_B\delta^{\nu A}) : \text{vector;} \end{cases}$$

- Then let's find transform. of generic field under Lorentz transform.

$$\tilde{\Phi}^A(x) = L^A{}_B\, \Phi^B(g^{-1}x) = \left(\delta^A{}_B - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^A{}_B\right)\left(\Phi^B(x^\mu - \omega^{\mu\nu}x_\nu)\right)$$

$$= \left(\delta^A{}_B - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^A{}_B\right)\left(\Phi^B(x) - \omega^{\mu\nu}x_\nu\partial_\mu\Phi^B(x)\right) =$$

$$= \Phi^A(x) - \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^A{}_B\,\Phi^B(x) - \frac{1}{2}\omega^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\Phi^A(x) +$$

$$+ O(\omega^2) \Rightarrow \delta\Phi^A(x) \equiv \tilde{\Phi}^A(x) - \Phi^A(x)$$

$$\Downarrow$$

$$\boxed{\delta\Phi^A(x) = -\frac{i}{2}\omega^{\mu\nu}\left(i(x_\mu\partial_\nu - x_\nu\partial_\mu)\delta^A_B + (S^{\mu\nu})^A{}_B\right)\Phi^B(x);}$$

② For the second approach we remined ourself that operators in QFT (as well as in QM) transform using unitary operators:

③

$$\tilde{\Phi}^A(\tilde{x}) = U^{-1}(g)\, \Phi^A(\tilde{x})\, U(g) = L^A{}_B(g)\, \Phi^B(x)$$

• $U(g)$ should also satisfy composition rule

$$U(g_1 g_2) = U(g_1)\, U(g_2);$$

• In particular for example in case of Lorentz we have $U(g) = e^{-\frac{i}{2}\omega^{\mu\nu}\hat{L}_{\mu\nu}}$, then

$$\hat{\tilde{\Phi}}(\tilde{x}) = e^{\frac{i}{2}\omega_{\mu\nu}\hat{L}^{\mu\nu}}\, \hat{\Phi}(x)\, e^{-\frac{i}{2}\omega_{\mu\nu}\hat{L}^{\mu\nu}} \underset{\substack{\uparrow \\ \text{infinites.} \\ \text{transform.}}}{\approx} \frac{i}{2}\omega^{\mu\nu}[\hat{L}_{\mu\nu}, \hat{\Phi}(x)] + \hat{\Phi}(x)$$

Hence

$$\boxed{\delta\hat{\Phi}(x) \equiv \hat{\tilde{\Phi}}(\tilde{x}) - \hat{\Phi}(x) = \frac{i}{2}\omega^{\mu\nu}[\hat{L}_{\mu\nu}, \hat{\Phi}(x)]}$$

So $\delta\Phi \sim \omega^a [G_a, \Phi(x)]$ for any other transform as well. Now we postulate action of the $\hat{L}_{\mu\nu}$ at $x=0$

$$\boxed{[\hat{L}_{\mu\nu}, \hat{\Phi}^A(0)] = -(S_{\mu\nu})^A{}_B\, \Phi^B(0)}$$

→ this is the same matrix repr. we have used before.

• Knowing action at zero we can deduce action on $\Phi^A(x) = e^{-iP\cdot x} \cdot \Phi^A(0)\, e^{iPx}$ where we have used spacetime translation on operator. $\Phi^A(x)$;

• Then the commutator we aim at is given by:

$$[\hat{L}_{\mu\nu}, \Phi^A(x)] = e^{-iPx}[e^{iPx}\hat{L}_{\mu\nu}e^{-iPx}, \Phi^A(0)]e^{iPx}$$

- Now we use Baker - Campbell - Hausdorff (BCH) formula:

$$e^{-A} B e^{A} = B + [B,A] + \frac{1}{2!}[[B,A],A] + \frac{1}{3!}[[[B,A],A],A]$$

Extras:

① One way to see it is to expand exponents in Taylor and it will result in commutators.

② Rigid (and my fav.) proof:

- consider function $f(t) = e^{-At} B e^{At}$
- this function satisfies dif. equation

$$\frac{df}{dt} = -e^{-At}[A,B]e^{At} = [f,A]$$

- expand f in series $f(t) = \sum_{n=0}^{\infty} f_n t^n \cdot \frac{1}{n!} \Rightarrow$

$$\Rightarrow \frac{df}{dt} = \sum_{n=1}^{\infty} f_n t^{n-1} \frac{1}{(n-1)!} = \sum_{n=0}^{\infty} f_{n+1} \frac{1}{n!} t^n = \sum_{n=0}^{\infty} \frac{t^n}{n!}[f_n, A] \Rightarrow$$

$$\Rightarrow \quad f_{n+1} = [f_n, A]$$

- $f_0 = B$ obviously. Then: $f_1 = [f_0, A] = [B,A]$;

$f_2 = [f_1, A] = [[B,A],A]$, ...

- finally function we are looking for is

$$f(1) = e^{-A} B e^{A} = \sum_{n=0}^{\infty} f_n \frac{1}{n!} = B + [B,A] + \frac{1}{2}[[B,A],A] + \ldots$$

QED.

- Now returning to our case

$$e^{iPx} \hat{L}_{\mu\nu} e^{-iPx} = \hat{L}_{\mu\nu} + (-ix^{\rho})[\hat{L}_{\mu\nu}, \hat{P}_{\rho}] + \frac{1}{2}(-ix^{\rho})(-ix^{\sigma})[[\hat{L}_{\mu\nu}, \hat{P}_{\rho}], \hat{P}_{\sigma}] + \ldots$$

⑤ Using previously derived algebra (see Lecture 2)

$$[\hat{P}_\rho, \hat{L}_{\mu\nu}] = i(\eta_{\rho\mu}\hat{P}_\nu - \eta_{\rho\nu}\hat{P}_\mu)$$

$$\Downarrow$$

$$e^{iPx} L_{\mu\nu} e^{-iPx} = L_{\mu\nu} + ix^\rho \cdot i(\eta_{\rho\mu}\hat{P}_\nu - \eta_{\rho\nu}\hat{P}_\mu) + \overset{\uparrow}{0}$$

all further correlators are 0

• Hence we obtain:

$$[\hat{L}_{\mu\nu}, \Phi^A(x)] = e^{-iPx}\Big(\underbrace{[\hat{L}_{\mu\nu}, \Phi^A(0)]}_{-(S_{\mu\nu})^A{}_B \Phi^B(0)} + x_\nu \underbrace{[P_\mu, \Phi(0)]}_{?} - x_\mu \underbrace{[P_\nu, \Phi(0)}_{?} \ldots$$

What is $[P_\mu, \Phi(0)]$?

Notice $\partial_\mu \Phi(x) = \partial_\mu(e^{-iPx} \cdot \Phi(0) e^{iPx}) = -i \cdot e^{-iPx}[P_\mu, \Phi(0)] \ldots$

which is precisely what we need. So we conclude:

$$\boxed{\begin{aligned} &[\hat{P}_\mu, \hat{\Phi}^A(x)] = i\partial_\mu \Phi^A(x); \\ &[\hat{L}_{\mu\nu}, \hat{\Phi}^A(x)] = \left(-(S_{\mu\nu})^A{}_B - i(x_\mu\partial_\nu - x_\nu\partial_\mu)\delta^A_B\right)\Phi^B(x); \end{aligned}}$$

• Let's compare with our previous results:

$$\delta\Phi^A(x) = -\frac{i}{2}\omega^{\mu\nu}\underbrace{\left((S^{\mu\nu})^A{}_B + i\delta^A{}_B(x_\mu\partial_\nu - x_\nu\partial_\mu)\right)\Phi^B(x)}_{-[\hat{L}_{\mu\nu}, \Phi^A(x)]} =$$

$$= \frac{i}{2}\omega^{\mu\nu}[L_{\mu\nu}, \Phi^A(x)] \text{ ✓}$$

for translations:

$$\delta\Phi^A(x) = \tilde{\Phi}(x) - \Phi(x) = \Phi(x-a) - \Phi(x) = -a^\mu\partial_\mu\Phi(x) =$$

$$= ia^\mu[\hat{P}_\mu, \hat{\Phi}^A(x)] \text{ ✓}$$

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- This approach is more preferable to us. Let's go for it.
- First of all let's postulate action of transformations leaving origin invariant in this origin:

$$\boxed{\begin{aligned}&[\hat{L}_{\mu\nu}, \Phi^A(0)] = -(S_{\mu\nu})^A{}_B\, \Phi^B(0);\\ &[\hat{\mathcal{D}}, \Phi^A(0)] = \hat{\Delta}\, \Phi^A(0)\\ &[K_\mu, \Phi^A(0)] = \underbrace{k_\mu \Phi^A(0)}_{\text{operator}}\end{aligned}}$$

$\hat{\Delta}$ → operator that in general transforms between various operators with the same Lorentz structure. One can introduce extra index $\hat{\Delta}^a{}_b \Phi^{AB}$, but we will not do it.

k_μ also acts by mixing various operators which we will see in the future.

- Now one can repeat calculation made for Lorentz to find how operators are transformed at arbitrary point.

- Dilations: $e^{iPx}\hat{\mathcal{D}}e^{-iPx} = \hat{\mathcal{D}} + (-ix^\rho)[\hat{\mathcal{D}}, P_\rho] + \frac{1}{2}(-ix^\rho)(-ix^\sigma) \times$

$\times \underbrace{[[\hat{\mathcal{D}}, \hat{P}_\rho]}_{0}, \hat{P}_\sigma] + \ldots = \hat{\mathcal{D}} + (-ix^\rho)\, i\hat{P}_\rho = \hat{\mathcal{D}} + x^\rho \hat{P}_\rho$

Hence

$$[\hat{\mathcal{D}}, \hat{\Phi}^A(x)] = e^{-iPx}\left([\hat{\mathcal{D}}, \hat{\Phi}^A(0)] + x^\rho\, [\hat{P}_\rho, \hat{\Phi}^A(0)]\right)e^{iPx} =$$

$$= \hat{\Delta}\, \hat{\Phi}^A(x) + i x^\mu \partial_\mu \hat{\Phi}^A(x)$$

- SCT: $e^{iPx}\hat{K}_\mu e^{-iPx} = \hat{K}_\mu + (-ix^\rho)[\hat{K}_\mu, \hat{P}_\rho] + \frac{1}{2}(-ix^\rho)(-ix^\sigma)[[\hat{K}^\mu, P_\rho], P_\sigma] +$

$= \hat{K}_\mu + (-ix^\rho)\cdot 2i(\eta_{\mu\rho}\hat{\mathcal{D}} - \hat{L}_{\mu\rho}) - \frac{1}{2} x^\rho x^\sigma \cdot 2i(\eta_{\mu\rho}[\hat{\mathcal{D}}, \hat{P}_\sigma] - [\hat{L}_{\mu\rho}, \hat{P}_\sigma])$

(7)

$$= \hat{K}_\mu + 2x_\mu \hat{D} - 2x^\rho \hat{L}_{\mu\rho} - i x^\rho x^\sigma \cdot i P_\sigma \eta_{\mu\rho} + i x^\rho x^\sigma i (\eta_{\sigma\rho} \hat{P}_\mu - \eta_{\sigma\mu} \hat{P}$$

So we obtain

$$e^{iPx} \hat{K}_\mu e^{-iPx} = \hat{K}_\mu + 2x_\mu \hat{D} + 2x^\nu \hat{L}_{\nu\mu} + x_\mu x^\nu \hat{P}_\nu + x_\mu x^\nu \hat{P}_\nu - x^2 \hat{P}_\mu$$

then:

$$[\hat{K}_\mu, \hat{\Phi}^A(x)] = (\hat{k}_\mu + 2x_\mu \hat{\Delta} - 2x^\nu \hat{S}_{\nu\mu} + 2i x_\mu x^\nu \partial_\nu - i x^2 \partial_\mu) \hat{\Phi}^A(x)$$

$$[\hat{D}, \hat{\Phi}^A(x)] = (\hat{\Delta} + i x^\mu \partial_\mu) \hat{\Phi}^A(x);$$

$$[\hat{P}_\mu, \hat{\Phi}^A(x)] = i\partial_\mu \hat{\Phi}^A(x); \quad [\hat{L}_{\mu\nu}, \hat{\Phi}^A(x)] = (-S_{\mu\nu} - i(x_\mu \partial_\nu - x_\nu \partial_\mu)) \hat{\Phi}^A(x)$$

~~• Now we define important class of operators, namely conformal primaries~~

• Comment: Notice peculiar property of SCT. Acting on the generic field it also generates dilation ($2x_\mu \hat{\Delta}$) and Lorenz transformation ($-2x^\nu \hat{S}_{\nu\mu}$)

• Now let's consider field at $x=0$

$$[\hat{D}, \hat{\Phi}^A(0)] = \hat{\Delta} \hat{\Phi}^A(0);$$

$$[\hat{P}_\mu, \hat{\Phi}^A(0)] = i\partial_\mu \hat{\Phi}^A(0);$$

$$[\hat{K}_\mu, \hat{\Phi}^A(0)] = k_\mu \cdot \hat{\Phi}^A(0);$$

just numb… scaling dim…

• Assume we choose operators diagonalizing $\hat{\Delta}$: $\hat{\Delta} \hat{\Phi}^A(0) = +i\Delta \times \hat{\Phi}^A(0)$

then $[\hat{D}, \hat{\Phi}^A(0)] = +i\Delta \hat{\Phi}^A(0)$

Now consider the field $\tilde{\Phi}^A(0) = [\hat{K}_\mu, \hat{\Phi}^A(0)] = k_\mu \Phi^A(0)$

⑧ Q: What scaling dim it has?

Let's find

$$[\hat{D},[\hat{K}_\mu,\Phi^A(0)]] \overset{\text{using Jacobi identity}}{=} -[\Phi^A(0),[\hat{D},\hat{K}_\mu]] - [\hat{K}_\mu,[\Phi^A(0),\hat{D}]] =$$

$$= i[\Phi^A(0),\hat{K}_\mu] + i\Delta[\hat{K}_\mu,\Phi^A(0)] = +i(\Delta-1)\hat{K}_\mu\Phi^A(0)$$

Hence we see that eigen value of Δ is decreased by one

- Now let's answer the same question but for the field $i\partial_\mu\hat{\Phi}(x)|_{x=0} = [\hat{P}_\mu,\Phi(0)]$:

$$[\hat{D},[\hat{P}_\mu,\Phi(0)]] = -[\hat{\Phi}^A(0),[\hat{D},\hat{P}_\mu]] - [\hat{P}_\mu,[\Phi^A(0),\hat{D}]] =$$

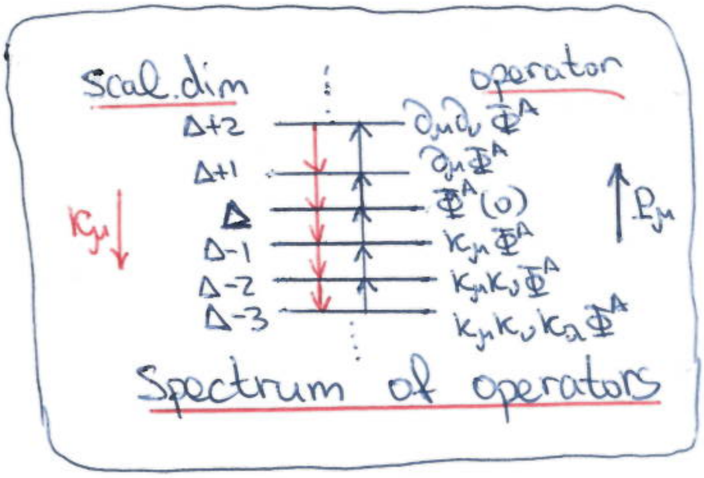
$$= -i[\Phi^A(0),P_\mu] + i\Delta[\hat{P}_\mu,\Phi(0)] = i(\Delta+1)[\hat{P}_\mu,\Phi^A(0)]$$

- Conclusion:

$[\hat{D},\hat{\Phi}^A(0)] = i\Delta\hat{\Phi}^A(0)$;

$[\hat{D},K_\mu\hat{\Phi}^A(0)] = i(\Delta-1)\hat{\Phi}^A(0)$; $\Rightarrow$

$[\hat{D},i\partial_\mu\hat{\Phi}^A(0)] = i(\Delta+1)\hat{\Phi}^A(0)$;



Analogy I: Harmonic oscillator.

Harmonic oscillator	CFT
Hamiltonian $\hat{H}=\hat{a}^+\hat{a}+\hat{a}\hat{a}^+$	Dilation operator $\hat{\Delta}$ (or $\hat{D}$)
Raising operator $\hat{a}^+$	Translations $\hat{P}_\mu$
Lowering operator $\hat{a}$	SCT: $\hat{K}_\mu$

⑨

Analogy II

- Consider $\mathbb{R}^d$ metric on S^{d-1}

$$ds^2 = dr^2 + r^2 \overbrace{d\Omega_{d-1}} = r^2\left(\frac{dr^2}{r^2} + d\Omega_{d-1}\right)$$

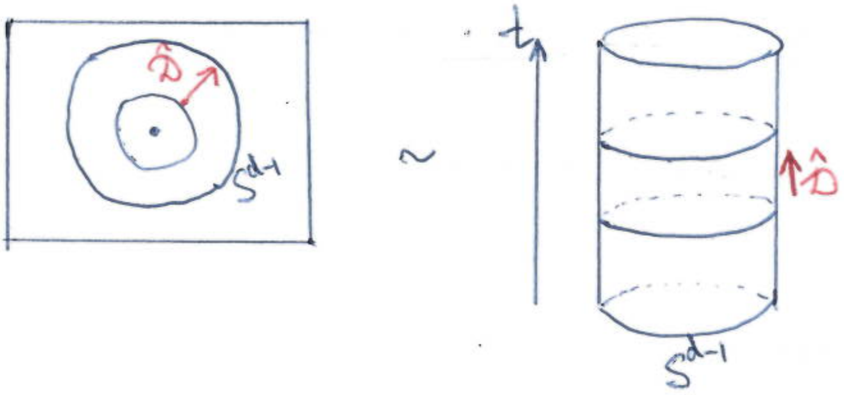
- Say that there is time coordinate $t = \log r \Rightarrow$

$$\Rightarrow \frac{dr^2}{r^2} + d\Omega_{d-1} = \underbrace{dt^2 + d\Omega_{d-1}}_{\text{metric on } \mathbb{R}_t \times S^{d-1}}$$

- CFT is inv. under rescaling metric

$$\boxed{\text{CFT on } \mathbb{R}^d \sim \text{CFT on } \mathbb{R}_t \times S^{d-1}}$$

- Dilation operator maps $S^{d-1} \to S^{d-1}$ on $\mathbb{R}^d \Rightarrow$
$\hat{D}$ = time translations on $\mathbb{R}_t \times S^{d-1} \Rightarrow \hat{D}$ is some Hamiltonian



- From this analogy we impose condition of lower boundary existence for Δ. In QM this comes from requiring theory to have energy bounded from below (otherwise theory does not have ground state and not stable). In CFT Δ has lowest bound known as unitarity bound

- Now we introduce conformal primary operator:

Operator annihilated by K_μ at $x=0$ (and that also diagonalizes Δ). Hence for the conformal primary:

$$[\hat{D}, \Phi^A(0)] = i\Delta \Phi^A(0);$$

$$[\hat{P}_\mu, \Phi^{(A)}_A(0)] = i\partial_\mu \Phi^A(0);$$

$$[\hat{L}_{\mu\nu}, \Phi^A(0)] = -(S_{\mu\nu})^A{}_B \Phi^B(0);$$

$$[K_\mu, \Phi^A(0)] = 0$$

↳ this is most important one.

- Starting from primary we can generate spectrum by acting with $\hat{P}_\mu$ (raising operators)

$$(\Phi^A; \Delta) \xrightarrow{P_\mu} (\partial_\mu \Phi^A, \Delta+1) \xrightarrow{P_\nu} (\partial_\mu \partial_\nu \Phi^A, \Delta+2) \longrightarrow \dots$$

This tower is infinite and oprerators are called descendants.

- Transformation rule

Let's now write down finite transformation of the primary operator. In particular let's find $\tilde{\Phi}^A(\tilde{x})$ in terms of $\Phi^A(x)$

① First notice that in case of Lorentz

$$\tilde{\Phi}^A(\tilde{x}) = \Phi^A(\tilde{x}) + \frac{i}{2}\omega^{\mu\nu}[\hat{L}_{\mu\nu}, \hat{\Phi}(\tilde{x})] = \Phi^A(x^\mu + \omega^{\mu\nu}x_\nu) +$$

$$+ \frac{i}{2}\omega^{\mu\nu}[L_{\mu\nu}, \Phi(x)] + O(\omega^2) = \Phi^A(x) + \frac{1}{2}\omega^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\Phi^A(x)$$

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$$+\frac{i}{2}\omega^{\mu\nu}[L_{\mu\nu}, \Phi(x)] = \Phi^A(x) + \frac{1}{2}\omega^{\mu\nu}(\cancel{x_\nu\partial_\mu} - x_\mu\partial_\nu)\Phi^A(x) +$$

$$+\frac{1}{2}\omega^{\mu\nu}(\cancel{x_\mu\partial_\nu} - x_\nu\partial_\mu)\Phi^A(x) - \frac{i}{2}\omega^{\mu\nu}(S_{\mu\nu})^A{}_B\Phi^B(x) =$$

$$= \left(\delta^A{}_B - \frac{i}{2}\omega^{\mu\nu}(S_{\mu\nu})^A{}_B\right)\Phi^B(x)$$

- Hence we drop generator of coordinate transformation

The same happens to other transformations:

- SCT

$$[K_\mu, \Phi^A(x)] \sim \left(2x_\mu i\Delta\delta^A_B - 2x^\nu(\hat{S}_{\nu\mu})^A{}_B\right)\Phi^B(x)$$

$$\tilde{\Phi}^A(\tilde{x}) = \left(\delta^A{}_B + i\,b^\mu\left(2x_\mu i\Delta\delta^A{}_B - 2x^\nu(S_{\nu\mu})^A{}_B\right)\right)\Phi^B(x) =$$

$$= \left(\delta^A{}_B + 2i\,b^\mu x^\nu(S_{\mu\nu})^A{}_B\right)\left(1 - 2(b\cdot x)\cdot\Delta\right)\Phi^B$$

- dilations

$$[D, \Phi^A(x)] \sim i\Delta\Phi^A(x)$$

$$\tilde{\Phi}^A(\tilde{x}) = \Phi^A(x) + i\alpha\cdot i\Delta\Phi^A(x)$$

- Let's exponentiate these transformations in the following way:

- $\left(\delta^A{}_B + i(b^\mu x^\nu - x^\mu b^\nu)(S_{\mu\nu})^A{}_B\right) \sim e^{i(b^\mu x^\nu - x^\nu b^\mu)S_{\mu\nu}} = (R(x))^A{}_B$

  this is x-dependent rotation matrix $\left(e^{\frac{i}{2}\Omega^{\mu\nu}S_{\mu\nu}};\ \Omega_{\mu\nu} = \omega_{\mu\nu} + 2(x_\nu b_\mu - x_\mu b_\nu)\right)$

- $(1 - 2(bx)\Delta) \approx (1 - 2(bx))^{+\Delta} \simeq \Lambda(x)^{+\Delta/2}$, where $\Lambda(x)$ is the scale factor of SCT $\Lambda(x) = (1 - 2(bx) + b^2x^2)^2$

- $(1 - \alpha\cdot\Delta) \approx (1-2\alpha)^{+\Delta/2} \simeq \lambda^{-\Delta} \approx \Lambda(x)^{+\Delta/2}$ where $\lambda \approx 1+\alpha$ is scaling factor corresponding to $\Lambda(x) = \lambda^{-2}$

Hence exponentiation in both cases gives:

$$\boxed{\tilde{\Phi}^A(\tilde{x}) = \Lambda(x)^{+\Delta/2} \cdot (R(x))^A{}_B \, \Phi^B(x);}$$

• Now notice that by definition of conf. transf.

$$\Lambda(x) \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} \cdot \eta_{\mu\nu} = \eta_{\alpha\beta}$$

Let's take the determinant of both parts

$$\Lambda(x)^d \cdot \det\left(\frac{\partial \tilde{x}}{\partial x} \cdot \frac{\partial \tilde{x}}{\partial x} \eta\right) = \det \eta \Rightarrow \underline{\det\left(\frac{\partial \tilde{x}}{\partial x}\right) = \Lambda(x)^{-d/2}}$$

Hence we can also write it in the form:

$$\boxed{\tilde{\Phi}^A(\tilde{x}) = \left\| \frac{\partial \tilde{x}}{\partial x} \right\|^{-\Delta/d} \cdot (R(x))^A{}_B \, \Phi^B(x)}$$

• In the simplest case of the scalar field $R(x) = \mathbb{1}$

and $$\boxed{\tilde{\Phi}^A(\tilde{x}) = \left\| \frac{\partial \tilde{x}}{\partial x} \right\|^{-\Delta/d} \cdot \Phi^A(x);}$$

<u>Comment</u>: Notice that if we consider field at arbitrary point: $\Phi^A(x) = \Phi^A(0) + x^\mu \partial_\mu \Phi^A(0) + \frac{1}{2} x^\mu x^\nu \partial_\mu \partial_\nu \Phi(0)$

then it contains descendants

But in our notations we still call it primary if $[K_\mu, \Phi(0)] = 0$.

(13) • Crucial property of primary field is it's nice transformation under conformal action. To understand this consider transform. of descendant

$\partial_\mu \Phi(x) = -i [P_\mu, \Phi(x)]$ under SCT:

$$\delta(\partial_\mu \Phi(x)) = i b^\nu [K_\nu, [P_\mu, \Phi(x)]] = -i b^\nu [\Phi, [K_\nu, P_\mu]]$$

$$-i b^\nu [P_\mu, [\Phi, K_\nu]] = -i b^\nu [\Phi, 2i(\eta_{\mu\nu} \hat{D} - \hat{L}_{\nu\mu})] -$$

$$+ i b^\nu \cdot [\hat{P}_\mu, 2x_\nu i\Delta \hat{\Phi}^A(x) - 2x^\alpha \hat{S}_{\alpha\nu} \Phi + 2i x_\nu x^\alpha \partial_\alpha \Phi - i x^2 \partial_\nu \Phi]$$

the problem comes from the first term in particular which results in:

$$\delta(\partial_\mu \Phi(x)) \sim 2 b^\nu \cdot (-\eta_{\mu\nu} [\hat{D}, \Phi] + [\hat{L}_{\nu\mu}, \Phi(x)]) =$$

$$= -2i b_\mu \Delta \Phi - 2 b^\nu (S_{\nu\mu})^A{}_B \Phi^B + \text{orbital part.}$$

Hence transforming $\partial_\mu \Phi(x)$ we obtain piece proportional to $\Phi(x)$. Hence formula like $\partial_\mu \tilde{\Phi}(\tilde{x}) = F[\partial_\mu \Phi(x)]$ is impossible !!!

Noether theorem and Energy-Momentum Tensor (EMT) (approx 2hrs)

Noether theorem

> To every continious symmetry of the action one may associate classically conserved current.

• Consider infinitesemal transform:

$$\tilde{x}^{\mu} = x^{\mu} + \omega_a \frac{\delta x^{\mu}}{\delta \omega_a}$$

acting on the field as follows $\underbrace{\tilde{\Phi}(\tilde{x}) = \mathcal{F}(\Phi(x))}$

$$\tilde{\Phi}(\tilde{x}) = \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}$$

we have derived form of this transf. for primaries before

• Let's write out transformation of the action under this coordinate transformation.

$$S = \int d^dx \cdot \mathcal{L}_\Phi(\Phi, \partial_\mu \Phi) \xrightarrow[x \to \tilde{x}]{} \int d^dx \cdot \mathcal{L}_\Phi(\tilde{\Phi}(x), \partial_\mu \tilde{\Phi}(x)) =$$

$$= \int d^d\tilde{x} \cdot \mathcal{L}_\Phi(\tilde{\Phi}(\tilde{x}), \tilde{\partial}_\mu \tilde{\Phi}(\tilde{x})) = \int d^dx \cdot \left\| \frac{\partial \tilde{x}}{\partial x} \right\| \cdot \mathcal{L}_\Phi\left(\tilde{\Phi}(\tilde{x}), \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \partial_\nu \tilde{\Phi}(\tilde{x})\right)$$

renaming variable

$$= \int d^dx \left\| \frac{\partial \tilde{x}}{\partial x} \right\| \cdot \mathcal{L}_\Phi\left(\Phi + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}, \left[\delta^\nu_\mu - \partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta \omega_a}\right)\right]\left(\partial_\nu \Phi + \partial_\nu\left[\omega_a\left(\frac{\delta \mathcal{F}}{\delta \omega_a}\right)\right]\right)\right)$$

assume ω_a depends on x.

• Let's consider elements of the expression above:

②

- determinant: $\left\|\frac{\partial \tilde{x}}{\partial x}\right\| = \det\left(\delta^\nu{}_\mu + \partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta \omega_a}\right)\right) =$

$= 1 + tr\left(\partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta\omega_a}\right)\right) = 1 + \partial_\mu\left(\omega_a \frac{\delta x^\mu}{\delta\omega_a}\right)$

- Lagrangian: $\mathcal{L}_\Phi\left(\Phi + \omega_a \frac{\delta \mathcal{F}}{\delta\omega_a}, \partial_\mu\Phi - \partial_\mu\left(\omega_a \frac{\delta x^\nu}{\delta\omega_a}\right)\partial_\nu\Phi + \right.$

$\left. + \partial_\mu\left(\omega_a \frac{\delta\mathcal{F}}{\delta\omega_a}\right)\right) =$

$= (\text{terms proportional to } \omega_a) + \frac{\partial \mathcal{L}_\Phi}{\partial(\partial_\mu\Phi)}\left[\frac{\delta x^\nu}{\delta\omega_a}\partial_\nu\Phi + \right.$

$\left. + \frac{\delta\mathcal{F}}{\delta\omega_a}\right]\partial_\mu\omega_a$

- We don't care about terms proport. to ω_a because if action is inv. under transformation with $\omega_a = \text{const}$ (rigid transform) then this terms vanish anyway.
- Summing up two contributions we obtain

$$\boxed{\delta S = -\int d^dx \cdot j^\mu_a \partial_\mu\omega_a}, \text{ where:}$$

$$\boxed{j^\mu_a = \left(\frac{\partial\mathcal{L}_\Phi}{\partial(\partial_\mu\Phi)} \cdot \partial_\nu\Phi - \delta^\mu{}_\nu\mathcal{L}_\Phi\right)\frac{\delta x^\nu}{\delta\omega_a} - \frac{\partial\mathcal{L}_\Phi}{\partial(\partial_\mu\Phi)}\frac{\delta\mathcal{F}}{\delta\omega_a};}$$

↓

Noether current.

- Integrating by parts $\delta S = \int d^dx \cdot (\partial_\mu j^\mu_a) \cdot \omega_a$
- On-shell action is inv. w.r.t. any variation

Hence δS should vanish for any x-dependent $\omega_a(x)$

③ $\Rightarrow$ $\partial_\mu j^\mu_a = 0$, i.e. conservation implied on-shell by existence of symmetry.

- Associated conserved charge:

$$Q_a = \int d^{d-1}x \cdot j^0_a \Rightarrow \dot{Q}_a = \int d^{d-1}x \, \partial_0 j^0_a \overset{\substack{\text{using conserv} \\ \partial_\mu j^\mu_a = 0}}{=} -\int d^{d-1}x \, \partial_i j^i_a =$$

$\downarrow$

this Q_a is conserved: $\dot{Q}_a = 0$

$$= -\underbrace{\int d\sigma^i}_{\infty} j^i = 0 \rightarrow$$ provided j^i_a vanishes at infinity.

surface integral over spatial infinity

- Noether current is not uniquely defined:

shift $j^\mu_a \rightarrow \tilde{j}^\mu_a = j^\mu_a + \partial_\nu B^{\nu\mu}_a$, where $B^{\nu\mu}_a = -B^{\mu\nu}_a$

then $\partial_\mu \tilde{j}^\mu_a = \partial_\mu j^\mu_a + \partial_\mu \partial_\nu B^{\nu\mu}_a$

0 due to conservation; 0 due to antisym. of $B^{\mu\nu}_a$.

Energy-momentum tensor.

Consider translations

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu + a^\mu$$

$$\Phi(x) \rightarrow \tilde{\Phi}(x+a) = \Phi(x), \text{ i.e.}$$

$$\frac{\delta x^\mu}{\delta a^\nu} = \delta^\mu{}_\nu \, ; \quad \frac{\delta \mathcal{F}}{\delta a^\nu} = 0$$

- Then associated noether current is

$$T_c{}^\mu{}_\nu = \left(\frac{\partial \mathcal{L}_\Phi}{\partial(\partial_\mu \Phi)} \partial_\alpha \Phi - \delta^\mu{}_\alpha \mathcal{L}_\Phi \right) \cdot \delta^\alpha{}_\nu \Rightarrow \boxed{T_c^{\mu\nu} = \frac{\partial \mathcal{L}_\Phi}{\partial(\partial_\mu \Phi)} \partial^\nu \Phi - \eta^{\mu\nu} \mathcal{L}}$$

- It is conserved $\partial_\mu T_c^{\mu\nu} = 0$

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$$P^\nu = \int d^{d-1}x \cdot T_c^{0\nu}$$

for example $P^0 = \int d^{d-1}x \cdot T_c^{00} = \int d^{d-1}x \cdot \left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\dot\phi - \mathcal{L}\right)$ is energy

• On it's own $T_c^{\mu\nu}$ is not symmetric and not gauge invariant (if there is gauge symmetry)

• It can be <u>made symmetric</u>:

$$T_c^{\mu\nu} \to T^{\mu\nu} = T_c^{\mu\nu} + \partial_\rho B^{\rho\mu\nu} \; ; \; B^{\rho\mu\nu} = -B^{\mu\rho\nu}$$

$\uparrow$

<u>Belinfante tensor</u> $T^{\mu\nu} = T^{\nu\mu}$

• In this way:

- <u>Conservation is not spoiled</u>:

$$\partial_\mu T^{\mu\nu} = \underbrace{\partial_\mu T_c^{\mu\nu}}_{0} + \partial_\mu\partial_\rho B^{\rho\mu\nu}$$

due to antisymmetry.

- Definition of <u>momentum is not affected</u>

$$P_c^\nu \to \overset{\text{new}}{P}{}^\nu = \int d^{d-1}x \cdot \left(T_c^{0\nu} + \partial_\rho B^{\rho 0\nu}\right) = P_c^\nu + \int d^{d-1}x \cdot \partial_i B^{i0\nu} =$$

$$= P_c^\nu + \int_\infty d\sigma^i B^{i0\nu}$$

$\to 0$ if $B^{i0\nu}$ decays at infinity fast enough.

5) • Now we can consider <u>x-dependent translations</u>

$\underline{x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x)}$ → this include all conformal transformations.

• Then according to

$$\delta S = -\int dx \cdot j^\mu_a \, \partial_\mu \omega_a$$

• Hence $T^{\mu\nu} \, \partial_\mu \xi_\nu$ → in case of x-dependent translations.

$$\underline{\underline{\delta S = -\int dx \, T^{\mu\nu} \, \partial_\mu \xi_\nu}}$$

• If we directly substitute Belinfante EMT we can use symmetry:

$$\underline{\delta S = -\frac{1}{2}\int dx \, T^{\mu\nu} (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)} \quad (1)$$

• <u>Comment:</u> Notice that metric transf. as follows

$$\tilde{g}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta} =$$

$$= (\delta^\alpha{}_\mu - \partial_\mu \xi^\alpha)(\delta^\beta{}_\nu - \partial_\nu \xi^\beta) g_{\alpha\beta} =$$

$$= g_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) \Rightarrow \underline{\underline{\delta g_{\mu\nu} = -\partial_\mu \xi_\nu - \partial_\nu \xi_\mu}}$$

• Hence <u>EMT has alternative definition:</u>

$$\delta S = \frac{1}{2}\int dx \, T^{\mu\nu} \delta g_{\mu\nu} \sqrt{g} \Rightarrow \underline{\underline{T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}}};$$

this EMT is symmetric by construction.

⑥ Let's now come back to eq(1)

Using Conformal Killing Equation we rewrite:

$$\delta S = -\frac{1}{d}\int dx\, T^{\mu\nu}\cdot\eta_{\mu\nu}\cdot\partial^{\alpha}\xi_{\alpha} = -\frac{1}{d}\int dx\, T^{\mu}{}_{\mu}\,\partial_{\alpha}\xi^{\alpha}$$

$\Rightarrow$ $T^{\mu}{}_{\mu}=0$ implies invariance under conformal transformation, but not vice versa.

Vice versa is not generally true due to yet unknown form of ξ_ν. Let's try particular ξ_ν in cases of scale and SCT

- In this two cases infinitesemal transform is
  - $x^\mu \to (1+\lambda)x^\mu \Rightarrow \xi^\mu = \lambda x^\mu$ for dilatation
  - $x^\mu \to x^\mu + 2x^\mu(b\cdot x) - x^2 b^\mu \Rightarrow \xi^\mu = 2x^\mu(b\cdot x) - x^2 b^\mu$ for SCT

- For dilatation

$$\delta S = -\frac{1}{d}\int dx\cdot T^{\mu}{}_{\mu}\cdot\lambda \Rightarrow \boxed{T^{\mu}{}_{\mu} = \partial_\nu K^\nu \Leftrightarrow \delta S = 0 \text{ theory is scale inv.}}$$

(I)

- For SCT

$$\delta S = -\frac{1}{d}\int dx\cdot T^{\mu}{}_{\mu}\cdot\left(2(b\cdot x) - 2(b\cdot x) + 2(b\cdot x)\right) = -\frac{2}{d}\int dx\, T^{\mu}{}_{\mu}\, b_\alpha x^\alpha$$

This is zero if $T^{\mu}{}_{\mu} = \partial_\alpha\partial_\beta A^{\alpha\beta}$

Indeed $\delta S = -\frac{2}{d}\int dx\, \partial_\alpha\partial_\beta A^{\alpha\beta}\cdot b_\mu x^\mu = \frac{2}{d}\int dx\cdot\partial_\beta A^{\alpha\beta}\cdot b_\alpha,$

which full derivative giving zero under integral.

- Hence $\boxed{T^{\mu}{}_{\mu} = \partial_\alpha \partial_\beta A^{\alpha\beta} \quad \text{(II)} \quad \Leftrightarrow \quad \delta S = 0 \text{ theory is SCT, and hence, conformally invariant.}}$

- Comments:

① Notice that SCT (II) is stronger then dilatation condition (I), which is reasonabl since SCT generates dilatation as well.

② In most of scale inv. physical theories EMT satisfies (II) as well so theory is fully conformally invariant.

③ Notice that we can add term $\partial_\lambda \partial_\rho X^{\lambda\rho\mu\nu}$ to the Belinfante EMT:

$$T_t^{\mu\nu} = T^{\mu\nu} + \partial_\lambda \partial_\rho X^{\lambda\rho\mu\nu}$$

, where $X^{\lambda\rho\mu\nu}$ does not have part fully sym in first three indices!

In this case conservation is not spoiled as

$$\partial_\mu \partial_\lambda \partial_\rho X^{\lambda\rho\mu\nu} = 0$$

- In this case $T_t{}^{\mu}{}_{\mu} = T^{\mu}{}_{\mu} + \partial_\lambda \partial_\rho X^{\lambda\rho\mu}{}_{\mu}$

hence if $T^{\mu}{}_{\mu} = \partial_\alpha \partial_\beta A^{\alpha\beta}$ as in the case of SCT inv. theory we can always make $T^{\mu\nu}$ traceless choosing $X^{\lambda\rho\mu}{}_{\mu} = A^{\lambda\rho}$

Example φ^4 theory in 4d (conformally inv. theory)

- Lagrangian $\mathcal{L}_\varphi = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{\lambda}{4!}\varphi^4$
- Canonical EMT: $T_c^{\mu\nu} = \frac{\partial\mathcal{L}_\varphi}{\partial(\partial_\mu\varphi)}\partial^\nu\varphi - \eta^{\mu\nu}\mathcal{L}_\varphi$

$$T_c^{\mu\nu} = -\partial^\mu\varphi\,\partial^\nu\varphi + \eta^{\mu\nu}\left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{\lambda}{4!}\varphi^4\right)$$

- Trace of EMT

$$T^\mu_{c\,\mu} = -\partial_\mu\varphi\partial^\mu\varphi + d\left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{\lambda}{4!}\varphi^4\right) = [d=4] =$$

$$= \partial_\mu\varphi\partial^\mu\varphi + \frac{\lambda}{3!}\varphi^4$$

on-shell $\left(\partial^2\varphi = \frac{\lambda}{3!}\varphi^3\right)$: $T^\mu_{c\,\mu} = \partial_\mu\varphi\partial^\mu\varphi + \varphi\partial^2\varphi \Rightarrow$

$$\Rightarrow T^\mu_{c\,\mu} = \underset{\substack{\downarrow \\ \text{scale inv.}}}{\partial_\mu(\varphi\partial^\mu\varphi)} = \underset{\substack{\downarrow \\ \text{SCT-inv}}}{\frac{1}{2}\partial^2\varphi^2}$$

- Improve EMT

$$T^{\mu\nu} = T_c^{\mu\nu} + A(\eta^{\mu\nu}\partial^2\varphi^2 - \partial^\mu\partial^\nu\varphi^2)$$

then $T^\mu_{\ \mu} = T^\mu_{c\,\mu} + 3A\partial^2\varphi^2 \Rightarrow$ if $\boxed{A = -\frac{1}{6} \;\downarrow\; T^\mu_{\ \mu} = 0\,!}$

- Improvement term

$A(\eta^{\mu\nu}\partial^2\varphi^2 - \partial^\mu\partial^\nu\varphi^2)$ can be seen from $T^{\mu\nu}$ defined by variation of metric.

① To vary w.r.t. metric we need to put theory on non-flat background.

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In general:

$$S = \int dx \sqrt{g}\left(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{\lambda}{4!}\varphi^4 + AR\varphi^2\right)$$

coupling to the curvature is arbitrary and correspond to adding refinement terms!

If one calculates $T^{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}}$

one obtains precisely refinement term written previously.

Trace anomaly.

- All previous results are derived classicaly (no quantum effects
- We can write down Noether current associated to dilatation $x^\mu \to \tilde{x}^\mu = x^\mu + \alpha x^\mu$; $\mathcal{F}(\Phi(x)) = (1-\Delta\alpha)\Phi(x)$

$$j^\mu_D = \underbrace{\left(\frac{\partial \mathcal{L}_c}{\partial(\partial_\mu\Phi)}\partial_\nu\Phi - \delta^\mu{}_\nu\mathcal{L}_c\right)}_{T^\mu{}_{c\,\nu}}\underbrace{\frac{\delta x^\nu}{\delta\alpha}}_{x^\nu} - \underbrace{\frac{\partial\mathcal{L}_c}{\partial(\partial_\mu\Phi)}\frac{\delta\mathcal{F}}{\delta\omega_a}}_{\text{can be cancelled by refinement in physical situations}}$$

$$\Downarrow$$

$$j^\mu_D = T^\mu{}_\nu x^\nu \Rightarrow \partial_\mu j^\mu_D = T^\mu{}_\mu = 0$$

↳ dilatation current is conserved!

- Q: How is conservation relation modified by quantum corrections?

(10) • Obviously amount of conservation violation should be proportional to β-function capturing energy scale generated in theory by quantum corrections.

$$\beta(g) = \frac{dg}{d\log\mu} \qquad g \text{ - coupling constant}$$

under dilatation $g \to g + \beta(g)\cdot\alpha$ ← infinitesemal parameter of scaling

• Action also changes $\delta \mathcal{L}_e = \alpha \cdot \beta(g) \frac{\delta}{\delta g} \mathcal{L}_e$

Now for Noether current expresson becomes

$$\delta S = \int dx \left(\partial_\mu j^\mu_D \alpha - \alpha\, \beta(g) \frac{\delta}{\delta g} \mathcal{L}_e \right)$$

So if we require $\delta S = 0$ j^μ_D is not conserved anymore

$$\boxed{\langle \partial_\mu j^\mu_D \rangle = \langle \beta(g) \frac{\delta}{\delta g} \mathcal{L}_e \rangle;}$$

and the amount of non-conserv. is β-function

$\langle\ \rangle$ is ← QFT average but eq. is in principle operator eq. meaning we can include it into any correlation fcts

"Trace anomaly" because $\partial_\mu j^\mu_D = T^\mu{}_\mu$;

• Another way of breaking conformal invariance → couple CFT to curved background!

Then $\boxed{\langle T^\mu{}_\mu \rangle = \frac{c}{16\pi^2} W^2_{ijkl} - \frac{a}{16\pi^2} R^2_{ijkl}}$ in 4d

Weyl density ↑ Euler density ↑

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$$W^2_{ijkl} = R^2_{ijkl} - 2R^2_{ij} + \frac{1}{3}R^2;$$

$$\tilde{R}^2_{ijkl} = R^2_{ijkl} - 4R^2_{ij} + R^2;$$

- In $d=2$ $\boxed{\langle T^{\mu}{}_{\mu} \rangle = -\frac{c}{12} R}$ conformal anomaly

- c measures d.o.f. of theory.
- $\frac{dc}{d\log\mu} < 0$: c is monotonically decreasing function (Zamolodchikov c-theorem)

- Numbers a and c are important also for flat theory as it appears in higher pt functions in flat space. You can see it by taking derivative w.r.t $g_{\mu\nu}$ and in the end putting $g_{\mu\nu} = \eta_{\mu\nu}$;