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Class N1 Flows on the line

Theory outline

Dynamical systems we study in this course can be separated into several classes:

* Iterated maps - this kind of dynamical systems reveal chaotic behaviour, and we will consider them in the end of the course.
* Differential equations. here instead of discrete beheviour we consider continious time. At the same time differential equations can be
** Partial (PDE) As examples: heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ or wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$;
** Ordinary (ODE) This we will deal with, because we are interested in evolution of system with time only.

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \dots, x_n) \\ &\dots \\ \dot{x}_n &= f_n(x_1, x_2 \dots x_n) \end{aligned}$$

→ all ODE's can be reduced to the system of first order ODE.
→ "n" here is order of the system

Systems of ODE can be

- linear (r.h.s is linear in $x_1, x_2, \dots, x_n$)
- nonlinear (r.h.s. is nonlinear - usally some polynom or transcendental function)
- autonomous (r.h.s. doesn't depend on time)
- non autonomous (r.h.s. depends on time. General recipe here is to introduce one more variable $x_{n+1} = t$ and equation $\dot{x}_{n+1} = 1$;)

③ * unstable flow tends to go away from this point.
We have understand how to determine positions of fixed points, but not the behaviour of flow near them. To determine this behaviour we should make linear stability analysis. Let x^* be a fixed point, and we are interested how fast do system aproach this point

$$\eta(t) = x(t) - x^* \; ; \; \dot{\eta} = \dot{x} \; ; \; f(x) = \underbrace{f(x^*)}_{0 \text{ by definition of fixed point } x^*} + \eta \cdot \frac{df}{dx}\Big|_{x=x^*} + O(\eta^2) \Rightarrow$$

$$\Rightarrow \dot{\eta} = \eta \cdot f'(x^*)$$

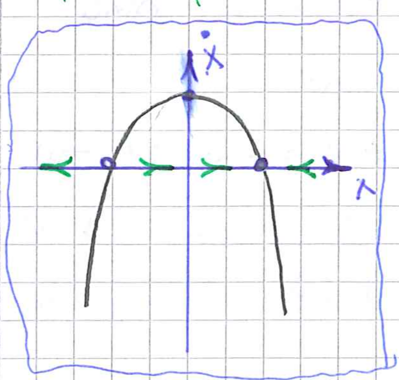
if $f'(x^*) > 0$ we get $\eta = e^{f'(x^*)t}$ - growing with time
if $f'(x^*) < 0 \Rightarrow \eta = e^{f'(x)t}$ is decaying with time, and because η is perturbation sign and magnitude of $f'(x^*)$ gives us picture of flow behaviour

$\boxed{\text{Problems}}$

Ex. N 2.2.2.

Consider $\dot{x} = 1 - x^{14}$ equation
- sketch the vector field
- find fixed points
- classify stability
- sketch $x(t)$ for different initial conditions
- find analytical solution.

phase portrait.



$-1 < x < 1 \Rightarrow \dot{x} > 0$ - flow to the right
$|x| > 1 \Rightarrow \dot{x} < 0$ - flow to the left
we can conclude that
$x = 1$ is stable fixed point
$x = -1$ is unstable fixed point
Now we are able to sketch graph of $x(t)$

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and flow is directed to the left.
x_1^* is stable fixed point
$x_2^* < x < x_1^* \Rightarrow \cos x < e^x \Rightarrow \dot{x} > 0 \rightarrow$ flow is directed to the right and x_2^* is unstable fixed point and we have this alternation of stable and unstable fixed points further.

Ex 2.2.13 (terminal velocity)
Skydiver is falling down with velocity $v(t)$ governed by $m\dot{v} = mg - kv^2$
this term describes resistence of the air

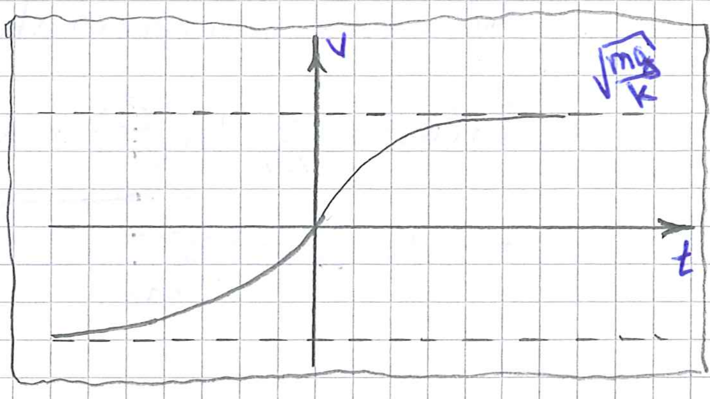
(a) Obtain analytical solution for $v(t)$, assuming $v(0) = 0$
Before solving this equation it is convinient to introduce dimensionless variables.

$$\frac{1}{g}\frac{dv}{dt} = 1 - \frac{k}{mg}v^2$$ if we introduce $\boxed{u = \sqrt{\frac{k}{mg}}\, v;}$

$$\sqrt{\frac{m}{kg}}\frac{du}{dt} = 1 - u^2$$ now we should introduce new time variable $\boxed{\tau = \sqrt{\frac{kg}{m}}\, t}$ and then we get equation $\frac{du}{d\tau} = 1 - u^2$ wich can be easily integrated

$$\int_0^u \frac{du}{1-u^2} = \int_0^\tau d\tau \Rightarrow \operatorname{arctanh}(u) = \tau; \quad u = \tanh\tau$$, or going back to our initial parameters we get:

$$\boxed{v(t) = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}\, t\right);}$$



(b) Find the limit of $v(t)$ as $t \to \infty$. This is terminal velocity.

as $t \to \infty$, $v(t) \to \sqrt{\frac{mg}{k}}$

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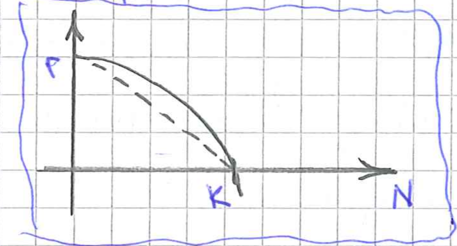
Population growth

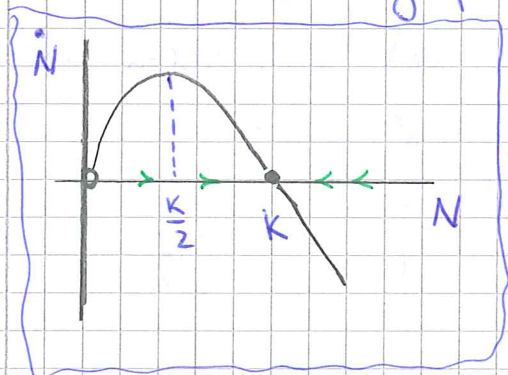
Simpliest model $\dot{N} = rN$ corresponds to exponential growth of population $N(t) = N_0 e^{rt}$;

This is not very realistic becaus eventually growth should stop. To include this idea into the model let's assume that rate of growth $r(N)$ has N dependence and decreases with population



if we take simpliest of possible dependencies, i.e. linear $\dot{N} = r(N - \frac{N^2}{K}) = rN(1 - \frac{N}{K})$ - this is calleg logistic equation.

Let's make graphical analysis.



We have stable fixed point at $N = K$

and

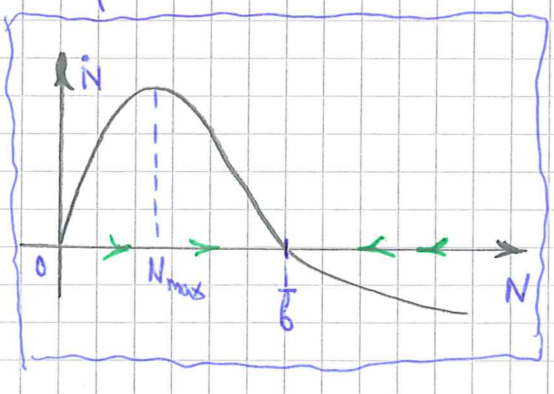
$N < \frac{K}{2}$ - accelaration of growth

$\frac{K}{2} < N < K$ - decelaration of growth

$N > K$ - population decreases towards K

Ex 2.3.3

Growth of cancerous tumors happens according to Gompertz law $\dot{N} = -aN\ln(bN)$, where $N(t)$ is proportional to the number of cells in tumor, and $a, b > 0$ are parameters



(a) interpret a and b

$\frac{1}{a}$ is characteristic time

$a \sim$ growth velocity;

b - characteristic number of cancer cells which plays role of carrying capacity.

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Classes 2 and 3 : Bifurcations

Let's assume that behaviour of 1d systems depends on parameters. Fixed point can be created or destroyed or it's classification (stable vs. unstable) can be changed.

Def Qualitative changes in the dynamics are called bifurcations, and the parameter values at which they occure are called bifurcation points

Saddle - Node Bifurcation

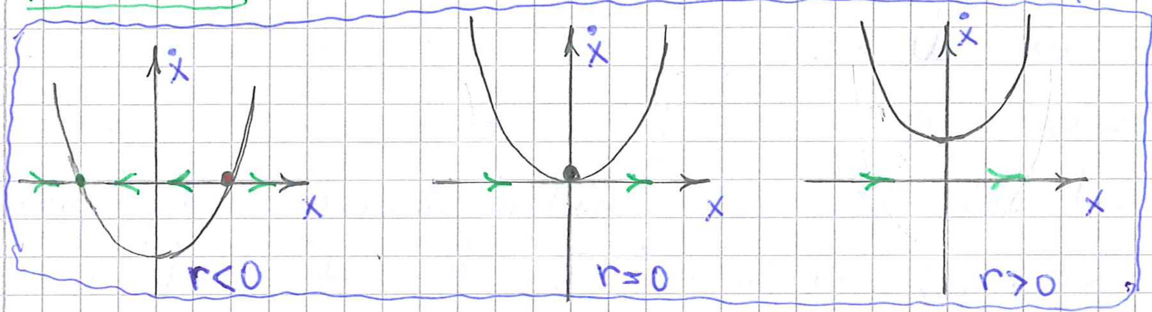
Mechanism of creation and destruction of fixed points

Let's consider simpliest example of this case of bifurcation. Namely:

$\boxed{\dot{x} = r + x^2}$

here • - unstable
• - stable
• - half-stable

!!! notations



$r<0 \rightarrow$ 2 fixed points (stable at $x=-\sqrt{r}$ and unstable at $x=\sqrt{r}$)

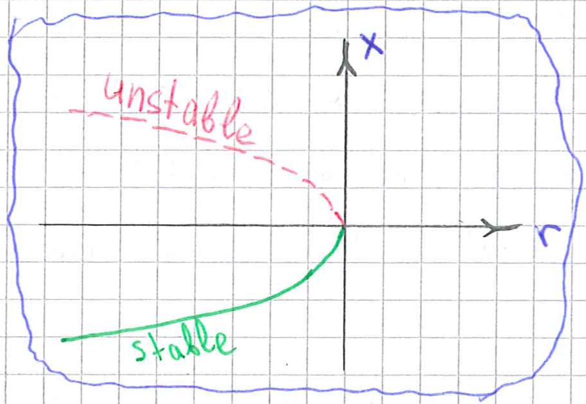
$r=0 \rightarrow$ 2 fixed points eventually meet at $r=0$, where we have one half-stable point

$r>0$ no fixed points occure.

Thus we can conclude that bifurcation occurs at $r=0$ point.

One usefull thing for the description of system that have bifurcation is bifurcation diagram on which you draw position of fixed points as the function of parameter "r". For example in our simpliest case

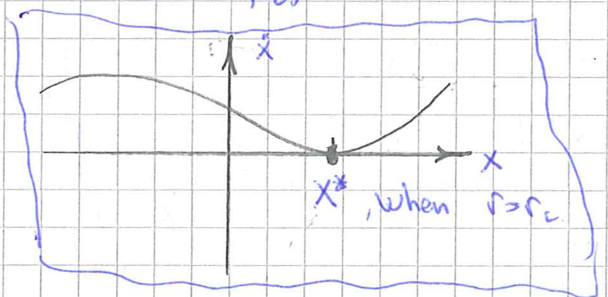
②



Let's consider more general case, when system is described by:
$\dot{x} = f(x, r)$. and let's concentrate on the point (x^*, r_c), where we can expand in series of $(x - x^*)$ and $(r - r_c)$:

$$\dot{x} = f(x^*, r_c) + (x - x^*) \frac{\partial f}{\partial x}\Big|_{(x^*, r_c)} + (r - r_c)\frac{\partial f}{\partial r} + (x - x^*)^2 \frac{\partial^2 f}{\partial x^2}\Big|_{(x^*, r_c)}$$

$f(x^*, r_c) = 0$ because we are in fixed point.
and if we take $\frac{\partial f}{\partial x}\Big|_{(x^*, r_c)} = 0$ i.e. function looks somehow like



i.e. tangent to x-axis ~~at~~ $x = x^*$ when $r = r_c$

and then we get system described by equation
$\dot{x} = (r - r_c) \cdot a + b\,(x - x^*)^2$ - this form of record is called normal form of saddle-node bifurcation.

Exercise 3.1.1

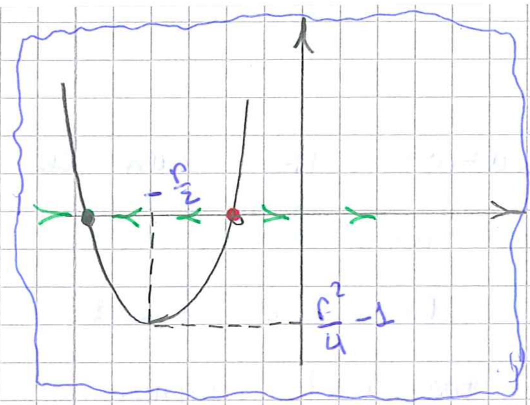
Let's consider system described by:
$\dot{x} = 1 + rx + x^2$

- sketch qualitatively different vector fields
- show that saddle node bifurcation occure and find bifurcation point r
- sketch bifurcation diagramm.

Let's rewrite equation in the following form
$\dot{x} = (1 - \frac{r^2}{4}) + (x + \frac{r}{2})^2$

There are several cases we can consider:

③



① $1-\frac{r^2}{4}<0$, $r>2$, $r<-2$;

2 fixed points.

② $1-\frac{r^2}{4}>0$; $-2<r<2$

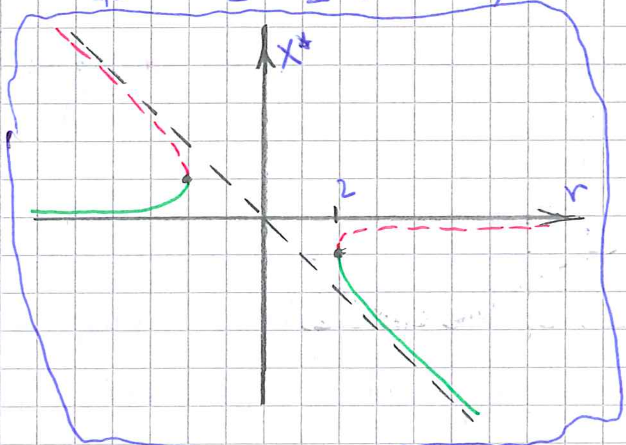
no fixed points

③ $r^* = \pm 2$ 1 fixed point (half-stable)

Position of fixed points are defined by

$\dot{x}=0$ or $(1-\frac{1}{4}r^2)+(x+\frac{1}{2}r)^2=0 \Rightarrow$

$x^*_{1,2} = -\frac{r}{2} \pm \frac{1}{2}\sqrt{r^2-4}$;



Fixed point that's bigger is unstable thus

$x^*_1 = -\frac{r}{2}+\frac{1}{2}\sqrt{r^2-4}$; - unstable f.p.

$r\to+\infty$; $x^*_1 \to 0$

$r\to-\infty$; $x^*_1 \to -r$

$x^*_2 = -\frac{r}{2}-\frac{1}{2}\sqrt{r^2-4}$; - stable f.p.

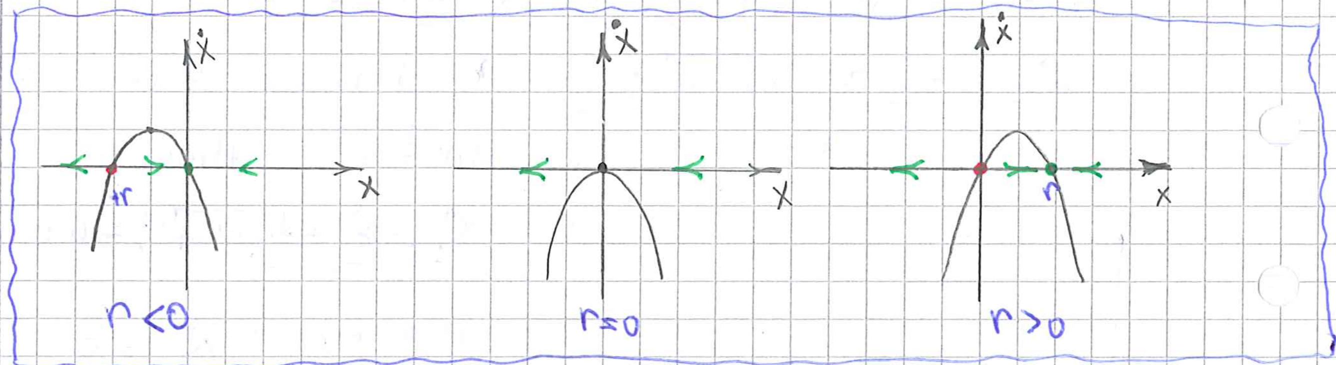
$r\to\infty$; $x^*_2 \to -r$;

$r\to+\infty$; $x^*_2 \to 0$;

④ Transcritical bifurcation

another type of bifurcations that can be observed in system is transcritical bifurcation. In this kind of bifurcation no new fixed points appears or disappear the just change stability. Simpliest example is

$\dot{x} = rx - x^2$;

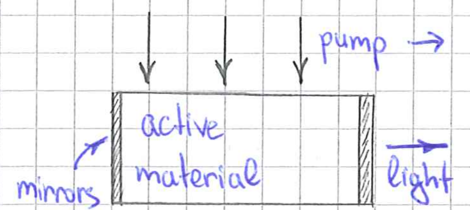


bifurcation diagram

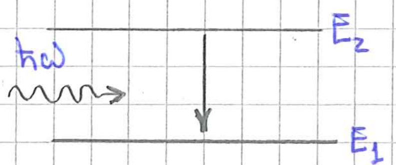
One interesting physical application of this is found in lasers.

Exercise 3.3.1 Laser threshold.

Let's consider some simple model of laser



pump → external source excite atoms in laser, i.e. it pumps it. Interaction of photons with excited atoms results in process of simulated emission



When pumping is weak emission of photons is not coherent and laser acts like ordinary lamp

When pumping is strong emission becomes coherent and we get laser. If we try to write down some simple mathematical model for the laser we get:

⑤ $$\boxed{\begin{aligned}\dot n &= GnN - kn; \\ \dot N &= -GnN - fN + p;\end{aligned}}$$

where
n is number of photons
N is the number of excited atoms
G – gain coefficient for simulated emission;
k – decay rate due to the loss in mirrors;
f – decay rate for spontaneous emission;
p – pump strength
All parameters except p are positive.

(a) adiabatic elimination if we suppose that N relaxes much more rapidly than n, we make quasi-static approximation $\dot N \approx 0$; Express N as function of n and then write down diff equation for n:

$N(f+Gn) = p \Rightarrow N = \frac{p}{f+Gn}$, substituting this into second equation we get: $\boxed{\dot n = \frac{Gnp}{f+Gn} - kn;}$

(b) Show that $n^* = 0$ becomes unstable for $p > p_c$, where p_c is to be determined.

fixed points are determined by equation $n\left(\frac{Gp}{f+Gn} - k\right) = 0$

$n^* = 0$; and $Gp = k(f + Gn^*) \Rightarrow n^* = \frac{1}{Gk}(Gp - fk)$

Here for analysis of stability it would be better to use linear stability analysis. If we take

$\dot n = F(n) = n\left(\frac{Gp}{f+Gn} - k\right)$; $F'(n) = \frac{Gp}{f+Gn} - k - \frac{G^2pn}{(f+Gn)^2}$

$F'(n) = \frac{fGp}{(f+Gn)^2} - k$; $F'(0) = \frac{G}{f}\left(p - \frac{kf}{G}\right)$;

* $F'(0) > 0$ for $p > p_c = \frac{kf}{G}$ – fixed point is unstable

* $F'(0) < 0$ for $p < p_c = \frac{kf}{G}$ – fixed point is stable

thus bifurcation point is $\boxed{p_c = \frac{kf}{G};}$

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ⓒ What kind of bifurcation is this?

Let's classify another stable point

$n^* = \frac{p}{k} - \frac{f}{G};$

We use linear stability again. We have found

$F'(n) = \frac{fGp}{(f+Gn)^2} - k;$ for n^* we have

$\frac{Gp}{f+Gn^*} - k = 0 \Rightarrow \frac{1}{f+Gn^*} = \frac{k}{Gp}$ thus

$F'(n^*) = \frac{fGpk^2}{(Gp)^2} - k = \frac{k}{p}\left(\frac{kf}{G} - p\right) = k\left(\frac{p_c}{p} - 1\right)$

$F'(n^*) = k\left(\frac{p_c}{p} - 1\right);$

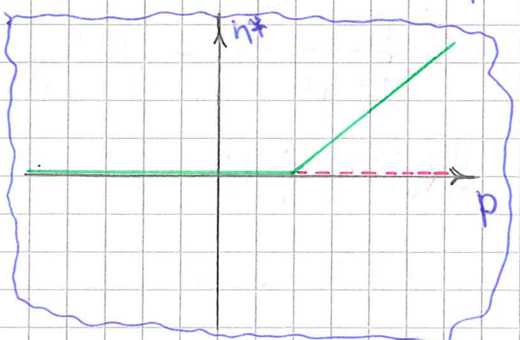
* $F'(n^*) > 0$ for $p < p_c$, unstable if $p > 0$
* $F'(n^*) < 0$ for $p > p_c$, stable and for $p < 0$

but as $p_c = \frac{kf}{G} > 0$ we are interested only in case of $p > 0$

$n^* = \frac{p}{k} - \frac{f}{G}$ is $\begin{cases} \text{stable for } p > p_c; \\ \text{unstable for } p < p_c; \end{cases}$

$n^* = 0$ is $\begin{cases} \text{stable for } p < p_c \\ \text{unstable for } p > p_c \end{cases};$

on bifurcation diagram we take $n^* \geq 0$ because this is number of photons and it can't be negative

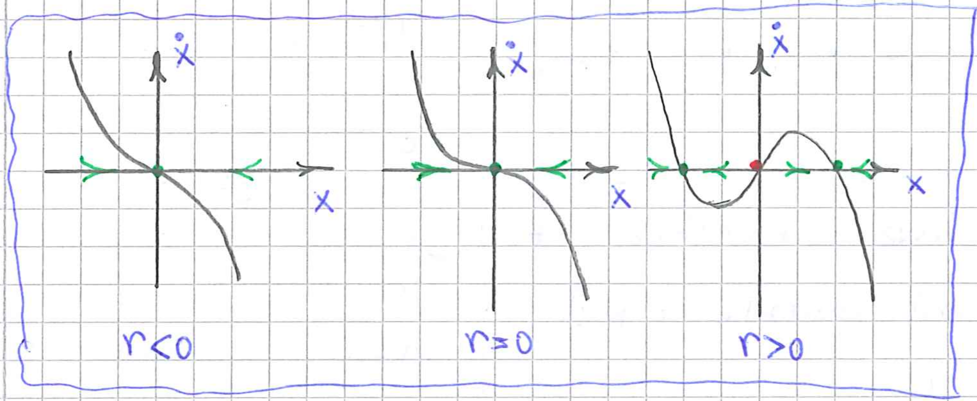


⑦ ## Pitchfork Bifurcation (seminar 3)

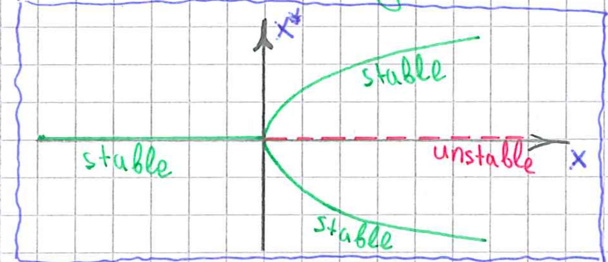
This kind of bifurcations appears in the systems with symmetry. There can be:

* supercritical pitchfork bifurcation

simpliest example $\dot{x} = rx - x^3$ this equation is invariant under reflection $x \to -x$

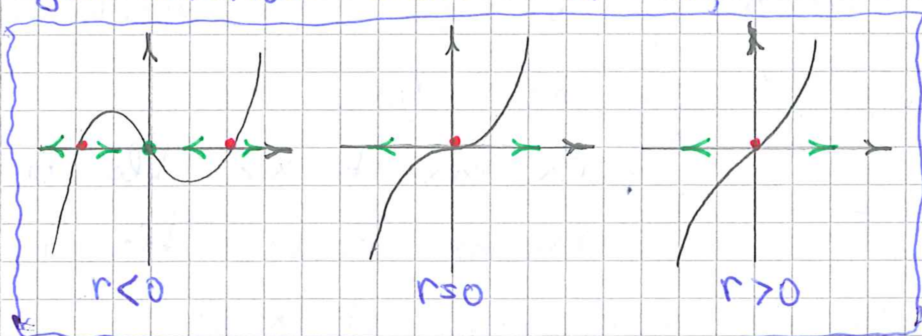


Bifurcation diagram looks like

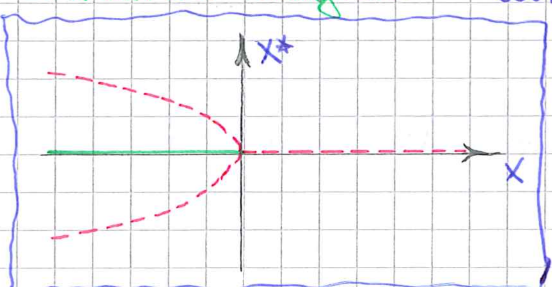


* subcritical pitchfork bifurcation

simpliest example is $\dot{x} = rx + x^3$ this equation is again invariant under $x \to -x$;



bifurcation diagram looks like



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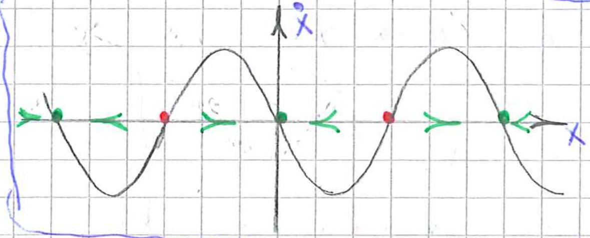
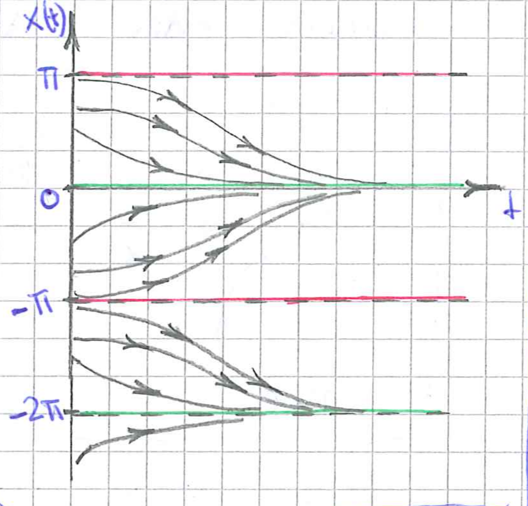
Exercise 3.4.11.

Let's consider system given by $\dot{x} = rx - \sin x$

ⓐ For the case $r=0$, find and classify all the fixed points, and sketch the vector field

if $r=0$; $\dot{x} = -\sin x$;





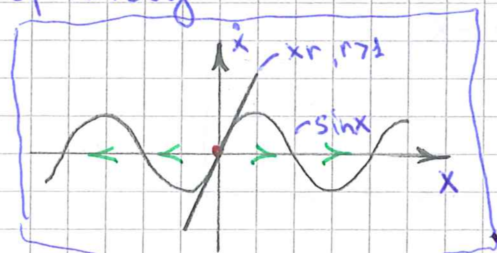
So we have alternation of stable and unstable fixed points, i.e.

$x^* = \pi(2n+1)$, $n \in \mathbb{Z}$ unstable

$x^* = 2\pi n$, $n \in \mathbb{Z}$ stable

ⓑ Show that when $r>1$, there is only one fixed point. What kind of fixed point is it?

It is difficult to analyse $rx - \sin x$ directly so we will make as usually - draw rx and $\sin x$ separately



we see that $r|x| > |\sin x|$ everywhere and in fact

$rx > \sin x$ for $x>0$ flow to the right

$rx < \sin x$ for $x<0$ flow to the left

we have unstable fixed point at $x=0$;

ⓒ as r decreases from ∞ to 0, classify all the bifurcations that occure

it is reasonable to consider several cases

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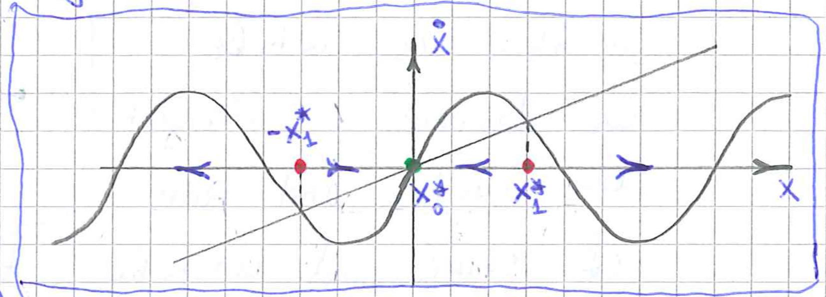
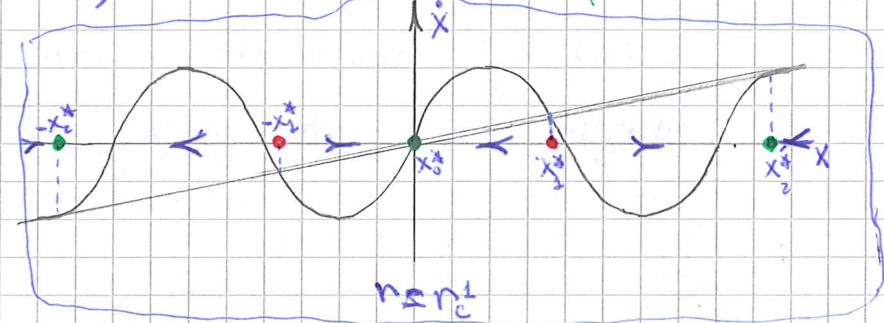
1. as we have seen for $r>1$ there is only one unstable fixed point at $x^*=0$;
2. now we decrease r and when it becomes smaller than 1 we get stable fixed point at $x_0^*=0$ and 2 unstable fixed points at x_1^* and $-x_1^*$ (i.e. situated symmetrically with respect to $x_0^*=0$) this is subcritical pitchfork bifurc.



3. next bifurcation occures when $r_c^1 x$ is tangent to $\sin x$ we get



two symmetricaly situated stable fixed points. and this process will be repeating again and again - we will obtain pairs of new fixed points whish will have alternating stability - this are saddle node bif.

4. For $0<r\ll 1$ find approximate formula for values of r at which bifurcations occur.

as we have already noticed bifurcation occurs, when $r_c x$ is tangent to $\sin x$, i.e. bifurcation point is given by following system of equations

$$\begin{cases} r_c = \cos x^* & \text{- tangent} \\ r_c \cdot x^* = \sin x^* & \text{- intersects.} \end{cases}$$

if r_c is small enough we conclude that

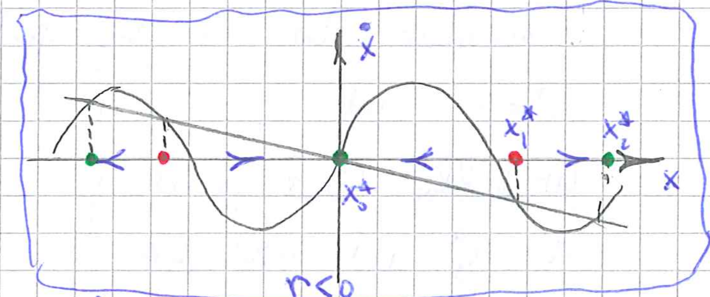
$\cos x^* \approx 0 \Rightarrow x^* = \frac{\pi}{2} + 2\pi n$, $n \gg 1$ and we can approximate $x^* = 2\pi n$, and we can say that $r_c^2(1+x^{*2}) = 1 \Rightarrow$

$$\Rightarrow r_c = \frac{1}{\sqrt{1+(x^*)^2}} = \frac{1}{\sqrt{1+\left(\frac{\pi}{2}+2\pi n\right)^2}} \approx \frac{2}{\pi(1+4n)}$$

$$\boxed{r_c = \frac{2}{\pi(1+4n)}}$$

(10) (e) Now classify all the bifurcations that occur as r decreases from 0 to $-\infty$

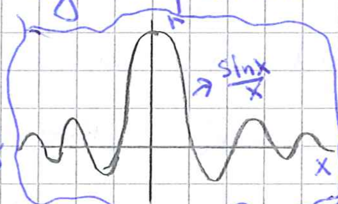
Now let's look what happens when we go to negative r



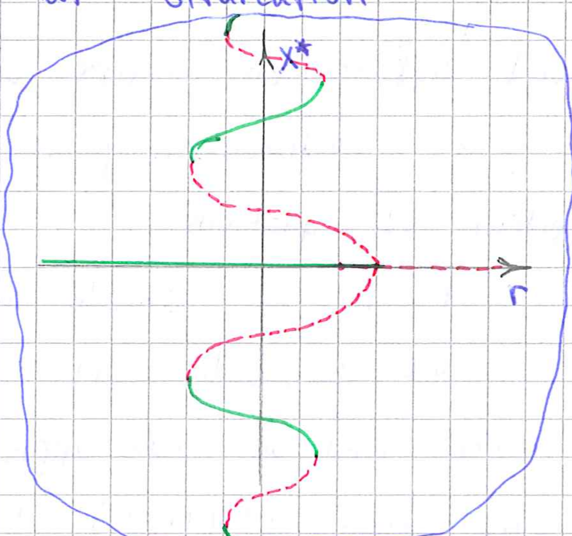
in this case while we decreas r, streight line rotates and pairs of stable and unstable fixed points (x_1^* and x_2^* on the picture) come closer and closer and eventully annihilate when streight line become tangent to sin x graph, thus we agan have saddle-node bifurcations.

(f) Bifurcation diagram

Bifurcation points are given by equation $rx^* = \sin x^*$, which looks like:



Now if we summarise our knowledge about classification of fixed points and bifurcations we can immideately conclude that bifurcation



Summary

$r>1$ - unstable at $x^*=0$

$0<r<1$ - new pairs of stable-unstable fixed points appear

$-1<r<0$ pairs of points start annihilating

$r<-1$ - only $x^*=0$ remains and it is stable now

$r_c=1$ - subcritical pitchfork bifurcation point

$r_c = r_c^1, r_c^2 \dots$ - saddle-node bifurcation points

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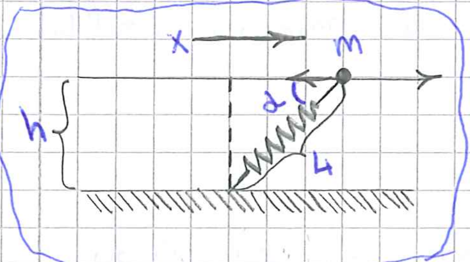
Problem 3.5.4 Bead on horizontal wire.



A bead of mass m is constrained to slide along a straight horizontal wire. A spring of relaxed length L_0 and spring constant k is attached to the mass and to a support point a distance h from the wire. Finally, suppose that the motion of the bead is opposed by a viscous damping force $b\dot{x}$

(a) Write Newton's law for the motion of the bead

$m\ddot{x} = -b\dot{x} - k(L-L_0)\cdot\cos\alpha$; $\cos\alpha = \frac{x}{L}$;

$m\ddot{x} = -b\dot{x} - k(L-L_0)\frac{x}{L}$; $L=\sqrt{x^2+h^2}$;

$$\boxed{m\ddot{x} = -b\dot{x} - kx + kL_0\frac{x}{\sqrt{x^2+h^2}};}$$

(b) Find all possible equilibria, i.e., fixed points, as functions of k, h, m, b, and L_0;

equilibria points are given by $\ddot{x}=0$ and $\dot{x}=0$, i.e.

$L_0 = \sqrt{x^2+h^2} \Rightarrow$ $\boxed{x = \pm\sqrt{L_0^2-h^2};}$ $\boxed{x=0}$

(c) Suppose $m=0$. Classify stability of fixed points and draw bifurcation diagram.

$-b\dot{x} - kx + kL_0\frac{x}{\sqrt{x^2+h^2}} = 0$; or

$$\dot{x} = F(x) = -\frac{k}{b}x + \frac{kL_0}{b}\frac{x}{\sqrt{x^2+h^2}};$$

fixed points are given by $F(x^*)=0$ i.e. points found previously

$$F'(x) = -\frac{k}{b} + \frac{kL_0}{b}\frac{1}{\sqrt{x^2+h^2}} - \frac{kL_0 x^2}{b(x^2+h^2)^{3/2}}$$

in particular:

$F'(0) = -\frac{k}{b} + \frac{kL_0}{bh} = \frac{k}{b}\left(\frac{L_0}{h}-1\right)$ from linear stability analysis we know that

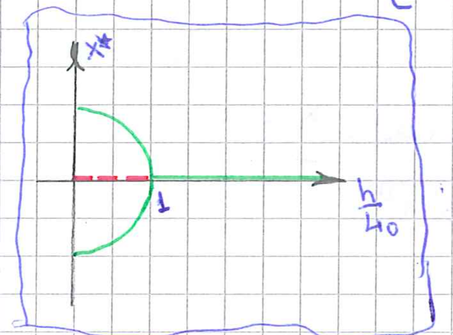
(12) $F'(0) > 0$ or $L_0 > h$ we get unstable fixed point

$F'(0) < 0$ or $L_0 < h \Rightarrow$ fixed poin is stable

$$F'(\pm\sqrt{L_0^2 - h^2}) = \frac{k}{b}\left(\frac{h^2}{L_0^2} - 1\right)$$

situation for this fixed point is just opposite with situation for $x^* = 0$; i.e.

$$x^* = \pm\sqrt{L_0^2 - h^2} = \begin{cases} L_0 < h \rightarrow \text{unstable} \\ L_0 > h \rightarrow \text{stable} \end{cases}$$



(d) In what sense m can be considered small?

Let's assume $m \neq 0$

$$m\ddot{x} = -b\dot{x} - kx\left(1 - \frac{L_0}{\sqrt{x^2+h^2}}\right); \quad \tau = \frac{t}{T} -$$

By this step we introduce explicitely time scale.

Then we get $$\frac{m}{T^2}\frac{d^2x}{d\tau^2} = -\frac{b}{T}\frac{dx}{d\tau} - kx\left(1 - \frac{L_0}{\sqrt{x^2+h^2}}\right)$$

Now if we say $\frac{b}{Tk} = O(1)$ $\quad T \approx \frac{b}{k}$;

Then we get equation

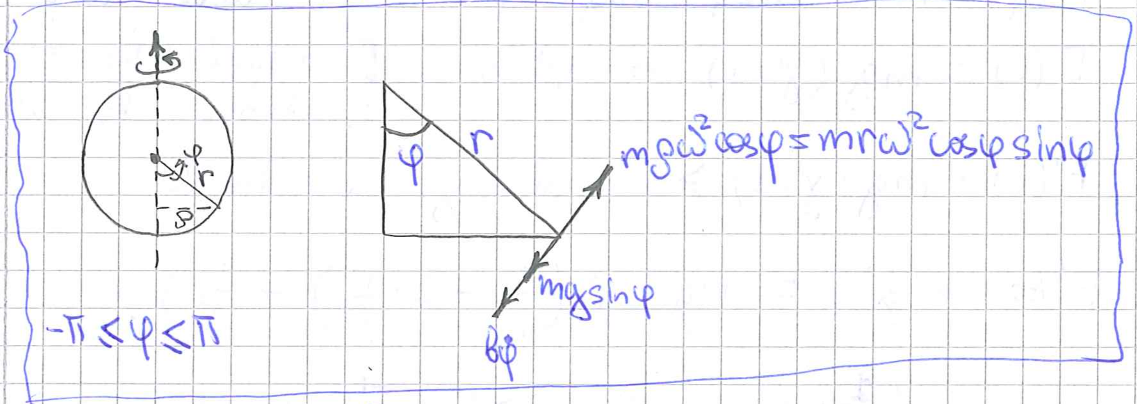
$$\frac{mk}{b^2}\frac{d^2x}{d\tau^2} = -\frac{dx}{d\tau} - x\left(1 - \frac{L_0}{\sqrt{x^2+h^2}}\right); \quad \frac{mk}{b^2} \ll 1$$ – this ~~is~~ condition that being satisfeed lead to negligable term with second derivative

condition is given by $\frac{mk}{b^2} \ll 1$;

(13)

Overdamped bead on a rotating hoop

A bead of mass "m" slides along a wire loop of a radius r, the loop is rotating with constant angular velocity ω about vertical axis.



mg – gravity force;

$m\rho\omega^2$ – centrifugal force;

$b\dot{\varphi}$ – damping force;

Newton's second law looks like

$$mr\ddot{\varphi} = -b\dot{\varphi} + mr\omega^2\sin\varphi\cdot\cos\varphi - mg\sin\varphi$$

We say that system is overdamped so that we can neglect term $mr\ddot{\varphi}$. In this case we get:

$$b\dot{\varphi} = mr\omega^2\sin\varphi\cdot\cos\varphi - mg\sin\varphi = mg\sin\varphi\left(\frac{r\omega^2}{g}\cos\varphi - 1\right)$$

Fixed points are given by $\dot{\varphi} = 0$

$\sin\varphi^* = 0$; $\cos\varphi^* = \frac{g}{r\omega^2}$; $\varphi^* = \pm\arccos\left(\frac{g}{r\omega^2}\right)$;

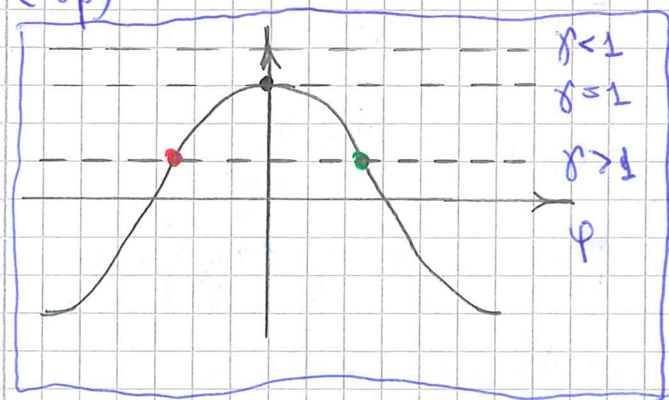
$\varphi^* = 0$ (bottom)

$\varphi^* = 0$ (top)

$\gamma = \frac{r\omega^2}{g}$; $\gamma > 1$ – 2 additional fixed points

$\gamma < 1$ – no fixed points

$\gamma = 1$ – 1 half stable point – here bifurcation occurs.



if $\gamma \to \infty$ $\varphi^* \to \pm\frac{\pi}{2}$;

(14) to analyse stability at this fixed points we should use linear stability analysis.

if we take function

$$F(\varphi) = \frac{1}{2} m r \omega^2 \sin 2\varphi - mg \sin\varphi$$

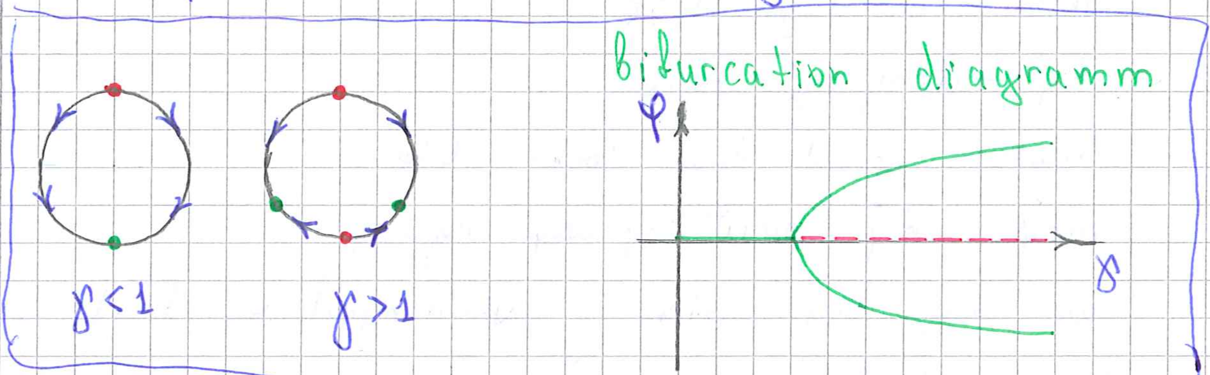
$$F'(\varphi) = m r \omega^2 \cos 2\varphi - mg\cos\varphi = mg(\gamma \cos 2\varphi - \cos\varphi)$$

$$F'(0) = mg(\gamma - 1) \rightarrow \varphi^* = 0 \text{ is } \begin{cases} \text{unstable for } \gamma > 1 \\ \text{stable for } \gamma < 1 \end{cases}$$

$$F'(\pi) = mg(\gamma + 1) > 0 - \text{always unstable}$$

$$F'\left(\pm\arccos\left(\frac{1}{\gamma}\right)\right) = mg\left(\gamma\left(\frac{1}{\gamma^2} - 1 + \frac{1}{\gamma^2}\right) - \frac{1}{\gamma}\right) =$$

$= mg\left(\frac{1}{\gamma} - 1\right)$ as $\gamma > 1$ $F'(\pm\arccos(\frac{1}{\gamma})) < 0$ and this points are stable always



(d) if $m \neq 0$ how small does m have to be to be considered negligible? In what sense is it negligible

Let's make the same trick as in previous problem. Let's introduce $\tau = \frac{t}{T} \rightarrow \dot{\varphi} = \frac{1}{T}\frac{d\varphi}{d\tau}$;

$$\frac{mr}{T^2}\frac{d^2\varphi}{d\tau^2} = -\frac{b}{T}\frac{d\varphi}{d\tau} - mg\sin\varphi + mr\omega^2 \sin\varphi \cdot \cos\varphi$$

$$\frac{r}{gT^2}\frac{d^2\varphi}{d\tau^2} = -\frac{b}{Tmg}\frac{d\varphi}{d\tau} - \sin\varphi + \frac{r\omega^2}{g}\sin\varphi \cdot \cos\varphi$$

if we take $\frac{b}{Tmg} = O(1)$ $T \approx \frac{b}{mg}$;

condition we need is $\frac{r}{gT^2} \ll 1 \Rightarrow \frac{r}{g}\left(\frac{mg}{b}\right)^2 \ll 1$ thus we get following condition $\boxed{b^2 \gg m^2 g r}$

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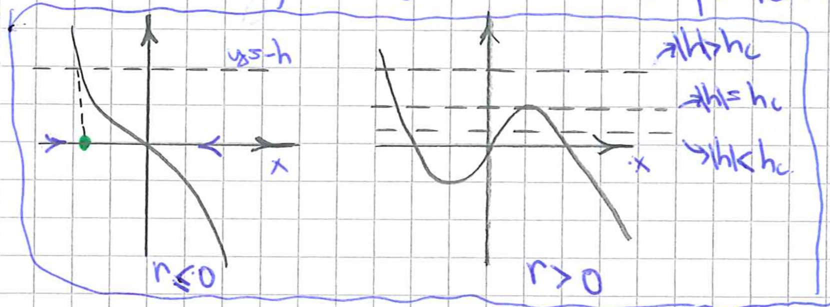
Some additional material on Imperfect Bifurcations

This kind of bifurcations occur when we have some imperfection term, i.e. term violating symmetry of typical system containing pitchfork bifurcation. Simpliest example of such system is

$\dot{x} = h + rx - x^3$; here h is imperfection parameter.



Here we should consider 2 cases

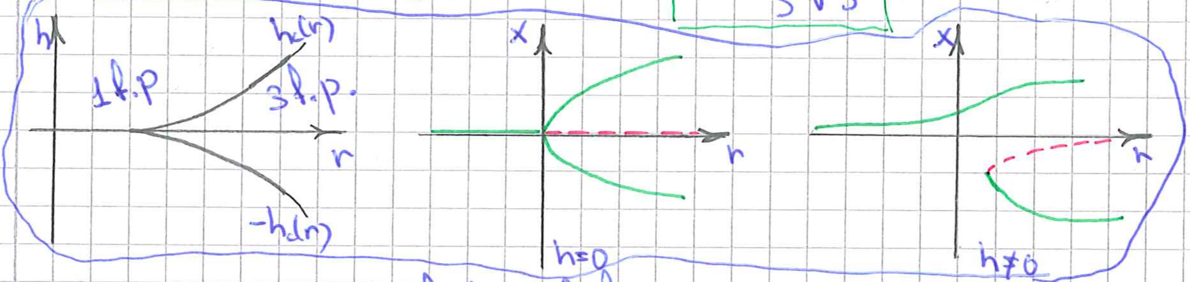
if $r \leq 0$ we always have only 1 intersection which gives 1 stable fixed point

if $r > 0$ we have 1 stable fixed point for $|h| > h_c$ when $|h| = h_c$ we have saddle-node bifurcation and when $h < h_c$ we get 3 fixed points

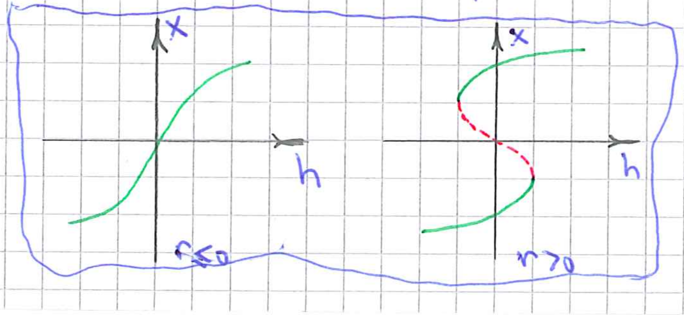
Bifurcation occurs in maximum of $rx - x^3$ i.e.

$\frac{d}{dx}(rx - x^3) = (r - 3x^2) \Rightarrow x^* = \pm\sqrt{\frac{r}{3}}$. Then h_c is given by

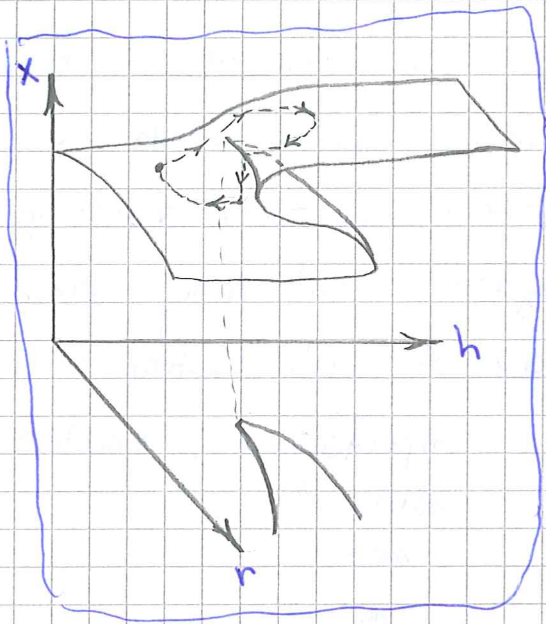
$h_c = rx^* - (x^*)^3 = \pm \frac{2r}{3}\sqrt{\frac{r}{3}}$, so $\boxed{h_c = \frac{2r}{3}\sqrt{\frac{r}{3}}}$;



Befor picturing full bifurcation diagram which in this case is 3-dimensional, we can draw 2 crossections



(16)



Here we see example of what is catastrophe, i.e. this histeresis-like jump which in life can lead to the destruction of bridge or building

1

Seminar 4 (Flows on a circle)

We know that on a line no oscillations are possible because flow is either ending on fixed points or goes away to infinity.. But if we make function periodic we allow oscilatory behaviour, and making function periodic is just the same as to consider function on the circle

So we have periodic function $f(\theta)$ and system wich is governed by equation

$\boxed{\dot{\theta} = f(\theta)}$ Here "θ" is the angle on the circle. This is the simpliest system that allows oscillations.

Exercise N4.1.4

As the example consider system:

$\dot{\theta} = \sin^3\theta$ (i.e. find and classify fixed points and sketch phase portrait)

fixed points : $\sin^3\theta = 0$; $\theta_k^* = k\pi$, $k \in \mathbb{Z}$;

To define what kind of fixed points are this we will use linear stability analysis

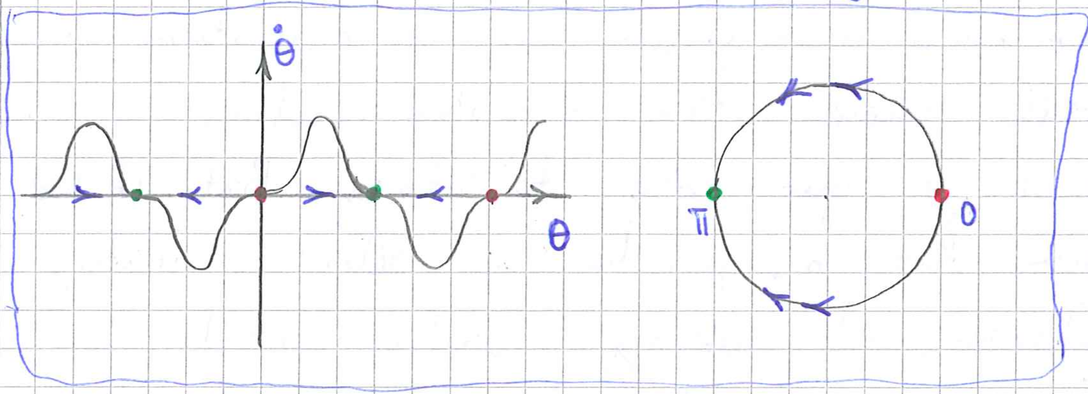
$f(\theta) = \sin^3\theta$; $f'(\theta) = 3\sin^2\theta \cdot \cos\theta$; $f'(\theta^*) = 0$ - If first derivative is 0 we should go further in Taylor expansion and look for the next derivative. Now growth (or decay) of the flow near fixed point won't be exponential any more and will be slower.

$f''(\theta) = 6\sin\theta \cdot \cos^2\theta - 3\sin^3\theta$; $f''(\theta^*) = 0$ thus we should again go further in Taylor expansion

$f'''(\theta) = 6\cos^3\theta - 12\sin^2\theta \cdot \cos\theta - 9\sin^2\theta \cdot \cos\theta$; $f'''(\theta_k^*) = (-1)^k \cdot 6$;

$$f'''(\theta^*) = (-1)^k \cdot 6 = \begin{cases} 6 > 0 & \text{if } k = 2n \text{ (even). f.p. is unstable} \\ -6 < 0 & \text{if } k = 2n+1 \text{ (odd) f.p. is stable.} \end{cases}$$

② The same results follow from graphical analysis



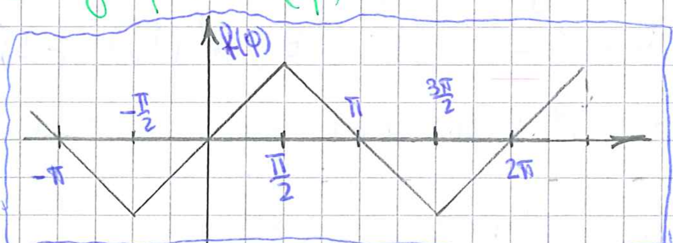
Example of such kind of system on a circle that we meet in nature are fire flies.
Fireflies flash on and of with the same frequency and in nature they synchronize their flashes with external stimulus (external stimulus is flashlignt in the lab or other fireflies in nature)
assume that $\dot\theta = \omega$ - uniform oscillators of fireflies
$\dot\Theta = \Omega$ - flashes of stimulus. Fireflies are able to change their frequencies to get synchronised with stimulus. One of mathematical models of fireflies is considered in

Exercise 4.5.1

$\dot\Theta = \Omega$ we introduce new variable $\varphi = \Theta - \theta$
$\dot\theta = \omega + Af(\Theta - \theta)$ i.e. difference of phase of stimulus and firefly.. we are given

$$f(\phi) = \begin{cases} \varphi, & -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2} \\ \pi - \varphi, & \frac{\pi}{2} \leq \varphi \leq \frac{3\pi}{2} \end{cases}$$

ⓐ graph $f(\varphi)$



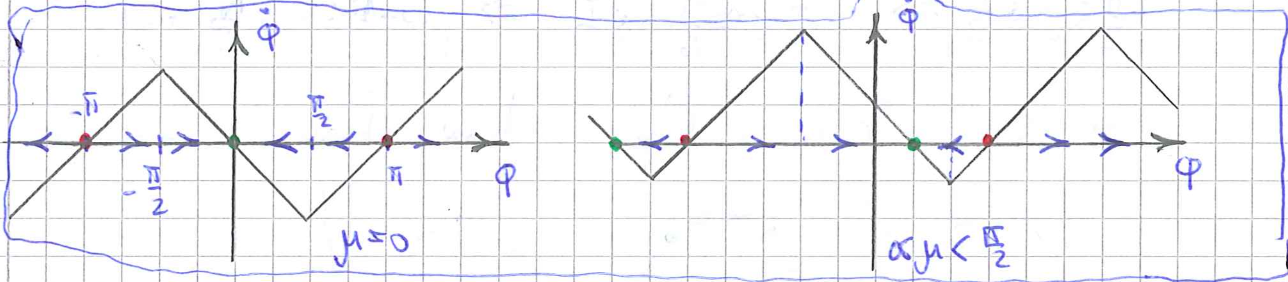
③ ③ Find range of entrainment.

Before understanding what is entrainment we consider dynamics of the system.

$\dot{\Theta} - \dot{\theta} = \dot{\varphi} = \Omega - \omega - A f(\varphi)$. Let's nondimensionalize system

$\frac{1}{A}\dot{\varphi} = \frac{\Omega - \omega}{A} - f(\varphi)$; $\tau = At$; $\frac{\Omega-\omega}{A} = \mu$, then

$$\boxed{\dot{\varphi} = \mu - f(\varphi);}$$



Let's consider several cases:

Ⅰ $\mu = 0$ - i.e. stimulus lights with the frequency equal to fireflies natural frequency. Here we have following fixed points:

$\varphi^* = 0$ - stable; $\varphi^* = 2\pi k$; $k \in \mathbb{Z}$ - stable; $\varphi^* = \pi(2k+1)$, $k \in \mathbb{Z}$ - unstable;

we see that $\varphi^* = 0$ is stable fixed point, i.e. firefly eventually synchronizes with stimulus.

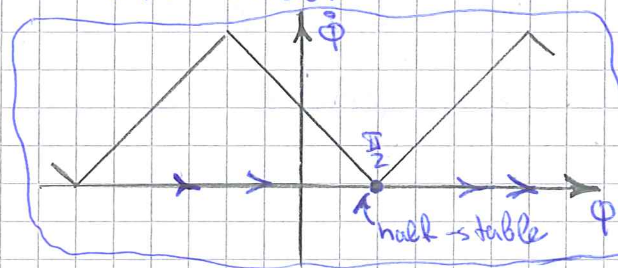
Ⅱ $0 < \mu < \frac{\pi}{2}$ stable fixed points exist, but not at $\varphi = 0$ namely:

$\varphi^* = \mu + 2\pi k$, $k \in \mathbb{Z}$ - stable fixed points

$\varphi^* = \pi - \mu + 2\pi k$, $k \in \mathbb{Z}$ - unstable fixed points.

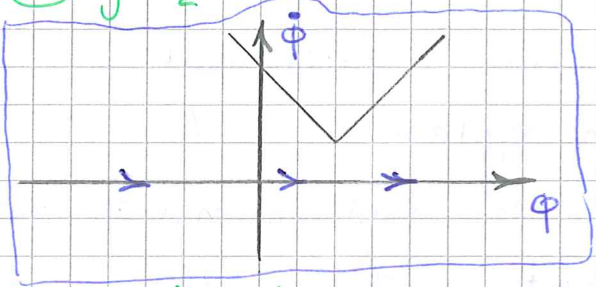
In this case firefly flashing is phase-locked to the stimulus which mean that firefly and stimulus run with the same instantoneous frequency, although they no longer flash in unison

Ⅲ $\mu = \frac{\pi}{2}$



two fixed points become one half-stable fixed point at $\varphi = \frac{\pi}{2}$ (saddle-node bifurcation)

④ IV $\mu > \frac{\pi}{2}$

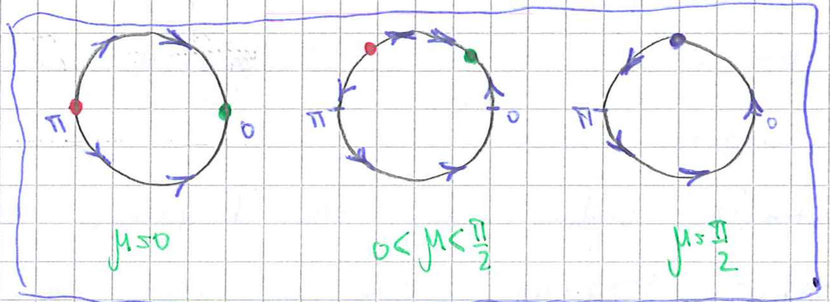


no fixed points, i.e. phase locking is lost. This state is called phase drift.

Range of entrainment, wich we are asked to find here is interval of μ where phase locking is possible, i.e.

$-\frac{\pi}{2} \leq \mu \leq \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \leq \frac{\Omega - \omega}{A} \leq \frac{\pi}{2} \Rightarrow \boxed{\omega - \frac{A\pi}{2} \leq \Omega \leq \frac{A\pi}{2} + \omega}$

We can "canonically" draw flow on a circle:



ⓒ in assumption of phase locking find phase difference $0 \leq \mu \leq \frac{\pi}{2}$

$\frac{d\varphi}{d\tau} = \mu - f(\varphi)$ fixed points are given by r.h.s. taken to "0", i.e. $\mu = f(\varphi^*)$ or $\boxed{\varphi^* = \mu;\ \varphi^* = \pi - \mu;}$

If you look on the pictures we have drawn you see then that $\varphi^* = \mu$ is stable one and give us required phase difference $\boxed{\varphi^* = \mu;}$ (the same result can be obtained from linear stability analysis)

ⓓ find T_{drift} - time required to change phase φ by 2π

$$\frac{d\varphi}{d\tau} = \mu - f(\varphi); \quad T_{drift} = \int dt = \frac{1}{A}\int d\tau = \frac{1}{A}\int_0^{2\pi} \frac{d\tau}{d\varphi}\, d\varphi = \frac{1}{A}\int_0^{2\pi} \frac{d\varphi}{\mu - f(\varphi)}$$

becaus function is 2π-periodic we can shift phase arbitrary.

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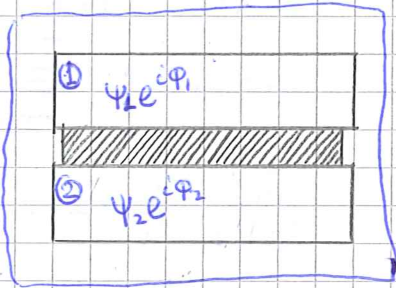
thus $T_{drift} = \frac{1}{A}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{d\varphi}{\mu - f(\varphi)} = \frac{1}{A}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\varphi}{\mu-\varphi} + \frac{1}{A}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{d\varphi}{(\mu-\pi)+\varphi} =$

$= -\frac{1}{A}\ln(\mu-\varphi)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{A}\ln(\mu-\pi+\varphi)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \frac{1}{A}\ln\left(\frac{(\mu+\frac{\pi}{2})(\mu+\frac{\pi}{2})}{(\mu-\frac{\pi}{2})(\mu-\frac{\pi}{2})}\right) =$

$= \frac{2}{A}\ln\left(\frac{\mu+\frac{\pi}{2}}{\mu-\frac{\pi}{2}}\right)$ as $\mu \to \frac{\pi}{2}$ then $T_{drift} \to \infty$ as is expected

$$\boxed{T_{drift} = \frac{2}{A}\ln\left(\frac{\mu+\frac{\pi}{2}}{\mu-\frac{\pi}{2}}\right);}$$

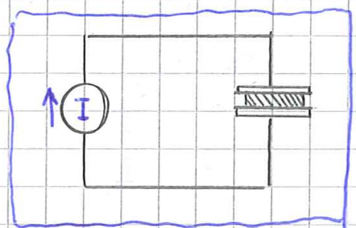
Josephson junctions



Suppose we have 2 superconductors separated by a weak connection. Cooper pairs are correlated and forms ground state wave functions with one single phase.

Josephson effect: no voltage between superconductors but current still exists (this happens due to quantum tunneling of cooper pairs across the junction)

Josephson relations



if $I < I_c$ – no voltage in junction is created and the phases of superconductors are driven apart to a constant phase difference $\varphi = \varphi_2 - \varphi_1$;

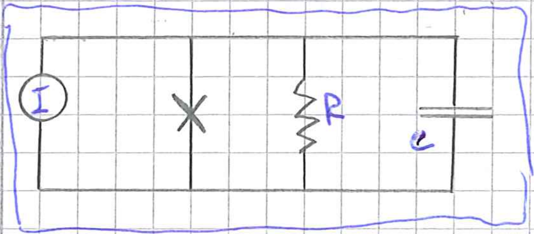
and we obtain Josephson current-phase relation

$$\boxed{I = I_c \sin\varphi;}$$

if $I > I_c$ phase difference is no more constant and voltage between superconductors appears. Voltage is described Josephson voltage-phase relation

$$\boxed{V = \frac{\hbar}{2e}\dot{\varphi};}$$

⑥ Except supercurrent we have displacement and ordinary currents. We represent ordinary current with resistor R and displacement current with capacitor C, and use equivalent parallel circuit.



$$I = C\dot{V} + \frac{V}{R} + I_c \sin\varphi \,; \quad V = \frac{\hbar}{2e}\dot{\varphi}$$

$$I = \frac{C\hbar}{2e}\ddot{\varphi} + \frac{\hbar}{2eR}\dot{\varphi} + I_c \sin\varphi$$

First of all we should make this problem dimensionless. Let's introduce

$$\tau = \frac{1}{T}t \,; \quad \frac{I}{I_c} = \frac{C\hbar}{2eI_c}\frac{1}{T^2}\ddot{\varphi} + \frac{\hbar}{2eRI_c}\frac{1}{T}\dot{\varphi} + \sin\varphi \,;$$

$$\frac{\hbar}{2eRI_c}\frac{1}{T} = 1 \,; \quad T = \frac{\hbar}{2eRI_c} \,; \quad \text{- this is characteristic time}$$

$$\beta = \frac{c\hbar}{2eI_c}\frac{1}{T^2} = \frac{C\hbar \cdot 4 \cdot e^2 R^2 I_c^2}{2eI_c\hbar^2} = \frac{2eR^2 I_c}{\hbar} \cdot C \,;$$

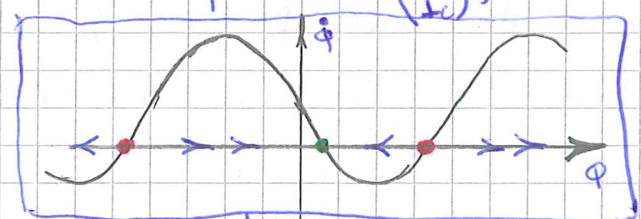
thus $\boxed{\beta\ddot{\varphi} + \dot{\varphi} + \sin\varphi = \frac{I}{I_c}}$ where $\boxed{\beta = \frac{2eR^2 I_c C}{\hbar} \,; \; T = \frac{\hbar}{2eRI_c} \,;}$

In real systems depending on geometry and material β can range in wide spectrum of parameters. $10^{-6} < \beta < 10^{6}$. Let's consider limit, in which dynamics of the system is described by simple first order equation.

$\dot{\varphi} = \frac{I}{I_c} - \sin\varphi \,;$ Fixed points are given by

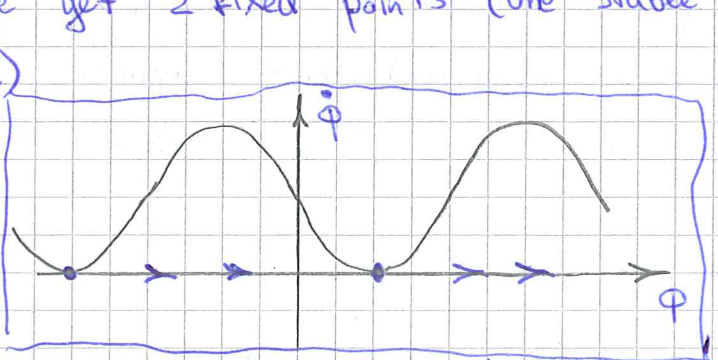
$$\varphi^* = \arcsin\left(\frac{I}{I_c}\right) \,;$$

I $I < I_c$



If we consider only interval $\varphi \in [0, 2\pi]$ then we get 2 fixed points (one stable and one unstable)

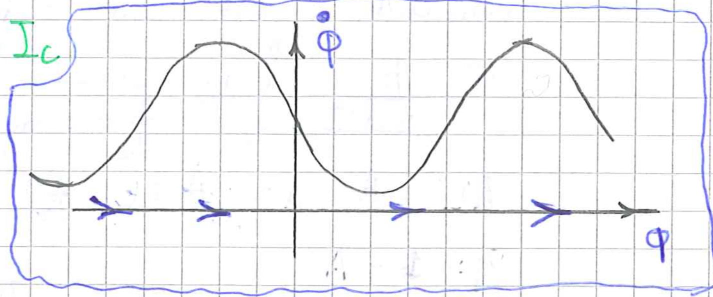
II $I = I_c$



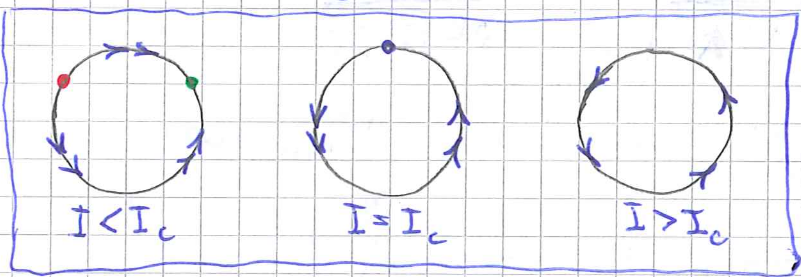
Stable and unstable fixed points come to

⑦ one half-stable fixed point (saddle-node bifurcation)

III Finally the case $I > I_c$ no fixed points (all of them dissapeared in saddle-node bifurcations)



So we conclude that solution tends to the stable one only for $I < I_c$ and periodic for $I > I_c$



Let's find current-voltage curve in overdamped limit, i.e. the average value of voltage $\langle V \rangle$ as a function of constant applied current I. As $V = \frac{\hbar}{2e}\dot{\varphi}$; $\langle V \rangle = \frac{\hbar}{2e}\langle \dot{\varphi} \rangle$

$$\langle \dot{\varphi} \rangle = \left\langle \frac{d\varphi}{dt} \right\rangle = \left\langle \frac{d\tau}{dt}\frac{d\varphi}{d\tau} \right\rangle = \frac{2eRI_c}{\hbar}\left\langle \frac{d\varphi}{d\tau} \right\rangle = \frac{2eRI_c}{\hbar}\langle \varphi' \rangle$$

Thus in terms of dimensionless time

$\langle V \rangle = RI_c \langle \varphi' \rangle$; Here we should consider 2 cases separately

I $I \leq I_c$ all solutions approach $\varphi^* = \arcsin\left(\frac{I}{I_c}\right)$; $-\frac{\pi}{2} \leq \varphi^* \leq \frac{\pi}{2}$; and then $\varphi' = 0$ and thus $\langle V \rangle = 0$ for $I \leq I_c$;

II Now if $\boxed{I > I_c}$ we have periodic evolment of the system $T = \int dt = \int d\varphi \frac{dt}{d\varphi}$ in the overdamped limit

$$\frac{d\varphi}{dt} = \frac{1}{T}\frac{d\varphi}{d\tau} = \frac{2eRI_c}{\hbar}\left(\frac{I}{I_c} - \sin\varphi\right); \quad T = \frac{\hbar}{2eRI_c}\int d\varphi \frac{1}{\frac{I}{I_c} - \sin\varphi}$$

Way to evaluate this integral.

make change of variables $u = \tan\frac{\varphi}{2}$; $\sin\varphi = 2\tan\frac{\varphi}{2}\cdot\cos^2\frac{\varphi}{2} =$

$= \frac{2\tan\frac{\varphi}{2}}{1+\tan^2\frac{\varphi}{2}}$; $\boxed{\sin\varphi = \frac{2u}{1+u^2}}$;

⑧ and $du = \frac{1}{2}\frac{1}{\cos^2\frac{\varphi}{2}}\,d\varphi \Rightarrow d\varphi = \frac{2}{1+u^2}\,du$ thus we conclude that our integral equal to:

$$T = \frac{\hbar}{2eRI_c}\int_{-\pi}^{\pi} d\varphi \frac{1}{\frac{I}{I_c} - \sin\varphi} = \frac{\hbar}{2eRI_c}\int_{-\infty}^{+\infty}\frac{2du}{Au^2 + A - 2u}\ ; \quad A = \frac{I}{I_c}$$

$$Au^2 + A - 2u = A\left(u - \frac{1}{A}\right)^2 + A\left(1 - \frac{1}{A^2}\right)\ ; \text{ thus}$$

$$T_{drift} = \frac{\hbar \cdot 2}{2eRI_cA}\int_{-\infty}^{+\infty}\frac{du}{\left(u - \frac{1}{A}\right)^2 + \left(1 - \frac{1}{A^2}\right)} =$$

$$= \frac{\hbar}{eRI\sqrt{1-\frac{1}{A^2}}}\arctan\frac{x}{\sqrt{1-\frac{1}{A^2}}}\Bigg|_{-\infty}^{+\infty} = \frac{\hbar\pi}{eR\sqrt{I^2 - I_c^2}}$$

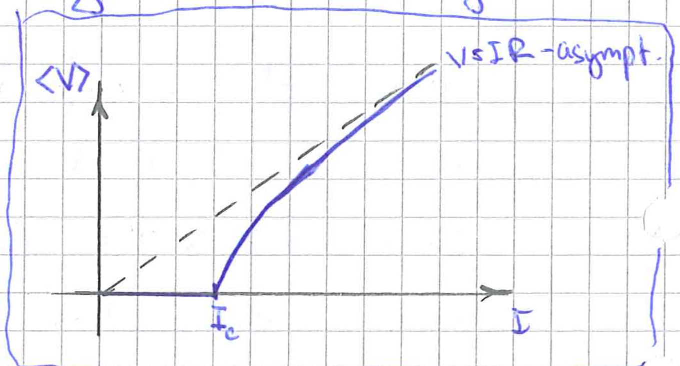
$$\boxed{T_{drift} = \frac{\hbar\pi}{eR\sqrt{I^2-I_c^2}}\ ;}$$ as $I \to I_c$, $T_{drift} \to \infty$;

$T_{drift} = T \cdot \frac{2\pi}{\sqrt{\left(\frac{I}{I_c}\right)^2 - 1}}$; now $\langle\varphi'\rangle = \frac{1}{T_{dr}}\int_0^{T_{dr}}\frac{d\varphi}{d\tau}d\tau = \frac{1}{T_{dr}}\int_0^{2\pi}d\varphi = \frac{2\pi T}{T_{drift}}$

and thus $\langle V\rangle = I_cR\langle\varphi'\rangle = \frac{2\pi I_cRT}{T_{drift}} = I_cR\sqrt{\left(\frac{I}{I_c}\right)^2 - 1}$;

$\langle V\rangle = R\sqrt{I^2 - I_c^2}$, and finally current-voltage curve is given by

$$\boxed{\langle V\rangle = \begin{cases} 0, & I \leq I_c \\ R\sqrt{I^2 - I_c^2}, & I \geq I_c \end{cases}}$$



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Seminar 5 (2d flow - linear systems) + Beginning of seminar 6

Let's consider system

$\dot{\bar{x}} = A\bar{x}$, where x is 2d vector and A is 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $\bar{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

This system is linear if $C_1 \cdot \bar{x}_1 + C_2 \cdot \bar{x}_2$ is a solution of equation and $\bar{x}_1 ; \bar{x}_2$ are solutions of equation too. Solutions of $\dot{\bar{x}} = A\bar{x}$ can be visualized as trajectories moving on the (x,y) plane, called phase plane.

Assume we have this system

$\dot{\bar{x}} = A\bar{x}$ and initial condition $\bar{x}(0) = \bar{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

Let's $\bar{x}(t) = \bar{v} e^{\lambda t}$; Then we get algebraic equation $A\bar{v} = \lambda \bar{v}$, i.e. we obtain eigenproblem.

we have 2 corresponding set of solutions

$\bar{v}_{1,2}$ - 2 eigenvectors ; $\lambda_{1,2}$ - 2 eigenvalues

general solution of system is then $\bar{x} = \sum_{i=1,2} C_i v_i e^{\lambda_i t}$;

where $C_{1,2}$ are constants defined from initial conditions. Eigenvalues are defined by secular equation $\det(A - \lambda I) = 0$

For 2×2 matricies

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det(A - \lambda I) = ad + \lambda^2 - \lambda(a+d) - bc =$

$= \lambda^2 - \tau\lambda + \Delta$, and thus solutions of secular equation are

$$\boxed{\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}}$$

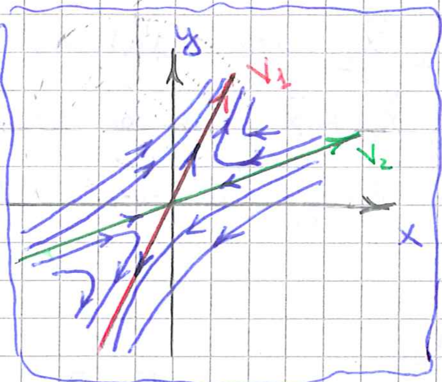
Here we have introduced notations τ - trace of A; Δ - determinant of A

② There are several cases we should consider

ⓐ $\Delta < 0$ In this case $\tau^2 - 4\Delta > \tau^2$ and $\lambda_{1,2}$ have different signs, and thus:

$\bar{x}(t) = C_1 \bar{v}_1 e^{\lambda_1 t} + C_2 \bar{v}_2 e^{\lambda_2 t}$, assume $\lambda_1 > 0$, $\lambda_2 < 0$

$t \to +\infty$ $\bar{x} \to C_1 \bar{v}_1$; as $t \to -\infty$, $\bar{x} \to C_2 \bar{v}_2$;



v_2 is stable manifold and behaviour of this system in this case is called saddle point at $\bar{x}^* = 0$;

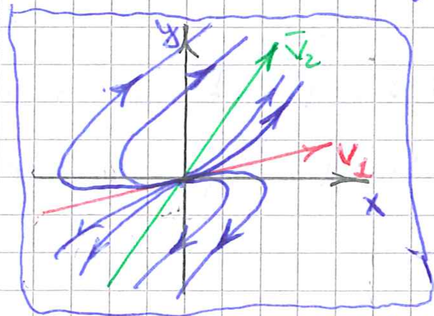
ⓑ $\Delta > 0$; $\tau^2 - 4\Delta > 0$ then eigenvalues are real and have the same signs

assuming $\lambda_2 > \lambda_1 > 0$ we get

$\bar{x}(t) = C_1 \bar{v}_1 e^{\lambda_1 t} + C_2 \bar{v}_2 e^{\lambda_2 t}$ - is unstable

$t \to +\infty$ $\bar{x} \parallel \bar{v}_2$ - fast direction

$t \to -\infty$ $\bar{x} \parallel \bar{v}_1$ - slow direction.



In this case $\bar{x}^* = 0$ is called node (unstable if $\tau > 0$ and stable if $\tau < 0$)

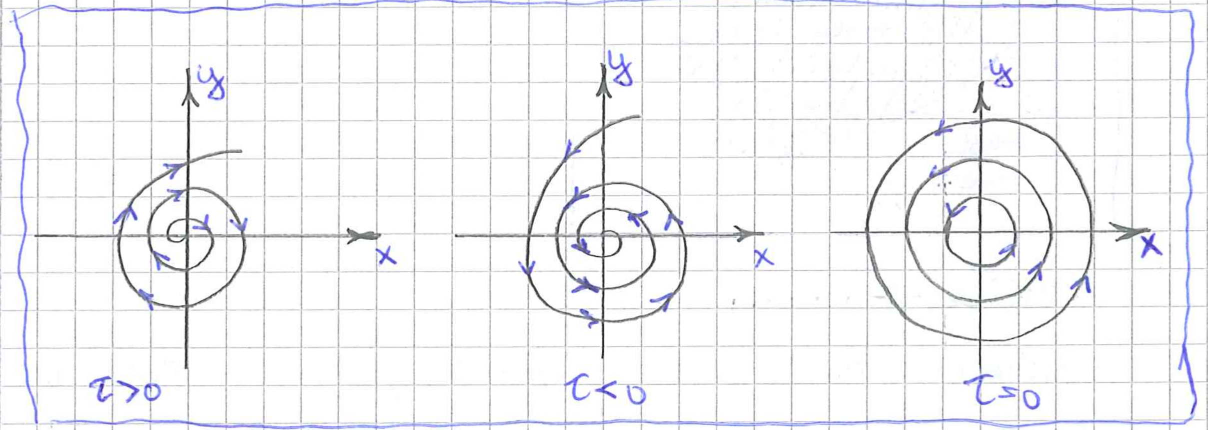
ⓒ $\Delta > 0$; $\tau^2 - 4\Delta < 0$ In this case we get complex conjugate solutions

$\lambda_{1,2} = \alpha \pm i\omega$; $\alpha = \frac{\tau}{2}$; $\omega = \frac{1}{2}\sqrt{4\Delta - \tau^2}$

$x(t) = C_1 e^{\alpha t} e^{i\omega t} \bar{v}_1 + C_2 e^{+\alpha t} e^{-i\omega t} \bar{v}_2$

as $e^{i\omega t} = \cos\omega t + i\sin\omega t$ we get oscillatory behaviour, and we get spiral at $\bar{x}^* = 0$;

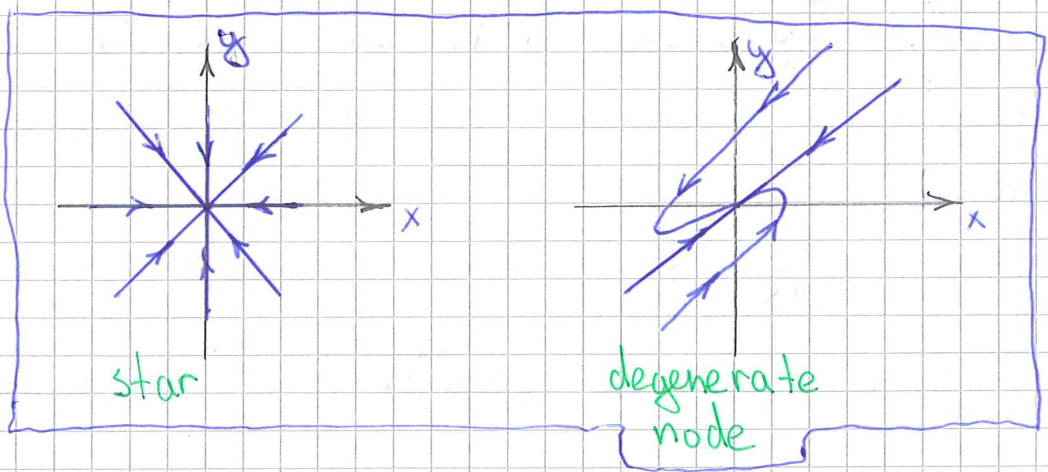
③ spiral can be stable if ($\tau<0$) and unstable if ($\tau>0$) If $\tau=0$ eigenvalues are purely imaginary and we get neutrally stable center.



ⓓ degenerate cases for which $\tau^2-4\Delta=0$;
in this case $\lambda_1=\lambda_2=\lambda$. There are 2 possible cases

Ⓘ $A=\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}$ - in this case all vectors are eigenvectors and we get star node

Ⓘⓘ $A=\begin{bmatrix}\lambda & b\\ 0 & \lambda\end{bmatrix}$ - in this case there is only one eigenvector and we get degenerate node



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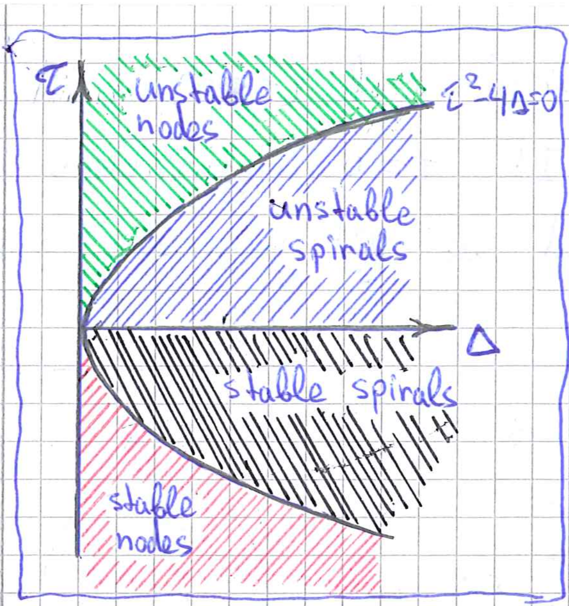

We can picture regions of parameters where we have different

Exercise 5.1.2.

Consider the system $\dot{x} = ax$, $\dot{y} = -y$, where $a < -1$;
Show that all trajectories become parallel to the y-direction as $t \to \infty$ and parallel to the x-direction as $t \to -\infty$

$$\begin{cases} \dot{x} = ax \quad \text{with } a<-1 \\ \dot{y} = -y \end{cases} \Rightarrow \begin{cases} \frac{dx}{x} = a\,dt \\ \frac{dy}{y} = -dt \end{cases} \Rightarrow \begin{cases} x(t) = x_0 e^{at} \\ y(t) = y_0 e^{-t} \end{cases}$$

$$\frac{dy}{dx} = -\frac{y}{ax} = -\frac{1}{a}\frac{y_0}{x_0} e^{-(a+1)t} \qquad a+1<0;$$

$t \to \infty$: $\frac{dy}{dx} \to -\infty$ → tangent is infinite, thus phase line is straight and parallel to y-axis, i.e. vertical.

$t \to -\infty$: $\frac{dy}{dx} \to 0$ → tangent is straight line that is horizontal, i.e. parallel to x-axis, q.e.d.

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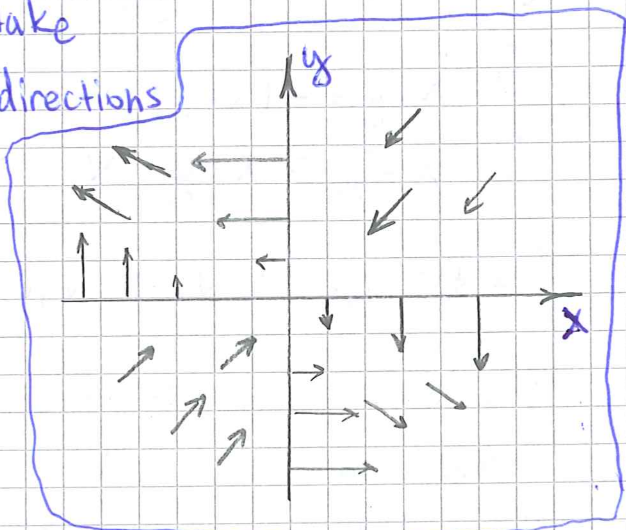
Exercise 5.1.9

$$\begin{cases} \dot{x} = -y \\ \dot{y} = -x \end{cases}$$

ⓐ sketch vector field.

When we sketch vector field first of all we take a look on some special directions



$y=0 \qquad (\dot{x}, \dot{y}) = (0, -x)$

$x=0 \qquad (\dot{x}, \dot{y}) = (-y; 0)$

and quarters of coordinates

(I) $x>0, y>0$; $\dot{x}<0, \dot{y}<0$;

(II) $x>0, y<0$; $\dot{x}>0; \dot{y}<0$;

(III) $x<0, y>0$; $\dot{x}<0, \dot{y}>0$;

(IV) $x<0, y<0$; $\dot{x}>0; \dot{y}>0$;

ⓑ show that trajectories are hyperbolas of the form $x^2-y^2=C$

$$\begin{cases} \dot{x} = -y \\ \dot{y} = -x \end{cases} \Rightarrow \frac{dx}{dy} = \frac{y}{x} \qquad x\,dx = y\,dy \Rightarrow \quad x^2 - y^2 = C \text{ after integration}$$

ⓒ The origin is saddle point, find equations for its stable and unstable manifolds. indeed $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$; $\Delta = -1 < 0$

thus $(\dot{x}; \dot{y}) = (0, 0)$ is saddle point indeed.

secular equation is

$$\begin{Vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{Vmatrix} = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1;$$

eigenvectors are given by equations:

$$\lambda = 1 > 0 \text{ - unstable direction } \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \underbrace{\boxed{x = -y}}_{\text{unstable manifold}}$$

(c) $\lambda = -1$ stable manifold $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$ $\boxed{x = y}$

stable manifold

(d) System can be decoupled and solved

Let's introduce new variables u and v:

$$\begin{cases} u = x + y \\ v = x - y \end{cases} \qquad \begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$$

$$\dot{u} = \dot{x} + \dot{y} = -(x+y) = -u$$
$$\dot{v} = \dot{x} - \dot{y} = x - y = v \Rightarrow \begin{cases} \dot{u} = -u \\ \dot{v} = v \end{cases}$$

$$\boxed{u(t) = u_0 e^{-t}\ ;\ v(t) = v_0 e^{t}}$$

(e) What are stable and unstable manifolds in terms of "u" and "v"

By definition stable manifold is set of initial points (u_0, v_0) such that $(u,v) \to (0,0)$ as $t \to \infty$. We see that this happens when $(u_0, v_0) = (u_0, 0)$ i.e. $v_0 = 0$ or $x - y = 0$ is stable manifold which coincides with the previous result. At the same time if $(u_0; v_0) = (0, v)$ then in the limit $t \to -\infty$ $(u,v) \to (0,0)$ thus $u_0 = 0$ or $x + y = 0$ is unstable manifold

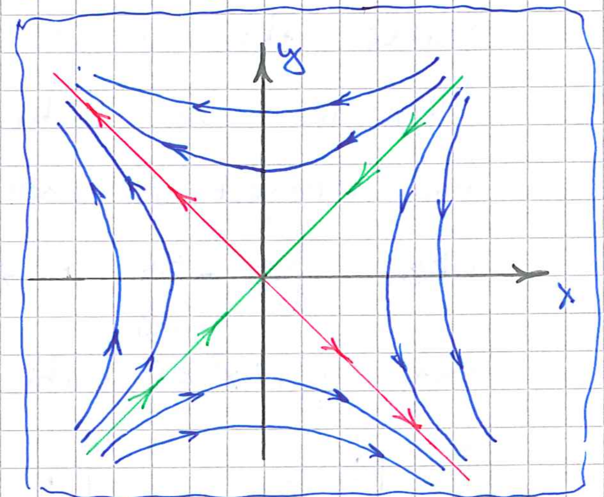
(f) find general solution for $x(t)$ and $y(t)$ starting from an initial condition (x_0, y_0)

we have $x_0 = \frac{1}{2}(u_0 + v_0)$ $y_0 = \frac{1}{2}(u_0 - v_0)$

$$x(t) = \frac{1}{2}u(t) + \frac{1}{2}v(t) = \frac{1}{2}(x_0 + y_0)e^{-t} + \frac{1}{2}(x_0 - y_0)e^{t} =$$
$$= x_0 \cosh t - y_0 \sinh t$$
$$y(t) = \frac{1}{2}u(t) - \frac{1}{2}v(t) = \frac{1}{2}(x_0 + y_0)e^{-t} - \frac{1}{2}(x_0 - y_0)e^{t} =$$
$$= y_0 \cosh t - x_0 \sinh t.$$

⑦ thus the solution is given by

$$\boxed{\begin{aligned} x &= x_0 \operatorname{ch} t - y_0 \operatorname{sh} t \\ y &= y_0 \operatorname{ch} t - x_0 \operatorname{sh} t \end{aligned}}$$



Exercise 5.1.10

Definitions of stability

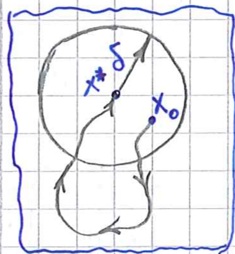
Def 1 $\bar{x}^*$ is attracting

if there is a $\delta > 0$ such that

$\lim\limits_{t\to\infty} \bar{x}(t) = \bar{x}^*$ whenever $\|\bar{x}(0) - \bar{x}^*\| < \delta$

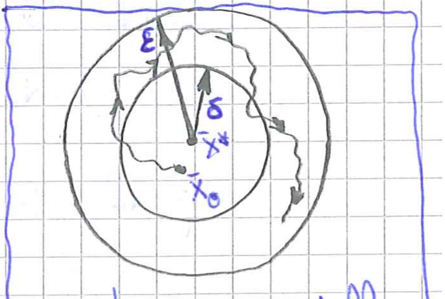
Meaning any trajectory that starts within a distance δ of $\bar{x}^*$ is in $\bar{x}^*$ eventually. (But in process of dynamics evolution it can go far away from $\bar{x}(0)$ and than come back)

x^* is attracting →



Def 2 $\bar{x}^*$ is Lyapunov stable if for each $\varepsilon > 0$ there is $\delta > 0$ such that $\|\bar{x}(t) - \bar{x}^*\| < \varepsilon$ whenever $t \geq 0$ and $\|\bar{x}(0) - \bar{x}^*\| < \delta$

Meaning Trajectories that start within δ of $\bar{x}^*$ remain within ε of $\bar{x}^*$ forever.



Lyapunov stable $\bar{x}^*$

* if point is both Lyapunov stable and attracting we say that it is asymptotically stable.
* if point is Lyapunov stable but not attracting we call it neutrally stable
* if point is not Lyapunov stable nor attracting we say that this point is unstable

Now let's exemine several systems for stabilities.

ⓐ $\begin{cases} \dot{x} = y \\ \dot{y} = -4x \end{cases}$ $A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$ $(x^*, y^*) = (0,0)$ is fixed point

$\det(A - \lambda I) = 0 \Rightarrow \lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$

eigenvalues are purely imaginary $\Rightarrow$ fixed point is

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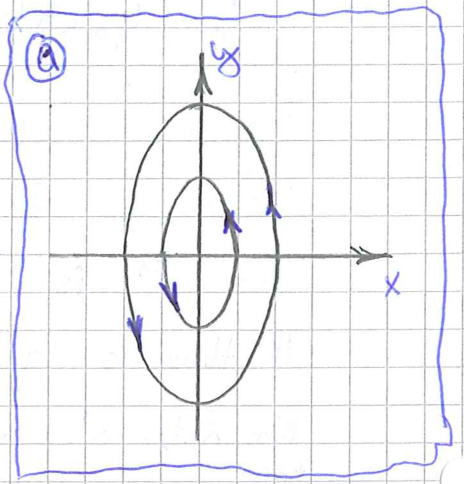
center which is not attracting but Lyapunov stable, thus it is neutrally stable

Here we can actually find out how trajectories explicitely look like $\frac{dy}{dx} = -\frac{4x}{y} \Rightarrow y\,dy + 4x\,dx = 0$

and $y^2 + 4x^2 = C$ - this is ellipse



b $\begin{cases} \dot{x} = 2y \\ \dot{y} = x \end{cases}$; $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$;

secular equation is $\begin{Vmatrix} -\lambda & 2 \\ 1 & -\lambda \end{Vmatrix} = \lambda^2 - 2 = 0 \Rightarrow \lambda = \pm\sqrt{2}$

$\Delta = -2$; $\tau = 0$; - saddle point.

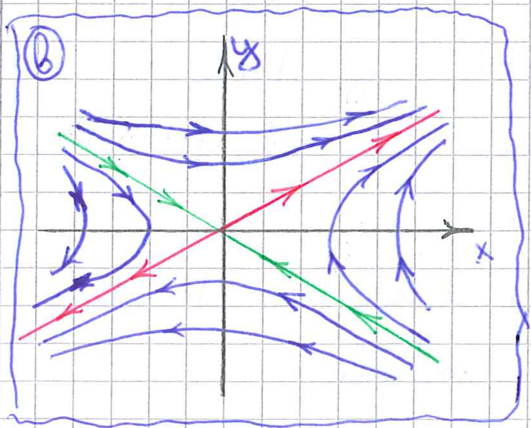
$\lambda = +\sqrt{2}$ unstable direction is given by $\begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{0} \Rightarrow$

$\Rightarrow \boxed{x = \sqrt{2}\, y}$ unstable manifold.

$\lambda = -\sqrt{2}$ stable direction is given by $\begin{bmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{0} \Rightarrow$

$\Rightarrow \boxed{x = -\sqrt{2}\, y}$ stable manifold

saddle point is not stable because it is not attractive nor Lyapunov stable



c $\begin{cases} \dot{x} = 0 \\ \dot{y} = x \end{cases}$ $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$\tau = 0$, $\Delta = 0$ $\lambda^2 = 0$

there is no isolated fixed point in this example, whole $x = 0$ is "fixed manifold"

We can easily find solution of this system

$\begin{cases} x(t) = x_0 \\ y(t) = x_0 t + y_0 \end{cases}$ thus each fixed point of $x = 0$ banch of fixed points is not stable.

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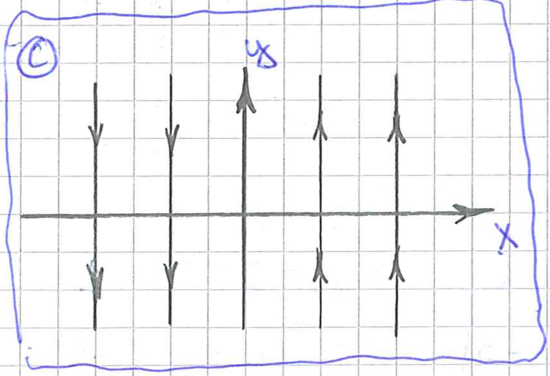
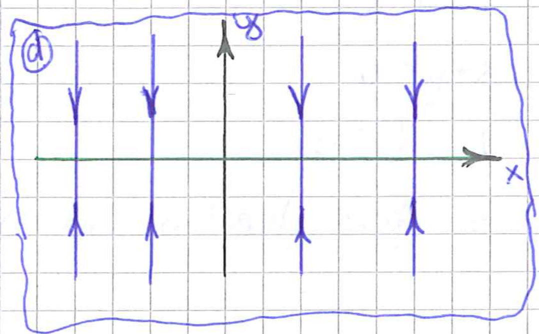


d $\begin{cases} \dot{x} = 0 \\ \dot{y} = -y \end{cases}$ $A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$

$\lambda_1 = 0$, $\lambda_2 = -1$ we meet here degenerate case which is not very good to analyze but it can be solved fortunately

$x(t) = x_0$; $y(t) = y_0 e^{-t}$. Fixed points are all points on $y=0$ axis



Lyapunov stable but not attractive, thus we get neutrally stable points.

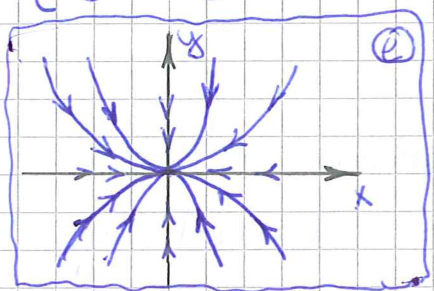
e $\begin{cases} \dot{x} = -x \\ \dot{y} = -5y \end{cases}$; $A = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}$

$\Delta = 5 > 0$; $\tau = -6 < 0$

node as $\tau^2 - 4\Delta > 0$ and it is stable

exact solution is

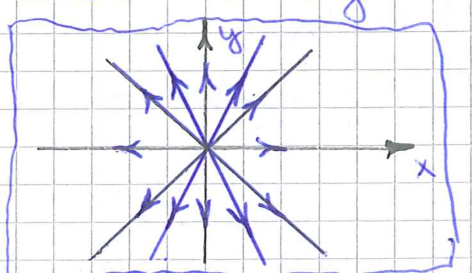
$\begin{cases} x(t) = x_0 e^{-t} \\ y(t) = y_0 e^{-5t} \end{cases}$ so that y is fast direction, x is slow direction



in this case origin is both Lyapunov stable and attracting so it is asymptotically stable.

f $\begin{cases} \dot{x} = x \\ \dot{y} = y \end{cases}$ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\lambda_1 = \lambda_2 = 1 >$ this is degenerate case when system has only one eigenvalue but all vectors are eigenvectors.

and we get unstable star eventually



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Exercise 5.2.3

Plot phase portrait and classify the fixed point of the following system.

$$\begin{cases} \dot{x} = y \\ \dot{y} = -2x - 3y \end{cases} \qquad A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad \Delta = 2;\ \tau = -3$$

$\Delta > 0$, but $\tau^2 - 4\Delta = 1 > 0$

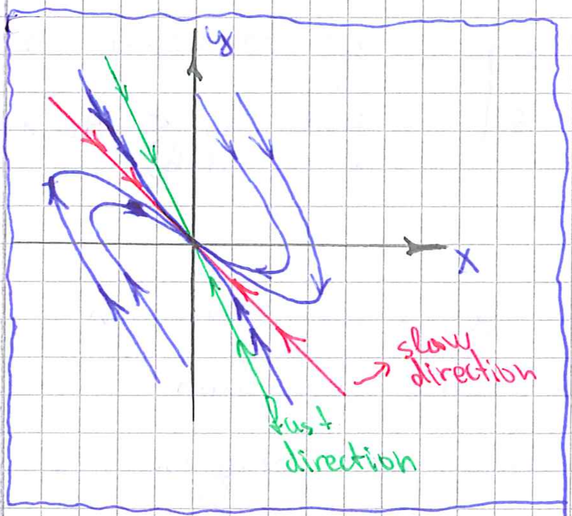
thus we have stable node

eigenvectors

$\lambda_1 = -2 < 0$; $\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{0}$ $\quad 2x + y = 0$, $\bar{v}_1 = (1; -2)$

$\lambda_2 = -1$; $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{0}$ $\quad x + y = 0$, $\bar{v}_2 = (1; -1)$

as $|\lambda_1| > |\lambda_2|$ thus $\bar{v}_1$ is fast direction and $\bar{v}_2$ is slow direction.



Exercise 5.2.13 (damped harmonic oscillator)

Damped harmonic oscillator is described by

$m\ddot{x} + b\dot{x} + kx = 0$, where $b > 0$ is the damping constant.

ⓐ Rewrite the equation as 2-dim. linear system.

For this we introduce

$$\begin{cases} \dot{x} = y \\ m\dot{y} + by + kx = 0 \end{cases}, \text{ or } \qquad \begin{cases} \dot{x} = y \\ \dot{y} = -\frac{b}{m} y - \frac{k}{m} x \end{cases}$$

(11) (B) Classify the fixed point at the origin and sketch the phase portrait. Be sure to show all different cases that can occur.

Fixed point is by definition given by $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \bar{0}$

$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}$; $\boxed{\tau = -\frac{b}{m} < 0; \quad \Delta = \frac{k}{m} > 0; \quad \lambda_{1,2} = -\frac{b}{2m} \pm \frac{1}{2}\sqrt{\frac{b^2-4km}{m^2}};}$

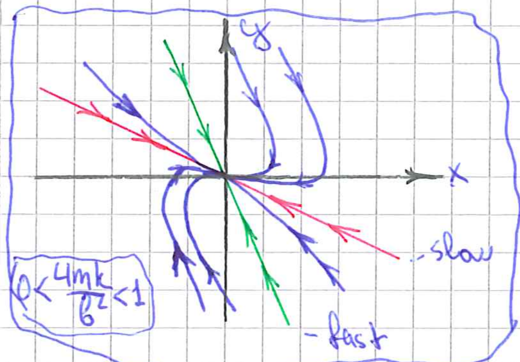
(I) $\tau^2 - 4\Delta > 0$ eigenvalues are real with the same sign, thus we get stable node (stable, because $\tau < 0$) in terms of oscillator parameters this reads as $\boxed{0 < \frac{4mk}{b^2} < 1}$ – this is overdamped case when solution fast enough goes toward fixed point. $\lambda_1 < \lambda_2$. Condition for eigenvectors is

$\begin{bmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{b}{m}-\lambda\end{bmatrix}\begin{bmatrix} v_1 \\ v_2\end{bmatrix} = \bar{0}$, thus $-\lambda v_1 + v_2 = 0$ $\quad v_2 = \lambda v_1$, thus

eigenvector corresponding to the eigenvalue λ_i is $v_i = \begin{bmatrix} 1 \\ \lambda_i\end{bmatrix}$

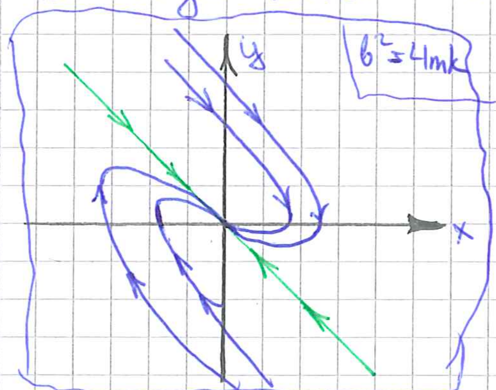
λ_1 – slow direction $\bar{v}_1 = (1; \lambda_1)$;

λ_2 – fast direction $\bar{v}_2 = (1; \lambda_2)$;



(II) $\boxed{b^2 = 4mk}$ or $\tau = 4\Delta$

this is degenerate case when $\lambda_1 = \lambda_2 = \frac{1}{2}\tau < 0$. The initial node is scissored and $(0,0)$ becomes degenerate node stable manifold is eigen vector $\bar{V} = (1; -\frac{b}{2m})$;



(III) Last case we should consider is $\tau^2 - 4\Delta < 0$ $\quad \frac{4mk}{b^2} > 1$;

$\lambda_{1,2} = -\frac{b}{2m}\left(1 \mp i\sqrt{\frac{4mk}{b^2} - 1}\right)$

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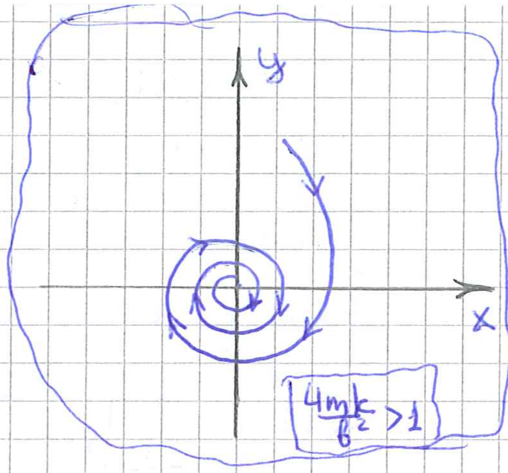
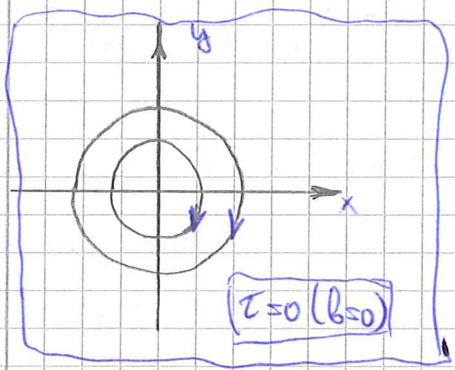
$$\begin{cases} \dot{x} = 0 \\ \dot{y} = -\frac{k}{m}x \end{cases} \Rightarrow \quad \dot{y} < 0 \quad \text{for } x > 0$$



If $b = 0$ (no damping at all)
we get $\tau = 0$, λ - imaginary
and solutions are centers



①

Seminars 6 and 7.

Phase plane - 2D nonlinear systems.

Previously we have considered linear 2d systems $\dot{\bar{x}} = A \cdot \bar{x}$ where A is some constant matrix and $\bar{x}$ is 2d vector $\bar{x} = \begin{bmatrix} x \\ y \end{bmatrix}$; Now we go to more general equation:

$\dot{\bar{x}} = \bar{f}(\bar{x})$ where $\bar{f}(\bar{x}) = \begin{bmatrix} f_1(\bar{x}) \\ f_2(\bar{x}) \end{bmatrix}$;

It is usually difficult to determine explicit behaviour of solution (in fact it is impossible) but we may determine the qualitative behaviour of the system, which includes:

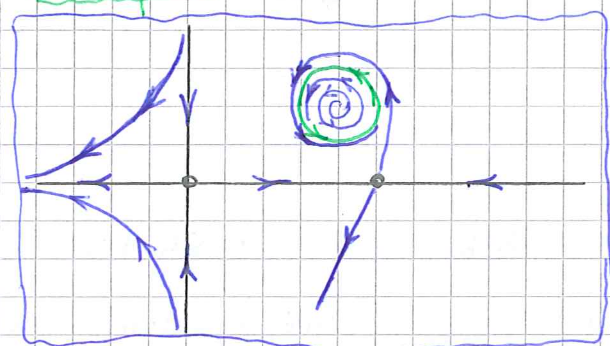
ⓐ Fixed points: $\bar{f}(\bar{x}^*) = 0$;

ⓑ closed orbits: $\bar{x}(t+T) = \bar{x}(t)$; $T > 0$;

ⓒ arrangement of trajectories near the fixed points and closed orbits.

ⓓ stability or unstability of fixed points and orbits.

Example



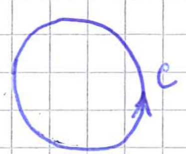
Existence and Uniqueness Theorem

$\dot{\bar{x}} = \bar{f}(x)$; $\bar{x}(0) = \bar{x}_0 \in D$ (open connected set $D \subset R^n$)

$\bar{f} \in C^1(D) \Rightarrow \exists\, \bar{x}(t)$ - solution on some interval $(-\tau, \tau)$ about $t=0$ and it's unique.

Consequence: Different trajectories never intersect (crossing point mean that solution is not unique)

Topological consequence assume C to be closed orbit.



Any trajectory inside C is bounded by C. (valid for 2d only)

(2)

Poincare - Bendixson theorem:

If trajectory is confined to a closed bounded region and there are no fixed points in the region, then the trajectory must eventually approach a closed orbit.

Ex 6.1.1

$$\begin{cases} \dot{x} = x - y \\ \dot{y} = 1 - e^x \end{cases}$$

fixed points

$(\dot{x};\dot{y}) = (0;0)$; $e^x = 1 \Rightarrow x=0, y=0$

so $(0;0)$ is fixed point.

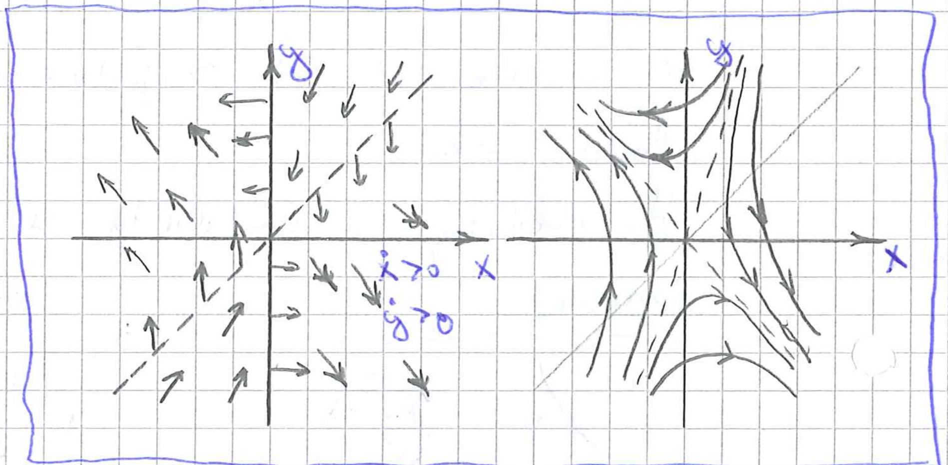
now we draw nullclines, i.e. lines for which $\dot{x}=0$ or $\dot{y}=0$

* $\dot{x}=0$ $x=y$; * $\dot{y}=0$; $x=0$

* $x>y, x>0 \Rightarrow \dot{x}>0, \dot{y}<0$; * $x>y, x<0 \Rightarrow \dot{x}>0; \dot{y}>0$;

* $x<y, x>0 \Rightarrow$
$\Rightarrow \dot{x}<0, \dot{y}<0$

* $x<y, x<0 \Rightarrow$
$\Rightarrow \dot{x}<0, \dot{y}>0$



To understand better behaviour of system around fixed point we use linearisation method

Linearization

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

What we will do now is similar to the linear stability analysis. Let's expand $f(x,y)$ and $g(x,y)$ functions into Taylor series. as (x^*,y^*) is fixed point we have $f(x^*,y^*)=0$; $g(x^*,y^*)=0$

③

Now if we introduce new variables

$\begin{cases} u = x - x^* \\ v = y - y^* \end{cases}$ i.e. deviations from the fixed points we get:

$$\dot{u} = \dot{x} = \underbrace{\cancel{f(x^*, y^*)}}_{0} + u \frac{\partial f}{\partial x}(x^*, y^*) + v \frac{\partial f}{\partial y}(x^*, y^*) + \dots$$

$$\dot{v} = \dot{y} = g(x^*, y^*) + u \frac{\partial g}{\partial x}(x^*, y^*) + v \frac{\partial g}{\partial y}(x^*, y^*) + \dots$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}\Bigg|_{(x^*, y^*)} \begin{bmatrix} u \\ v \end{bmatrix}$$ or written in canonical form:

$$\boxed{\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = J(x^*, y^*) \begin{bmatrix} u \\ v \end{bmatrix} \text{ where } J(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}}$$

The question that comes here is whether it is justified to use this linearized system. The answer is as long as the fixed point for the linearized system is not one of the borderline cases (star, degenerate node) then qualitative picture is correct.

As example let's linearize previously considered system.

$\begin{cases} \dot{x} = x - y \\ \dot{y} = 1 - e^x \end{cases}$ $\quad f(x,y) = x - y$, $\quad g(x,y) = 1 - e^x$ $\quad J(x,y) = \begin{bmatrix} 1 & -1 \\ -e^x & 0 \end{bmatrix}$

$J(0,0) = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$ $\quad \Delta = -1 < 0$ – thus we get saddle point.

Secular equation is given by $\begin{Vmatrix} 1-\lambda & -1 \\ -1 & -\lambda \end{Vmatrix} = \lambda^2 - \lambda - 1 = 0$

$\lambda = \frac{1}{2}(1 \pm \sqrt{5})$

stable direction (negative λ) $\quad \begin{bmatrix} \frac{1}{2}(1+\sqrt{5}) & -1 \\ -1 & \frac{1}{2}(\sqrt{5}-1) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{0} \Rightarrow 2y = (1+\sqrt{5})x \approx 3{,}2x$

$\underline{y \approx 1{,}6x}$

unstable direction is given by $2y = (1 - \sqrt{5}x) \Rightarrow \underline{y \approx -0{,}6x}$

④

Exercise 6.3.6

Find the fixed points, classify them, sketch the neighboring trajectories and try to fill the rest of the phase portrait.

$$\begin{cases} \dot{x} = xy - 1 \\ \dot{y} = x - y^3 \end{cases}$$

ⓐ fixed points

$xy = 1$ $\quad$ $y = \pm 1$

$x = y^3$ $\quad \Downarrow$ $\quad x = \pm 1$

fixed points are $\boxed{(x,y) = (1,1) \,;\, (x,y) = (-1;-1)}$

ⓑ let's linearize system.

$$J(x,y) = \begin{bmatrix} y & x \\ 1 & -3y^2 \end{bmatrix}$$

* $(x,y) = (1,1)$ fixed point

$$J(1,1) = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \; ; \quad \Delta = -4, \quad \tau = -2$$

thus $(1;1)$ is saddle point. eigenvalues are given by

$$\lambda_{1,2} = \frac{1}{2}\tau \pm \frac{1}{2}\sqrt{\tau^2 - 4\Delta} = \frac{1}{2}\left(-2 \pm 2\sqrt{5}\right) = -1 \pm \sqrt{5}$$

eigenvectors are

stable direction $\quad \lambda_1 = -1 - \sqrt{5}$; $\begin{bmatrix} 2+\sqrt{5} & 1 \\ 1 & -2+\sqrt{5} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{0}$

$y = -(2+\sqrt{5})x \Rightarrow y \approx -4{,}2x$

unstable direction $\quad \lambda_2 = -1 + \sqrt{5}$; $\begin{bmatrix} 2-\sqrt{5} & 1 \\ 1 & -2-\sqrt{5} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{0}$

$y = (\sqrt{5} - 2)x \Rightarrow y = 0{,}2x$

* $(x,y) = (-1;-1)$ fixed point

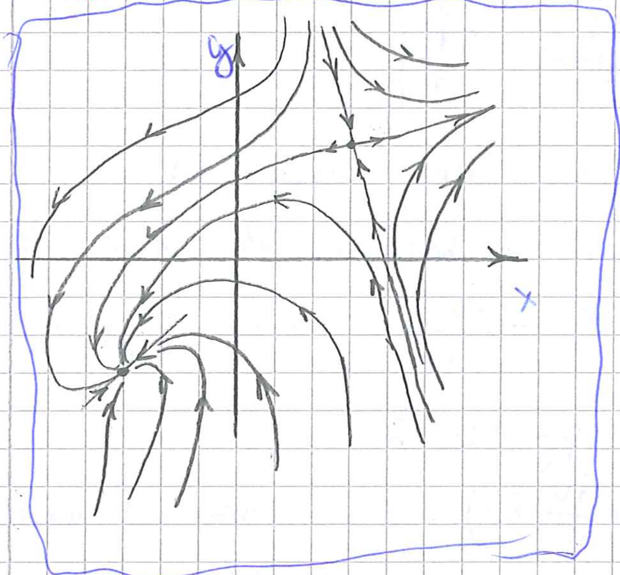
$$J(-1;-1) = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \quad \Delta = 4, \quad \tau = -4 \quad \tau^2 - 4\Delta = 0$$

degenerate case

secular equation is $\begin{vmatrix} -1-\lambda & -1 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2$

$\lambda_1 = \lambda_2 = -2$ eigenvectors are defined by equation

⑤ $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{0} \Rightarrow x = y$ - only one eigenvector and we get stable degenerate node



Exercise 6.3.3

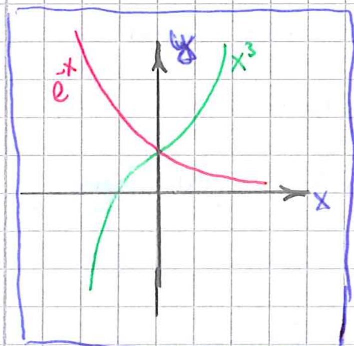
The same problem as previous. But now for the system

$$\begin{cases} \dot{x} = 1 + y - e^{-x} \\ \dot{y} = x^3 - y \end{cases}$$

fixed points

$x^3 = y$

$1 + y = e^{-x} \Rightarrow e^{-x} = 1 + x^3$



$x = 0$ is the only solution

Let's calculate Jacobian:

$$J(x,y) = \begin{bmatrix} e^{-x} & 1 \\ 3x^2 & -1 \end{bmatrix}.$$

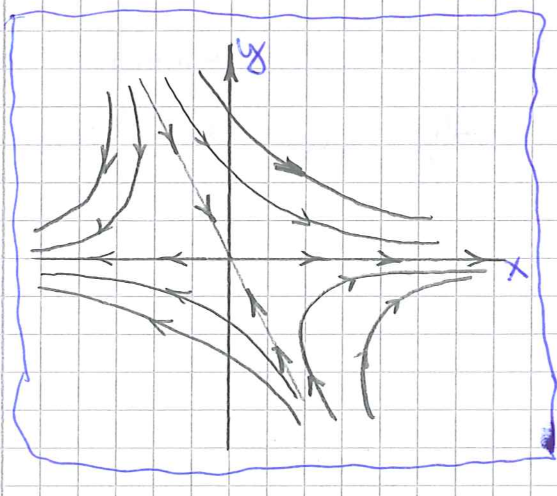
$$J(0;0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$\Delta = -1$, $\tau = 0$ → saddle point.

$\lambda = \pm 1$

stable direction $\lambda = -1$; $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{0} \Rightarrow y = -2x$

unstable direction $\lambda = +1$; $\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{0} \Rightarrow y = 0$



⑥

Conservative systems

Def Systems for which conserved quantaty exists. A conserved quantaty is a real-valued function $E(\bar{x},t)$ such that $\left.\frac{dE}{dt}\right|_{traject.} = 0$ and

$E(\bar{x})$ is non constant on any open set.

Example is energy in mechanics. We have Newton law $m\ddot{\bar{x}} = \bar{F}(\bar{x})$; $\bar{F}(\bar{x}) = -\frac{dV(\bar{x})}{d\bar{x}}$, $V(\bar{x})$ is potential energy then $m\ddot{\bar{x}} + \frac{dV}{d\bar{x}} = 0 \,|\cdot \dot{\bar{x}} \Rightarrow$

$$m\ddot{\bar{x}}\dot{\bar{x}} + \underbrace{\frac{dV}{d\bar{x}}\dot{\bar{x}}}_{\frac{d}{dt}V(\bar{x})} = \frac{d}{dt}\left(\frac{1}{2}m\dot{\bar{x}}^2 + V(\bar{x})\right) = 0 \Rightarrow \text{conserved quantaty}$$

here is $E = \frac{1}{2}m\dot{\bar{x}}^2 + V(\bar{x})$ which is well known expression for energy

Some properties of conserved systems

* a concervative system couldn't have any attracting fixed points. Otherwise $E(\bar{x}^*)$ should be constant in the basin, which contradicts to the definition of conserved quantaty.
* trajectories of conserved system are closed curves (different curves correspond to different energy value)
* Nonlinear centers for conservative systems

$\dot{\bar{x}} = \bar{f}(\bar{x})$, $\bar{x} \in \mathbb{R}^2$; $f \in C^1(\mathbb{R}^2)$

$E(\bar{x})$ - conserved quantaty. $\bar{x}^*$ - isolated fixed point (no others in small neighbourhood) if $\bar{x}^*$ is a local minimum of E then all trajectories sufficiently close to $\bar{x}^*$ are closed.

7

Exercise 6.5.11

Consider system $\ddot{x} = -x^3 + x - b\dot{x}$; Sketch the basin of attraction for fixed point at $(x^*, y^*) = (1, 0)$;

To go to the 2dimensional system we, as usually set $\dot{x} = y$; thus $\dot{y} = -x^3 + x - by$

Before going into details of damped oscillator dynamics let's refresh what we know about usual oscillator.

$$\ddot{x} = x - x^3$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 \end{cases}$$

fixed points are given by equations: $y = 0$ $\quad x - x^3 = 0$ thus fixed points are $(0,0)$; $(\pm 1, 0)$; Jacobian of the system

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{bmatrix}$$

① $(0;0)$ $\quad J(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\Delta = -1 < 0$ - saddle point. $\tau = 0$

stable manifold is one corresponding to negative eigenvalue $\lambda = -1$; equation for manifold is

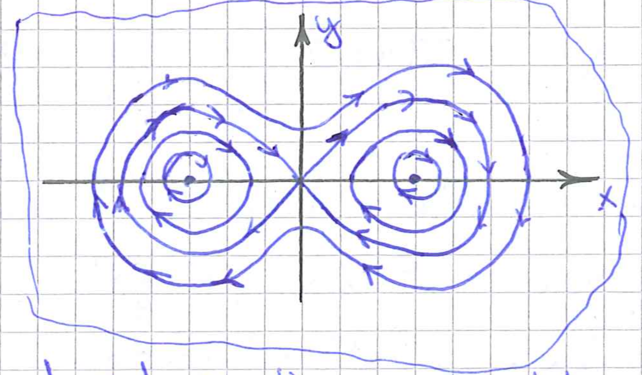
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{0} \Rightarrow \boxed{x = -y}$$ stable manifold.

unstable manifold is one corresponding to positive eigenvalue $\lambda = +1$; equation is $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{0}$ $\boxed{x = y}$ - unstable direction

② $(\pm 1, 0)$ point $\quad J(\pm 1; 0) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ $\quad \Delta = 2$, $\tau = 0$ - this points are centers

$$E = \frac{1}{2} y^2 - \frac{1}{2} x^2 + \frac{1}{4} x^4$$

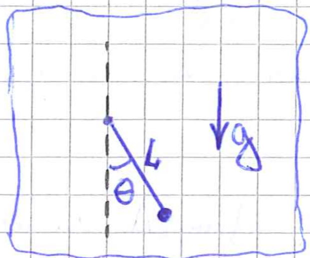
is conserved quantaty (energy in fact)



Here we observe for the first time homoclinic trajectories i.e. one that starts and end on the same point.

⑧ off topic : Where oscillator equations come from?

Let's consider simple pendulum



$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

$\omega = \sqrt{\frac{g}{L}}$ is frequency of this oscillator. now, if we introduce dimensionless time $\tau = \omega t$ we get equation

$\ddot{\theta} + \sin\theta = 0$; Now for small enough angles $\theta \ll 1$

$\sin\theta = \theta - \frac{1}{6}\theta^3 + O(\theta^3)$ and we get equation similar to the one given to us. But in our case we shouldn't think much about physical meaning.., and should just analize given 2d system.

Now let's move towards damped oscillator.

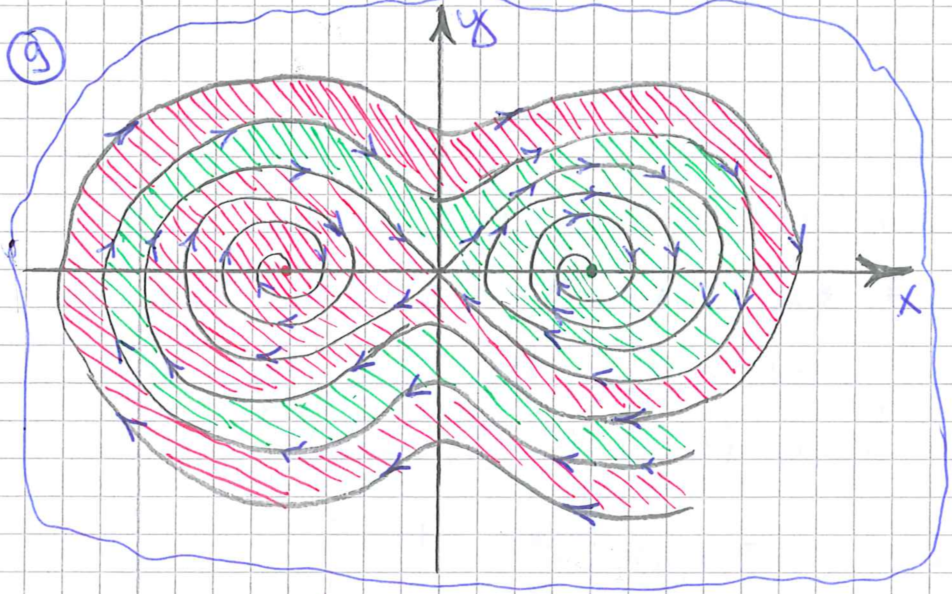
we assume $b \ll 1$ so it shouldn't effect position of fixed points and won't change saddle point behaviour, but as we will see further what it will affect is behaviour of centers.

the system corresponding to the damped oscillator is $\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 - by \end{cases}$ Jacobi matrix is $J(x,y) = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & -b \end{bmatrix}$

in particular $J(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & -b \end{bmatrix}$ $\Delta = -1$, $\tau = -b$ this is still saddle point whith approximately the same stable and unstable manifolds. Now if we look on $(\pm 1, 0)$ points we get: $J(\pm 1, 0) = \begin{bmatrix} 0 & 1 \\ -2 & -b \end{bmatrix}$ $\Delta = 2$, $\tau = -b$ $\tau^2 - 4\Delta > 0$ thus we

get stable spirals. Energy is not concerved anymore $E = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$;

$$\frac{dE}{dt} = \dot{x}\ddot{x} - x\dot{x} + x^3\dot{x} = \dot{x}\underbrace{(\ddot{x} - x + x^3)}_{-b\dot{x}} = -b\dot{x}^2 \leq 0;$$



Reversible systems

Some systems have time-reversal symmetry. For example any nondissipative mechanical systems are time reverseble $\ddot{x} = \frac{F(x)}{m}$ is indeed invariant under $t \to -t$ reflection. For 2d form of systems we usually have notation $\dot{x} = y$ and thus when we do $t \to -t$ reflection we should simultaneausly do $y \to -y$ reflection. So 2d system is reversible if it is invarian under $t \to -t$ and $y \to -y$ reflection, or

$$\begin{cases} \dot{x} = f(x,y) & f \text{ is odd in } y : f(x,-y) = -f(x,y); \\ \dot{y} = g(x,y) & g \text{ is even in } y : g(x,-y) = g(x,y); \end{cases}$$

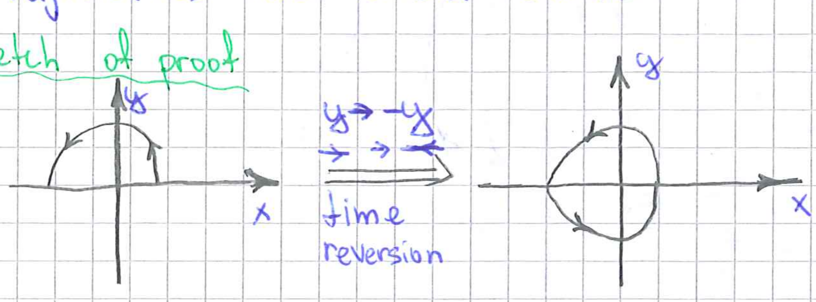
Nonlinear centers for reversible systems

$\bar{x}^* = 0$ - linear center for continiously differentiable system.

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

If the system is reversible, then sufficiently close to the origin, all trajectories are closed curves

sketch of proof



(10)

Ex 6.6.1

Show that each of the following systems is reversible, and sketch the phase portrait.

$$\begin{cases} \dot{x} = y(1-x^2) \\ \dot{y} = 1-y^2 \end{cases}$$

* fixed points are given by $y=\pm 1$; $x=\pm 1$; thus we get 4 fixed points: $\boxed{(1,1);\ (1,-1);\ (-1;1);\ (-1;-1);}$

* Jacobi matrix is $J(x,y) = \begin{bmatrix} -2xy & 1-x^2 \\ 0 & -2y \end{bmatrix}$

$J(1;1) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ $\Delta=4$, $\tau=-4$; $\lambda_1=\lambda_2=-2$ this is stable star at $(1;1)$;

$J(-1;-1) = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ $\Delta=-4$; $\tau=0$; $x^*-x=0$ is unstable manifold; $y^*-y=0$ is stable manifold; we get saddle point at $(-1;-1)$;

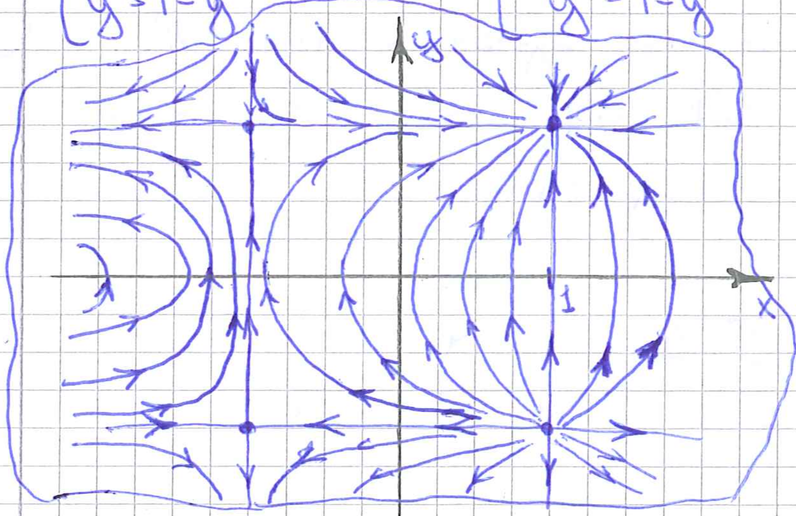
$J(1;-1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\Delta=4$, $\tau=4$; $\lambda_1=\lambda_2=2$ this is unstable star

$J(-1;1) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ $\Delta=-4$, $\tau=0$; $x-x^*=0$ is stable manifold; $y-y^*=0$ is unstable manifold.

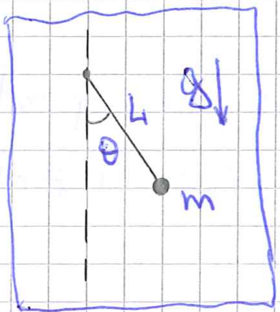
To show that system is reversible we should make change $y \to -y$, and $t \to -t$; then $\dot{x} \to -\dot{x}$, $\dot{y} \to \dot{y}$ and thus we get:

$$\begin{cases} \dot{x} = y(1-x^2) \\ \dot{y} = 1-y^2 \end{cases} \Longrightarrow \begin{cases} -\dot{x} = -y(1-x^2) \\ \dot{y} = 1-y^2 \end{cases}$$

the system is indeed invariant.



(11) Let's come back to our pendulum



as we have already shown dimensionless formulation of motion equations for pendulum is $\ddot{\theta} + \sin\theta = 0$ then

$$\begin{cases} \dot{\theta} = \nu \\ \dot{\nu} = -\sin\theta \end{cases}$$

Fixed point is given by $\nu^* = 0$ $\sin\theta^* = 0 \Rightarrow (\theta^*, \nu^*) = (\pi k, 0)$, $k \in \mathbb{Z}$

Let's look on points 0 and π because system is 2π-periodic and we can reproduce all other fixed points by simple shift. by 2π.

at $(\theta^*, \nu^*) = (0, 0)$ $J(\theta, \nu) = \begin{bmatrix} 0 & 1 \\ -\cos\theta & 0 \end{bmatrix}$

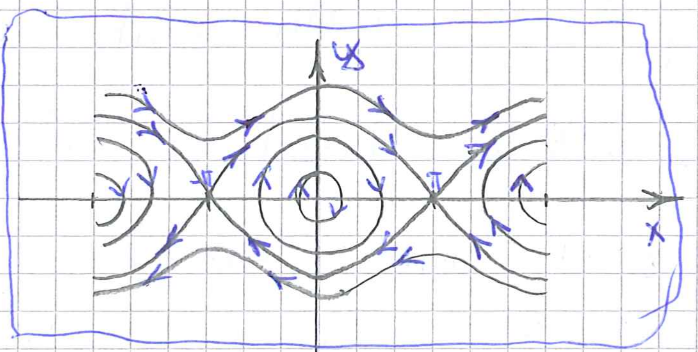
$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; $\Delta = 1$, $\tau = 0$ – at $(0,0)$ we get center.

now at $(\theta^*, \nu^*) = (\pi, 0)$ we get $J(\pi, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

thus $\Delta = -1$, $\tau = 0$ and we get saddle points (the same is valid for $(-\pi, 0)$ point)

stable direction $\lambda = -1$ the corresponding eigenvector is defined by $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{0} \Rightarrow \boxed{\bar{V}_s = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$

unstable direction $\lambda = 1$ eigenvector is defined by equation $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{0} \Rightarrow \boxed{\bar{V}_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$;



(12) The energy of the pendulum is given by
$E(\theta, \nu) = \frac{1}{2}\nu^2 - \cos\theta$

$E=-1$ - the lowest energy state corresponding to equlibrium point in $(\theta^*, \nu^*) = (0,0)$ (pendulum is hanging down)

$E=1$ - this is homoclinic orbit becase $E=1$ for $\theta = \pi$, $\nu = 0$ which for sure lies on this orbit. This is when pendulum makes half of rotation and then stops in the saddle point (pendulum is upside down). Oscillations for $-1 < E < 1$ are called librations.

For $E>1$ pendulum just whirls repeadetly over the top.

Now we can damp the oscillator just by adding damping force "$-b\dot{\theta}$" to the equations of motion. Then we get:

$\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$; $b>0$ then in similar way as in previous example

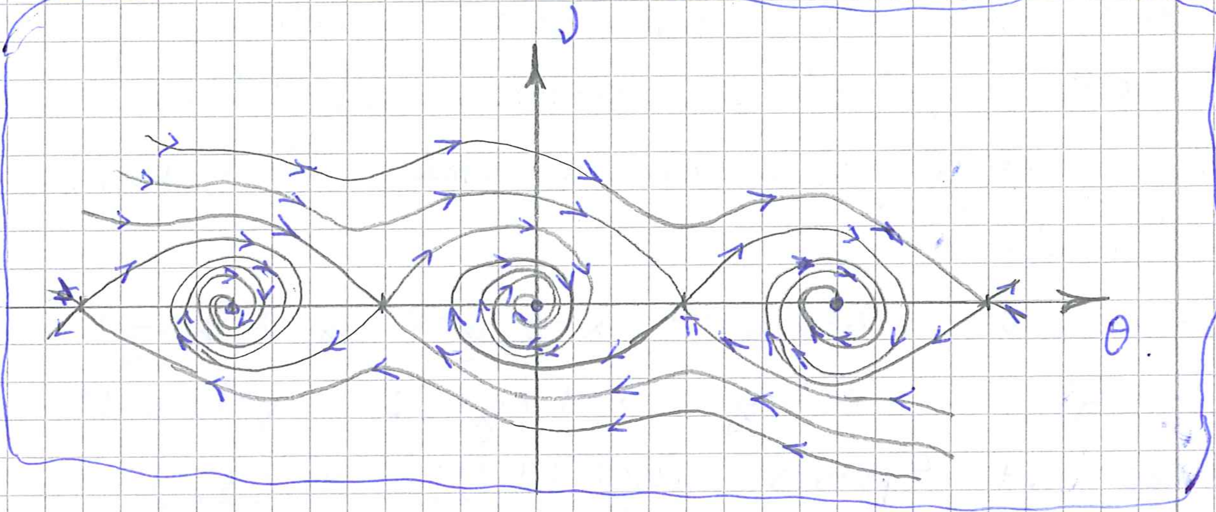
$$\begin{cases} \dot{\theta} = \nu \\ \dot{\nu} = -b\nu - \sin\theta \end{cases} \Rightarrow J(\theta, \nu) = \begin{bmatrix} 0 & 1 \\ -\cos\theta & -b \end{bmatrix}$$

if we take $b \ll 1$ position of fixed points is not changed very much and saddle point at $\theta^* = \pm\pi$ remains saddle point with approximately the same stable and unstable directions. Indeed

$J(\pi, 0) = \begin{bmatrix} 0 & 1 \\ 1 & -b \end{bmatrix}$ $\Delta = -1 < 0$ - saddle point.

But now centers will be changed to stable spirals.

$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & -b \end{bmatrix}$ $\Delta = +1$ but $\tau^2 - 4\Delta < 0$ still.
$\tau = -b < 0$ we get stable spiral.



The ~~ene~~ energy is no more conserved

$$\frac{dE}{d\tau} = \frac{d}{d\tau}\left(\frac{1}{2}\dot{\theta}^2 - \cos\theta\right) = \dot{\theta}(\ddot{\theta} + \sin\theta) = -b\dot{\theta}^2 < 0.$$

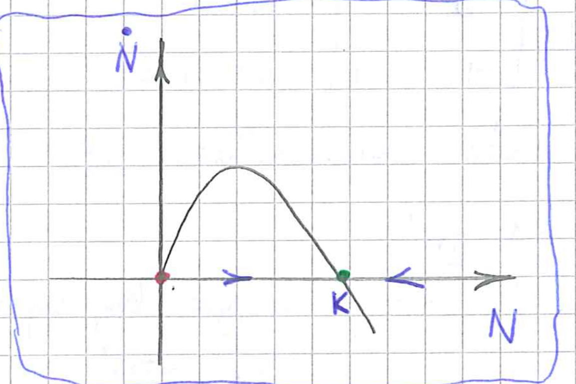
and the system becomes nonconservative

Rabbits versus sheep

Some words about logistic map.

Usually popolation of some species is described by so called logistic equation

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$



First term on the r.h.s. corresponds to exponential growth of population.

Second term creates limit of this growth - when $N=K$ growth stops. K is called carrying capacity. Now we consider rabitts and sheep which compete for the same food.

Ex 6.4.2

$$\begin{cases} \dot{x} = x(3-2x-y) \\ \dot{y} = y(2-x-y) \end{cases}$$

Thus because of conflicts for food rate of growth of both species is reduced.

Let's find fixed points, investigate stability, draw nullclines, sketch phase portraits and indicate the basins of attraction of any stable fixed point.

* Fixed points

$$\begin{cases} x(3-2x-y)=0 \\ y(2-x-y)=0 \end{cases} \quad x=0, y=0; \quad x=0, y=2; \quad y=0, x=\tfrac{3}{2}; \quad x=2-y, \quad 3-4+y=0; \quad y=1, x=1;$$

So we get 4 points $(0,0); (1,1); (0,2); (\frac{3}{2},0)$;

Now let's classify this fixed points

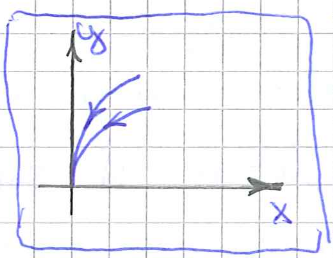
$$J(x,y)=\begin{bmatrix} 3-4x-y & -x \\ -y & 2-x-2y \end{bmatrix}; \quad J(0,0)=\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}; \quad \Delta=6>0, \quad \tau=5>0, \quad \tau^2-4\Delta=1>0$$

$\downarrow$ we get unstable node

$\lambda_1=3$; $\lambda_2=2$

fast direction $\lambda=3 \rightarrow y=0$

slow direction $\lambda=2 \rightarrow x=0$



(1,1) point

$$J(1,1)=\begin{bmatrix} -2 & -1 \\ -1 & -1 \end{bmatrix} \quad \Delta=1>0, \quad \tau=-3<0, \quad \tau^2-4\Delta=5>0$$ this point is stable node

eigenvalues are $\lambda=\frac{\tau}{2}\pm\frac{1}{2}\sqrt{\tau^2-4\Delta}$

$\lambda=-\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$;

fast ~~stable~~ direction is given by equation $\begin{bmatrix} -\frac{1}{2}+\frac{\sqrt{5}}{2} & -1 \\ -1 & \frac{1}{2}+\frac{\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}=\bar{0}$

$\lambda=-\frac{3+\sqrt{5}}{2}$

$$(y-y^*)=\frac{\sqrt{5}-1}{2}(x-x^*) \quad \text{or} \quad (y-y^*)\approx 0{,}6(x-x^*)$$

slow direction is given by equation $\begin{bmatrix} -\frac{1}{2}-\frac{\sqrt{5}}{2} & -1 \\ -1 & \frac{1}{2}-\frac{\sqrt{5}}{2} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}=\bar{0}$

$$(y-y^*)=-\frac{1+\sqrt{5}}{2}(x-x^*); \quad (y-y^*)=-1{,}6(x-x^*)$$

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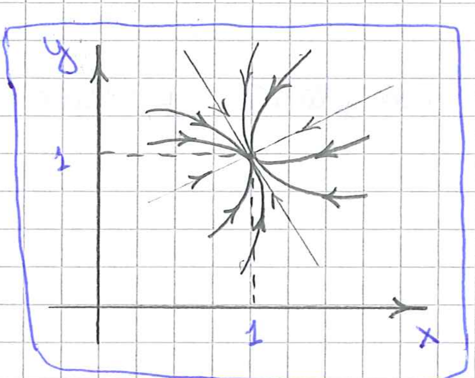


$\left(\frac{3}{2},0\right)$ point

$$J\left(\tfrac{3}{2},0\right) = \begin{bmatrix} -3 & -\frac{3}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \quad \Delta = -\frac{3}{2}; <0 - \text{saddle}, \quad \tau = -2{,}5 = -\frac{5}{2}$$

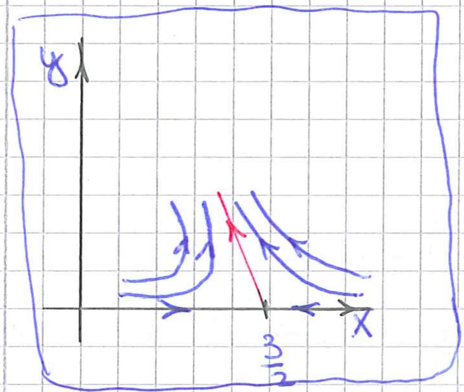
$$\tau^2 - 4\Delta = \frac{49}{4} = \left(\frac{7}{2}\right)^2$$

eigen values are $\lambda_{1,2} = \frac{1}{2}\left(-\frac{5}{2} \pm \frac{7}{2}\right) = \frac{1}{2}, -3$;

stable direction $\lambda = -3$; $(y - y^*) = 0$

unstable direction $\lambda = \frac{1}{2}$; $\begin{bmatrix} -\frac{7}{2} & -\frac{3}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0$ thus

unstable direction is $7(x - x^*) = -3(y - y^*)$;



$(0,2)$ point

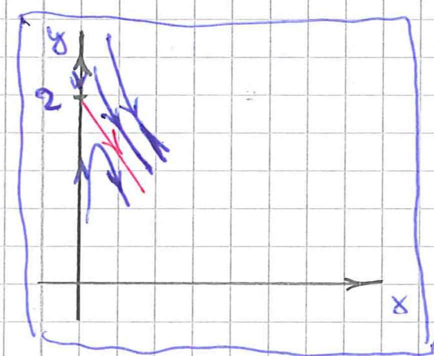
$$J(0,2) = \begin{bmatrix} 1 & 0 \\ -2 & -2 \end{bmatrix} \quad \Delta = -2 < 0, \quad \tau = -1 < 0$$

$\tau^2 - 4\Delta = 9 = 3^2$ this is saddle point

$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$; $\lambda = 1$; $\lambda = -2$;

stable dir. $\lambda = -2$ $(x - x^*) = 0$

unstable direction $\lambda = 1$ $\begin{bmatrix} 0 & 0 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0 \Rightarrow (y - y^*) = -\frac{3}{2}(x - x^*)$



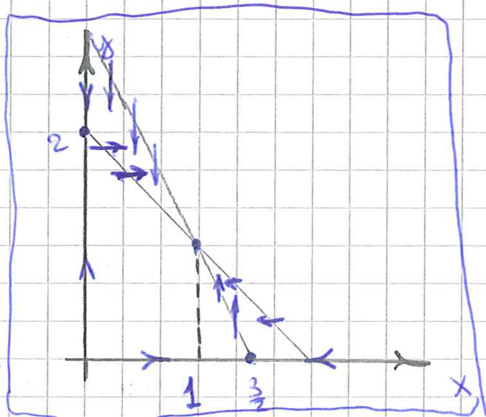
Nullclines

vertical $\dot{x} = 0$: $x = 0$ the $\dot{y} = y(2 - y)$

$+$ $y = 3 - 2x$ then $\dot{y} = (3 - 2x)(x - 1)$

horizontal $\dot{y} = 0$: $y = 0$ $\dot{x} = x(3 - 2x)$

$+$ $y = 2 - x$, $\dot{x} = x(1 - x)$



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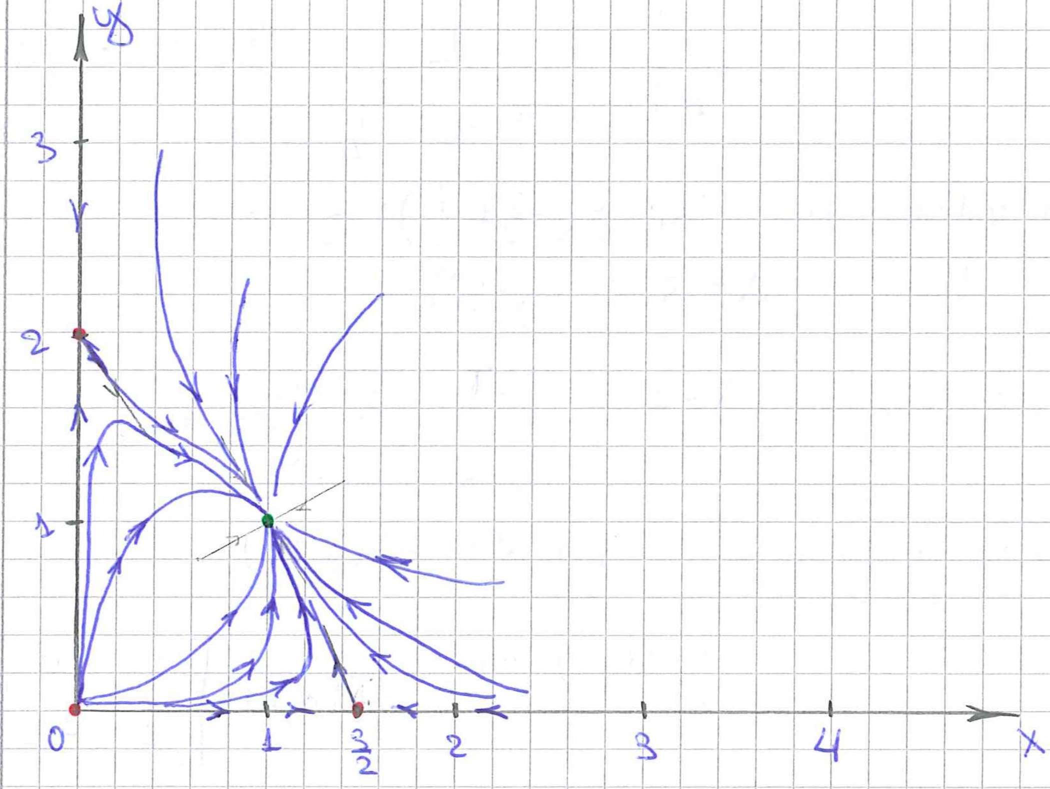
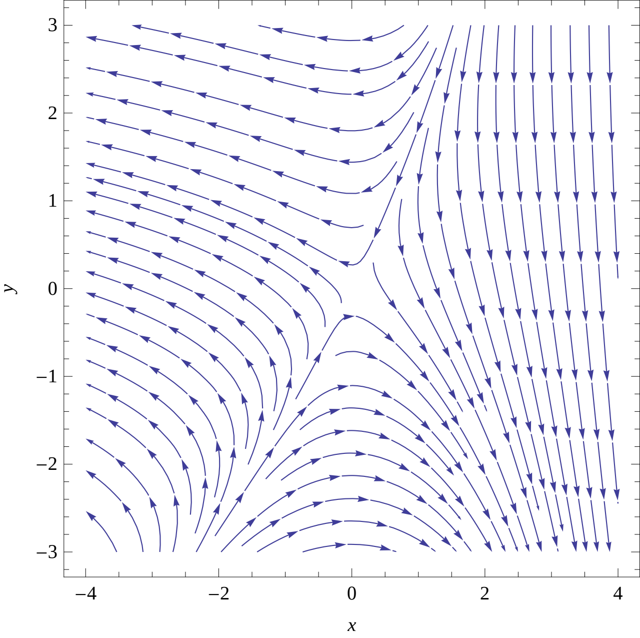
Finally we can draw complete picture of flow.



Basins of attraction is by definition set of initial conditions $\bar{x}_0$ such that $\bar{x}(t) \to \bar{x}^*$ as $t \to \infty$, all trajectories that don't start on the axis exactely end up on $(1,1)$ fixed point and thus eventually populations of sheeps and rabbits are equall.



3
2
1
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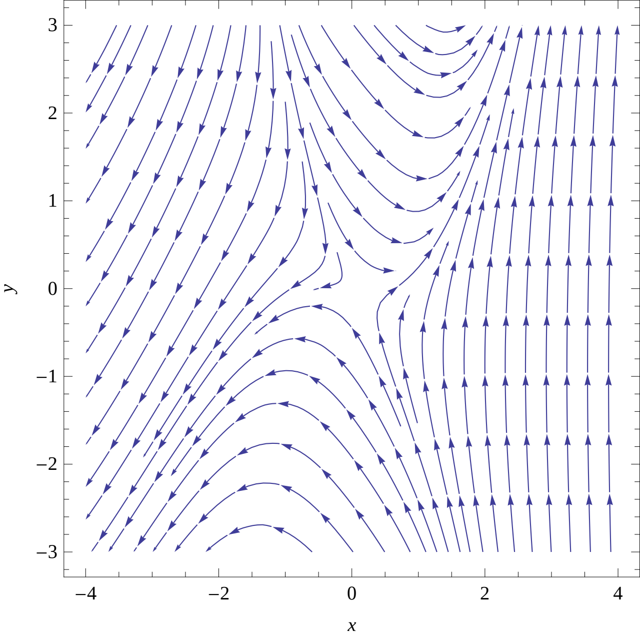
−1
−2
−3
y
−4
−2
0
2
4
x



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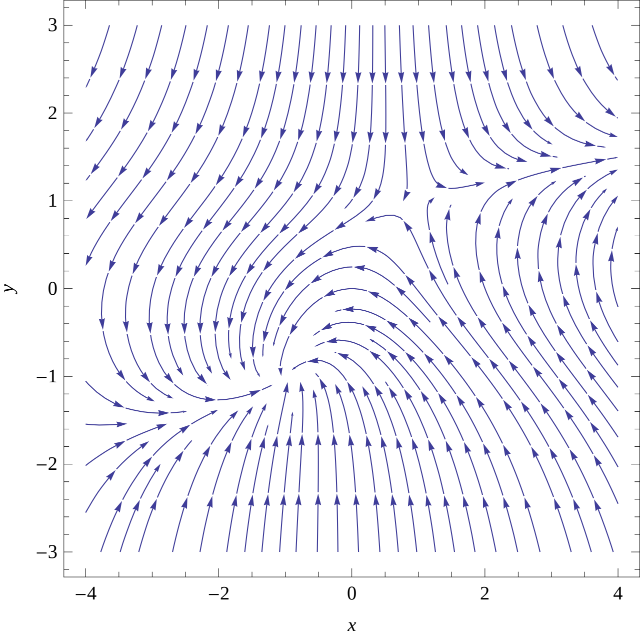

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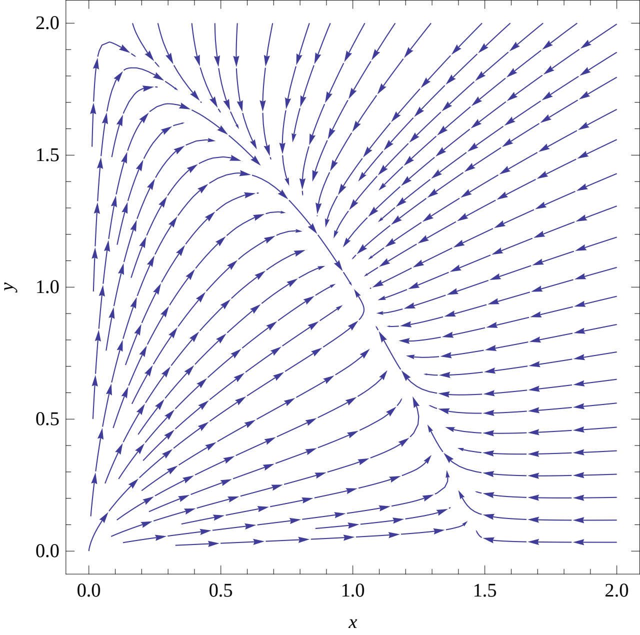
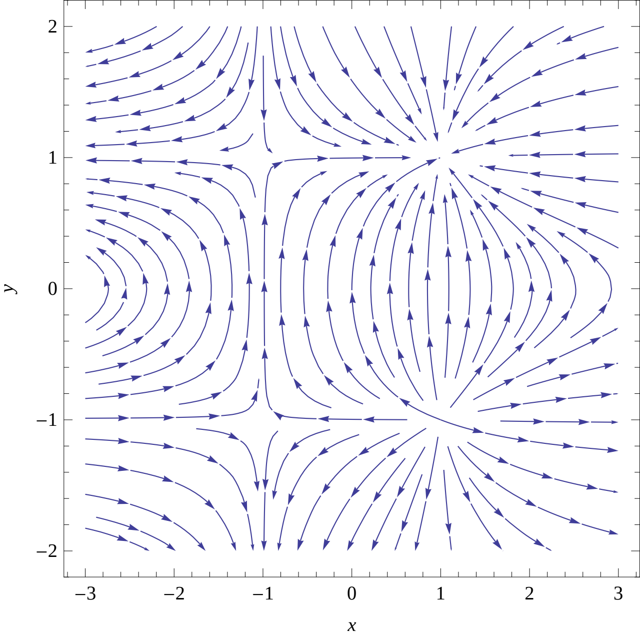


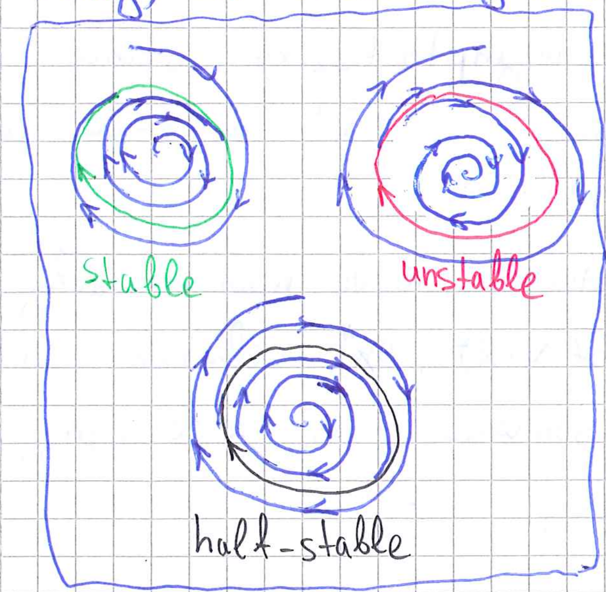
2.0
1.5
1.0
0.5
0.0
y
0.0
0.5
1.0
1.5
2.0
x



2
1
0
−1
−2
y
−3
−2
−1
0
1
2
3
x

① Limit cycles (classes 8 and 9)

Def Limit cycles are isolated closed trajectories (neighbouring trajectories are not closed)
Physical meaning - self-sustained oscillations (systems oscillate even in absence of external periodic forcing) In linear systems closed orbits don't exist as $\bar{x}(t) = c\bar{x}_0(t)$ is solution too and all perturbations exists forever We get whole family of closed orbits (so there are no isolated closed orbits)



How to rule out?
$\dot{\bar{x}} = -\bar{\nabla} V$ - suppose that system can be written in such form, where $V(\bar{x})$ is some continiously differentiable function (it's called potential function) In this case we call gradient system.

Ⓘ Theorem Closed orbits are impossible in gradient systems. Let's assume that we have closed orbit and consider change of $V(\bar{x})$ along it

$$\Delta V = \int_0^T \frac{dV}{dt}\,dt = \int_0^T (\bar{\nabla} V \cdot \dot{\bar{x}})\,dt = -\int_0^T |\dot{\bar{x}}|^2 dt < 0 \quad \text{thus } \Delta V \neq 0$$

(and it should be 0 for closed orbit as V is single valued function) only if $\dot{\bar{x}} = 0$ which is defenition of fixed point rather then closed orbit.

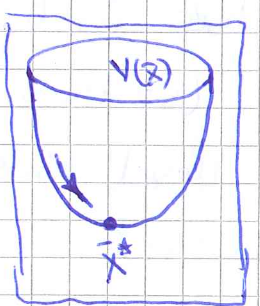
Ⓘⓘ Second way to rule out closed orbit is to construct Lyapunov Functions.

② It is possible to construct an energy-like function that decreases along trajectory. $\dot{\bar{x}} = \bar{f}(\bar{x})$ is our system, $\bar{x}^*$ is fixed point. There exists such real-valued function (Lyapunov function) such that:

ⓐ $V(\bar{x}) > 0$ for all $\bar{x} \neq \bar{x}^*$ and $V(\bar{x}^*) = 0$;

ⓑ $\dot{V}(\bar{x}(t)) < 0$ for all $\bar{x} \neq \bar{x}^*$.

Then $\bar{x}^*$ is globally asympthotically stable, i.e. for all initial conditions $\bar{x}(t) \to \bar{x}^*$ as $t \to \infty$ and system has no closed orbits.



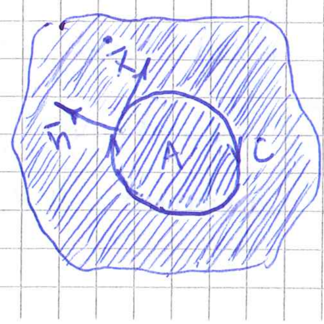
All trajectories move monotonically down the graph of $V(\bar{x})$, and solution can't get stuck somwhere as $\dot{V} < 0$ everywhere except $\bar{x} = \bar{x}^*$;

Dulacs criterion

Let $\dot{\bar{x}} = \bar{f}(\bar{x})$ be a continiously differentiable vector field defined on a simply connected subset R of the plane. If there exists continiously differentiable real-valued function $g(\bar{x})$ such that $\bar{\nabla} \cdot (g \dot{\bar{x}})$ has one sign throughout R, then there are no closed orbits lying entirely in R.

Proof C - closed orbit lying entirely in R region

A is region inside C $\iint_A \bar{\nabla}(g\dot{\bar{x}})dA = \oint_C g\dot{\bar{x}}\bar{n}\,dl$ which should be equal 0 if C is closed trajectorie as $\dot{\bar{x}} \perp \bar{n}$ and $\dot{\bar{x}}\bar{n} = 0$ but at the same time we know that $\bar{\nabla}(g\dot{\bar{x}}) > 0$ everywhere and thus $\iint_A \bar{\nabla}(g\dot{\bar{x}})dA \neq 0$



We came to contradiction which

③ lead us to the conclusion that there are no such closed orbit C.

Problems with Lyapunov function and Dulac's criterion is that there is no recepie to find $V(\bar{x})$ or $g(\bar{x})$ usually

- Lyapunov function usually consists of squares of x and y variables.
- Dulacs criterion usually use $\frac{1}{x^a y^b}$, e^{ax}, e^{ay} functions.

Examples

Ex 7.2.6

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

assume we want to obtain gradient system. For this we want the following system to be satisfied:

$$\begin{cases} f(x,y) = -\frac{\partial}{\partial x} V(x,y) \\ g(x,y) = -\frac{\partial}{\partial y} V(x,y) \end{cases}$$

For this system to have solutions the following condition should be satisfied.

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial x} g(x,y) \quad \left(\text{because } \frac{\partial^2}{\partial x \partial y} V(x,y) = \frac{\partial^2}{\partial y \partial x} V(x,y)\right)$$

ⓐ $\begin{cases} \dot{x} = y^2 + y\cos x \\ \dot{y} = 2xy + \sin x \end{cases}$

First of all let's check if this system can be written as at all.

$$\frac{\partial}{\partial y} f(x,y) = 2y + \cos x \, ; \quad \frac{\partial}{\partial x} g(x,y) = 2y + \cos x = \frac{\partial}{\partial y} f(x,y)$$

thus this system is gradient indeed. now we can integrate.

$$\frac{\partial V}{\partial x} = -f(x,y) = -y^2 - y\cos x \Rightarrow \quad V(x,y) = -xy^2 - y\sin x + C_1(y)$$

$$\frac{\partial V}{\partial y} = -g(x,y) = -2xy - \sin x \quad V(x,y) = -xy^2 - y\sin x + C_2(x)$$

Comparing this expressions we see that $C_1(y) = C_2(x) = C$

constant term is not relevant for the potential and thus we get:

$$\boxed{V(x,y) = -y^2x + y\sin x.}$$

Ex 7.2.12 (Lyapunov function)

Show that $\dot{x} = -x+2y^3-2y^4$; $\dot{y} = -x-y+xy$ has no periodic solutions (Hint: choose a, m and n such that $V = x^m + ay^n$ is a Lyapunov function)

Let's see what a, m, n make function $V(x,y) = x^m + ay^n$ satisfying Lyapunov function conditions.

$\dot{V} = m\dot{x}x^{m-1} + an\dot{y}y^{n-1} = m(2y^3-2y^4-x)x^{m-1} +$
$+ an(xy-y-x)y^{n-1}$. First of all for this term to be sign-determined let's take away cross terms, i.e. one containing both x and y, i.e:

cross terms $2my^3(1-y)x^{m-1} - anx(1-y)y^{n-1} = (1-y)(2my^3x^{m-1} -$
$- anxy^{n-1}) = 0$ this condition leads to

- same powers of x: $m=2$; - same coefficients: $2m = an$
- same powers of y: $n=4$; $a=1$

thus Lyapunov function $V(x) = x^2+y^4 > 0$ expcept $(0,0)$ point.

$\dot{V} = -mx^m - any^n = -2x^2 - 4y^4 < 0$ - Lyapunov function is indeed $V(x,y) = x^2+y^4$, thus there are no periodic solutions.

Ex 7.2.13

Let's recall the competition model.

$$\begin{cases} \dot{x} = r_1\left(x - \frac{x^2}{K_1}\right) - b_1xy \\ \dot{y} = r_2y\left(1-\frac{y}{K_2}\right) - b_2xy \end{cases}$$

Using Dulac's criterion with the weighting function

$g(x,y) = \frac{1}{xy}$; Let's find $\vec{\nabla}(g\dot{\vec{x}}) = \frac{\partial}{\partial x}\left(\frac{1}{xy}\dot{x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{xy}\dot{y}\right)$

$\frac{\partial}{\partial x}\left(\frac{\dot{x}}{xy}\right) = \frac{\partial}{\partial x}\left(r_1\cdot\frac{1}{y}\left(1-\frac{x}{K_1}\right)\right) - \frac{\partial}{\partial x}b_1 = -\frac{r_1}{K_1y}$;

(5) $\frac{\partial}{\partial y}\left(\frac{\dot{y}}{xy}\right) = \frac{\partial}{\partial y}\left(r_2 \frac{1}{x}\left(1-\frac{y}{k_2}\right)\right) - \frac{\partial}{\partial y} b_2 = -\frac{r_2}{x k_2}$

thus

$\bar{\nabla}(g\dot{\bar{x}}) = -\frac{r_1}{k_1 y} - \frac{r_2}{k_2 x} < 0$ when $x, y > 0$ thus in firs

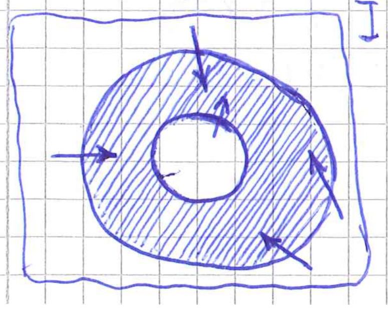
quadrant Dulac's criterion is working.

Poincare – Bendixson theorem

We have studied how to rule out closed orbits. Now natural question is how to establish the existence of closed orbit. Let's suppose that

1. R is a closed bounded subset of the plane.
2. $\dot{\bar{x}} = \bar{f}(\bar{x})$ is a continuously differentiable vector field on an open set containing R.
3. R doesn't contain any fixed points
4. There exists trajectory C that is confined in R in R (starts in R and stays there forever)

Then either C is a closed orbit or it spirals toward a closed orbit as $t\to\infty$; In either case, R contains a closed orbit. or it spirals toward a closed orbit as $t\to\infty$ (in either case R contains closed orbit) First 3 points can be easily satisfied But how can we conclude about existence of trapped trajectory? Standart trick is to construct a trapping region R, i.e. a closed connected set such that the vector field points "inward" everywhere on the boundary of R



It is usually used when the system has a simple representation in polar coordinates

(6) Ex 7.3.1

Consider

$$\begin{cases} \dot{x} = x - y - x(x^2+5y^2) \\ \dot{y} = x + y - y(x^2+y^2) \end{cases}$$

(a) classify fixed points in the origin.

$$J(x,y) = \begin{bmatrix} 1-3x^2-5y^2 & -1-10xy \\ 1-2xy & 1-3y^2-x^2 \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}; \quad \Delta = 2 > 0 \quad \tau^2 - 4\Delta = -4 < 0 \text{ and we}$$
$$\tau = 2 \quad \text{get unstable spiral.}$$

(b) rewrite system in polar coordinates

For this we should rewrite time derivatives of r and θ:

$r^2 = x^2 + y^2 \Rightarrow \boxed{r \cdot \dot{r} = x\dot{x} + y\dot{y}}$;

$\cos\theta = \frac{x}{r}$; $\sin\theta \cdot \dot{\theta} = \frac{y}{r}$ $\dot{\theta} = -\frac{\dot{x}}{r} + \frac{x}{r^2}\dot{r} =$

$= -\frac{1}{r^3}(\dot{x} \cdot r^2 - x r \cdot \dot{r}) = -\frac{1}{r^3}(\dot{x} r^2 - x^2\dot{x} - xy\dot{y}) =$

$= -\frac{1}{r^3}(y^2\dot{x} - xy \cdot \dot{y})$ and thus $\boxed{\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}}$

Now we can use our system of equations to write down $\dot{r}$ and $\dot{\theta}$

$r \cdot \dot{r} = x(x - y - x(x^2+5y^2)) + y(x + y - y(x^2+y^2)) =$

$= x^2 - xy - x^4 - 5x^2y^2 + xy + y^2 - y^2x^2 - y^4 = x^2 + y^2 - (x^2+y^2)^2 - 4x^2y^2 =$

$= r^2 - r^4 - r^4\sin^2 2\theta$;

$$\boxed{\dot{r} = r - r^3(1 + \sin^2 2\theta);}$$

$r^2\dot{\theta} = x(x + y - y(x^2+y^2)) - y(x - y - x(x^2+5y^2)) =$

$= x^2 + y^2 - xy^3 - x^3y + x^3y + 5y^3x = r^2 + 4xy^3 =$

$= r^2 + 4r^4\cos\theta \cdot \sin^3\theta = r^2 + r^4\sin 2\theta(1 - \cos 2\theta)$

$$\boxed{\dot{\theta} = 1 + r^2\sin 2\theta(1 - \cos 2\theta);}$$

⑦ $\begin{cases} \dot r = r - r^3\left(1+\sin^2(2\theta)\right) \\ \dot\theta = 1 + r^2 \sin 2\theta(1-\cos 2\theta) \end{cases}$

ⓒ Determine the circle of maximum radius r_1, centered on the origin such that all trajectories have a radially outward component on it.

$\dot r > 0 \quad r - r^3\left(1+\sin^2(2\theta)\right) > 0 \Rightarrow \quad r^2 < \frac{1}{1+\sin^2 2\theta} < \frac{1}{2}$

$r < \frac{1}{\sqrt{2}}$ thus $\boxed{r_1 = \frac{1}{\sqrt{2}};}$

ⓓ Determine the circle of minimum radius r_2, centered on the origin such that all trajectories

$\dot r < 0 \quad r - r^3\left(1+\sin^2(2\theta)\right) < 0 \quad r^2 > \frac{1}{1+\sin^2(2\theta)} > 1$

$\boxed{r_{min} = 1}$

ⓔ Prove that the system has a limit cycle somewhere inside this region

To prove this we should use Poincare-Bendixson theorem. The last thing that should be proven is that there are no fixed points inside the region

Fixed points are given by equations:

$\begin{cases} 1 - r^2\left(1+\sin^2 2\theta\right) = 0 \\ 1 + r^2 \sin 2\theta\left(1-\cos 2\theta\right) = 0 \end{cases} \qquad r^2 = \frac{1}{1+\sin^2 2\theta}$

thus equation for θ^* is given by

$1 + \sin^2 2\theta + \sin 2\theta\left(1-\cos 2\theta\right) = 0$

Let's understand how do function $f(\theta) = 1 + \sin^2\theta + \sin\theta - \sin\theta \cdot \cos\theta$ looks like? let's find minimum of this function

$\frac{\partial f}{\partial \theta} = \sin 2\theta + \cos\theta - \cos 2\theta$

$\frac{\partial^2 f}{\partial \theta^2} = 2(\cos 2\theta + \sin 2\theta) - \sin\theta$

(8) this strategy is streight forward but difficult to put in life by hands thus what we need is another strategy of solving. Let's express our expression in the following form:

$$1+4\sin^2\theta\cos^2\theta+4\sin^3\theta\cdot\cos\theta=1+4\sin^2\theta\left(\cos^2\theta+\frac{1}{2}\sin 2\theta\right)=$$

$$=1+2\sin^2\theta\,(1+\cos 2\theta+\sin 2\theta)$$

$-\sqrt{2}<\cos 2\theta+\sin 2\theta<\sqrt{2}$ and $\cos 2\theta+\sin 2\theta=-\sqrt{2}$ when $2\theta=\frac{7\pi}{4}$; when $\cos 2\theta=-\frac{1}{\sqrt{2}}$; $2\sin^2\theta=1-\cos 2\theta=\frac{\sqrt{2}+1}{\sqrt{2}}$

Then our expressio in this point becomes:

$$1+2\sin^2\theta\cdot(1+\cos 2\theta+\sin 2\theta)=1+\frac{\sqrt{2}+1}{\sqrt{2}}(1-\sqrt{2})=1+\frac{1-2}{\sqrt{2}}=$$

$=1-\frac{1}{\sqrt{2}}=\frac{\sqrt{2}-1}{\sqrt{2}}=\frac{0{,}4}{1{,}4}>0$ and this is the minimal value function takes, and it is still positive that means that it never goes to 0.

Note This proof is not actually rigid enough. What you should really do is find minimum of the function streght forward.

⑨ Ex 7.3.3

Show that

$$\begin{cases} \dot{x} = x - y - x^3 \\ \dot{y} = x + y - y^3 \end{cases}$$ has a periodic solution

Here again it will be useful to go to radial coordinates assuming that

$$\begin{cases} r\dot{r} = x\dot{x} + y\dot{y} \\ r^2\dot{\theta} = x\dot{y} - y\dot{x} \end{cases}$$ thus

$$r\dot{r} = x(x-y-x^3) + y(x+y) - y^4 = x^2+y^2-xy+xy-x^4-y^4 =$$
$$= r^2 - (x^2+y^2)^2 + 2x^2y^2 = r^2 - r^4 + 2r^4 \cdot \sin^2\theta \cdot \cos^2\theta.$$
$$r^2\dot{\theta} = x(x+y-y^3) - y(x-y-x^3) = x^2+y^2+xy-xy-xy^3+yx^3 =$$
$$= r^2 + xy(x^2-y^2) = r^2 + r^2 \cdot \sin\theta \cdot \cos\theta\,(-\sin^2\theta + \cos^2\theta) =$$
$$= r^2\left(1 + \frac{1}{2}\sin 2\theta \cdot \cos 2\theta\right) = r^2\left(1 + \frac{1}{4}\sin 4\theta\right);$$

thus we get following system.

$$\begin{cases} \dot{r} = r - r^3 + \frac{1}{2} r^3 \sin^2 2\theta \\ \dot{\theta} = 1 + \frac{1}{4}\sin 4\theta \end{cases}$$ now we use $1 - \frac{1}{2}\sin^2 2\theta = 1 - \frac{1}{4} + \frac{1}{4}\cos 4\theta = \frac{1}{4}(3 + \cos 4\theta)$

and finally we get the following system.

$$\begin{cases} \dot{r} = r - \frac{1}{4} r^3 (3 + \cos 4\theta) \\ \dot{\theta} = 1 + \frac{1}{4} r^2 \sin 4\theta \end{cases}$$

* First we find maximal radius circuit with vectors going outward ($\dot{r} > 0$)

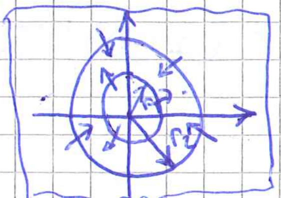
$\dot{r} > 0 \Rightarrow 1 - \frac{1}{4} r^2 (3 + \cos 4\theta) > 0 \Rightarrow r^2 < \frac{4}{3 + \cos 4\theta} < 1$, thus

$\boxed{r_1 = 1}$

* Now we find minimal radius circuit with vectors going inwards ($\dot{r} < 0$)

$\dot{r} < 0 \Rightarrow 1 - \frac{1}{4} r^2 (3 + \cos 4\theta) < 0 \Rightarrow r^2 > \frac{4}{3 + \cos 4\theta} > 2 \qquad r > \sqrt{2}$

$\boxed{r_2 = \sqrt{2}}$



(10) * finally the last thing we should check here is that there are no fixed points inside traping region $1 < r < \sqrt{2}$

$$\begin{cases} \dot{r} = 0 \\ \dot{\theta} = 0 \end{cases} \Rightarrow \begin{cases} 1 - \frac{1}{4} r^2 (3 + \cos 4\theta) = 0 \\ 1 + \frac{1}{4} r^2 \sin 4\theta = 0 \end{cases} \Rightarrow$$

$\Rightarrow \frac{1}{4}\sin 4\theta + \frac{1}{4}(3 + \cos 4\theta) = 0 \Rightarrow \sin 4\theta + \cos 4\theta + 3 = 0$ - obviously this equation have no solutions because $\sin 4\theta + \cos 4\theta < \sqrt{2}$ always, thus indeed we get limit cycle somwhere inside traping region $1 < r < \sqrt{2}$

Exercise 7.3.11 Cycle graphs

Suppose $\dot{\bar{x}} = \bar{f}(\bar{x})$ is a smooth vector field on R^2
An improved Poincare-Bendixson theorem states that if a trajectory is trapped in a compact region, then it must approach a fixed point, a closed orbit or a cycle graph. (an invariant set containing a finite number of fixed points connected by a finite number of trajectories all oriented either clockwise or counterclockwise)

(a) Plot phase portrait for the system.

$$\begin{cases} \dot{r} = r(1-r^2)\left[r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2\right] \\ \dot{\theta} = r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2 \end{cases}$$

$r^2\sin^2\theta + (r^2\cos^2\theta - 1)^2 = F(r,\theta)$

$$\begin{cases} \dot{r} = r(1-r^2)F(r,\theta) \\ \dot{\theta} = F(r,\theta) \end{cases}$$

in cartesian coordinate $F(r,\theta) = F(x,y) = y^2 + (x^2-1)^2 \geq 0$
$F(x,y) = 0$ where $\boxed{(x,y) = (\pm 1, 0)}$ Thus $r = 0$ is not fixed point

(9) Ex 7.3.3

Show that

$$\begin{cases} \dot{x} = x - y - x^3 \\ \dot{y} = x + y - y^3 \end{cases}$$ has a periodic solution

Here again it will be useful to go to radial coordinates assuming that

$$\begin{cases} r\dot{r} = x\dot{x} + y\dot{y} \\ r^2\dot{\theta} = x\dot{y} - y\dot{x} \end{cases}$$ thus

$$r\dot{r} = x(x-y-x^3) + y(x+y) - y^4 = x^2 + y^2 - xy + xy - x^4 - y^4 =$$
$$= r^2 - (x^2+y^2)^2 + 2x^2y^2 = r^2 - r^4 + 2r^4 \cdot \sin^2\theta \cdot \cos^2\theta.$$
$$r^2\dot{\theta} = x(x+y-y^3) - y(x-y-x^3) = x^2 + y^2 + xy - xy - xy^3 + yx^3 =$$
$$= r^2 + xy(x^2-y^2) = r^2 + r^2 \cdot \sin\theta \cdot \cos\theta \, (-\sin^2\theta + \cos^2\theta) =$$
$$= r^2\left(1 + \frac{1}{2}\sin 2\theta \cdot \cos 2\theta\right) = r^2\left(1 + \frac{1}{4}\sin 4\theta\right);$$

thus we get following system.

$$\begin{cases} \dot{r} = r - r^3 + \frac{1}{2}r^3\sin^2 2\theta \\ \dot{\theta} = 1 + \frac{1}{4}\sin 4\theta \end{cases}$$ now we use

$$1 - \frac{1}{2}\sin^2 2\theta = 1 - \frac{1}{4} + \frac{1}{4}\cos 4\theta = \frac{1}{4}(3 + \cos 4\theta)$$

and finally we get the following system.

$$\begin{cases} \dot{r} = r - \frac{1}{4}r^3(3 + \cos 4\theta) \\ \dot{\theta} = 1 + \frac{1}{4}r^2\sin 4\theta \end{cases}$$

* First we find maximal radius circuit with vectors going outward ($\dot{r} > 0$)

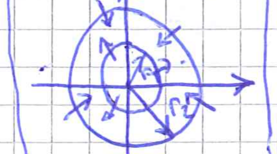
$\dot{r} > 0 \Rightarrow 1 - \frac{1}{4}r^2(3 + \cos 4\theta) > 0 \Rightarrow r^2 < \frac{4}{3 + \cos 4\theta} < 1$, thus

$\boxed{r_1 = 1}$

* Now we find minimal radius circuit with vectors going inwards ($\dot{r} < 0$)

$\dot{r} < 0 \Rightarrow 1 - \frac{1}{4}r^2(3 + \cos 4\theta) < 0 \Rightarrow r^2 > \frac{4}{3 + \cos 4\theta} > 2 \qquad r > \sqrt{2}$

$\boxed{r_2 = \sqrt{2}}$



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* finally the last thing we should check here is that there are no fixed points inside traping region $1<r<\sqrt{2}$

$$\begin{cases} \dot{r}=0 \\ \dot{\theta}=0 \end{cases} \Rightarrow \begin{cases} 1-\frac{1}{4}r^2(3+\cos 4\theta)=0 \\ 1+\frac{1}{4}r^2\sin 4\theta=0 \end{cases} \Rightarrow$$

$\Rightarrow \frac{1}{4}\sin 4\theta+\frac{1}{4}(3+\cos 4\theta)=0 \Rightarrow \sin 4\theta+\cos 4\theta+3=0$ - -obviously this equation have no solutions because $\sin 4\theta+\cos 4\theta<\sqrt{2}$ always, thus indeed we get limit cycle somewhere inside traping region $1<r<\sqrt{2}$

Exercise 7.3.11 Cycle graphs

Suppose $\dot{\bar{x}}=\bar{f}(\bar{x})$ is a smooth vector field on R^2. An improved Poincare - Bendixson theorem states that if a trajectory is trapped in a compact region, then it must approach a fixed point, a closed orbit or a cycle graph. (an invariant set containing a finite number of fixed points connected by a finite number of trajectories all oriented either clockwise or counterclockwise)

ⓐ Plot phase portrait for the system.

$$\begin{cases} \dot{r}=r(1-r^2)\left[r^2\sin^2\theta+(r^2\cos^2\theta-1)^2\right] \\ \dot{\theta}=r^2\sin^2\theta+(r^2\cos^2\theta-1)^2 \end{cases}$$

$$r^2\sin^2\theta+(r^2\cos^2\theta-1)^2=F(r,\theta)$$

$$\begin{cases} \dot{r}=r(1-r^2)F(r,\theta) \\ \dot{\theta}=F(r,\theta) \end{cases}$$

in cartesian coordinate $F(r,\theta)=F(x,y)=y^2+(x^2-1)^2\geqslant 0$

$F(x,y)=0$ where $\boxed{(x,y)=(\pm 1,0)}$ Thus $r=0$ is not fixed point

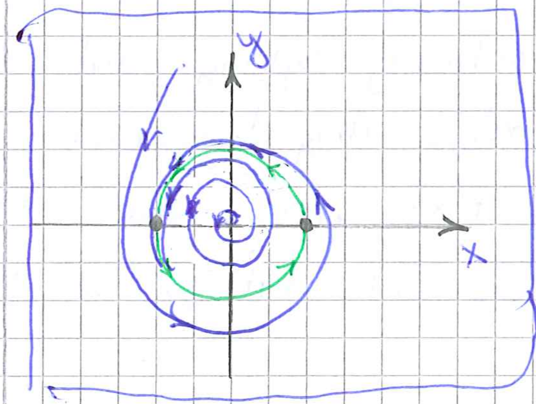
⑪ So fixed points are $(x,y)=(\pm1,0)$ or

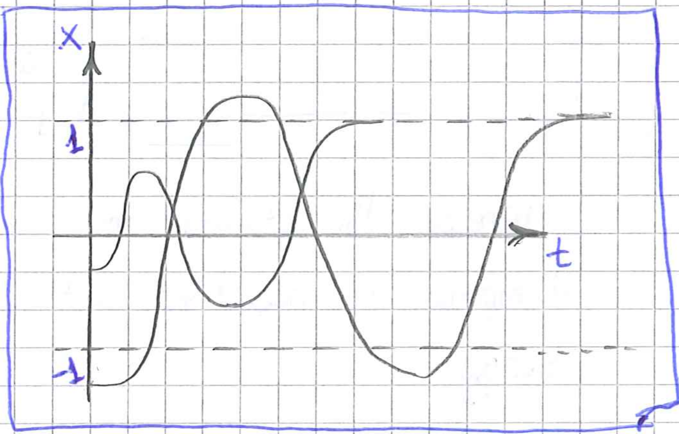
$$\boxed{(\theta,r)=(\pi,1);\ (\theta,r)=(0,1)}$$

as $F(r,\theta)>0$ everywhere except fixed points we get

$\dot r>0$ for $r<1$ and $\dot r<0$ for $r>1$ for any θ

$\dot\theta\geq0\ \forall\theta$ $\dot\theta=0$ for fixed points.





Lienard systems

$\ddot x+f(x)\cdot\dot x+g(x)=0$ - Lienard's equation. This is very general second order equation describing nonlinear oscillatons (generalization of the Van der Pol oscillatior

$\ddot x+\mu(x^2-1)\dot x+x=0$;

$$\begin{cases}\dot x=y\\ \dot y=-f(x)y-g(x)\end{cases}$$

Lienards theorem

Suppose that

① $f(x)$ and $g(x)$ are continiously differentiable for all x;

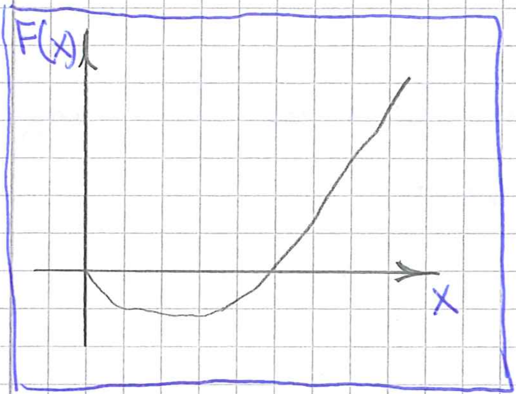
② $g(-x)=-g(x)$ for all x (i.e. $g(x)$ is an odd function);

③ $g(x)>0$ for $x>0$;

④ $f(-x)=f(x)$ for all x (i.e. $f(x)$ is even function)

⑤ The odd function $F(x)=\int\limits_0^x f(u)\,du$ has exactly one

(12) positive zero at $x=a$, is negative $0<x<a$, is positive and nondecreasing for $x>a$, and $F(x)\to\infty$ as $x\to\infty$, then system has unique, stable limit circle around 0.



Physical meaning

assumptions for restoring force $g(x)$ mean that the restoring force acts like an ordinary spring, and tends to reduce any displacement, whereas the assumptions on $f(x)$ imply that the damping is negative at small $|x|$ and positive at large.

Ex 7.4.2

Consider equation $\ddot{x}+\mu(x^4-1)\dot{x}+x=0$

(a) Prove that the system has a unique stable limit cycle if $\mu>0$

The system written here is Lienards system. Indeed

$$\begin{cases}\dot{x}=y \\ \dot{y}=-\mu(x^4-1)y-x\end{cases} \qquad \begin{array}{l} f(x)=\mu(x^4-1) \\ g(x)=x \end{array}$$

(1) $f(x), g(x)\in C^1\ \forall x$, ok.

(2) $g(x)$ is indeed odd $g(-x)=-x=-g(x)$ - ok

(3) $g(x)>0$ for $x>0$ - ok

(4) $f(x)$ - even, $f(x)=\mu(x^4-1)$ - ok.

(5) $F(x)=\int_0^x f(u)du=\mu\left(\frac{1}{5}x^5-x\right)\Big|_0^x=\frac{1}{5}\mu x(x^4-5)$;

$x^*=\sqrt[4]{5}=a$ for $x<a$ it is negative and for $x>a$ $F(x)$ is positive and $F'(x)=\mu(x^4-1)>0$ - ok

So all criterias of Lienards theorem are satisfied

(13) and system has unique stable limit cycle.

Relaxation oscillations

Assume we have understood somehow that closed trajectory exists. Now what we can say about it's shape and period?

Relaxation oscillations are specific limit cycles that consists of extremely slow build up followed by a sudden discharge, followed by another slow build up and so on (i.e. the "stress" accumulated during the slow build up is "relaxed" during sudden discharge)

Consequence Limit cycle has two widely separated time scales (slow build up is followed by a fast discharge)

Ex. 7.5.3

$\ddot{x} + k(x^2-4)\dot{x} + x = 1$ for $k \gg 1$;

We can introduce here usual phase plane variables

$$\begin{cases} \dot{x} = y \\ \dot{y} = 1 - k(x^2-4)y - x \end{cases}$$

But the system appears to be much easier to describe if we introduce some other variables. To motivate this variables let's make the following observation

$\ddot{x} + k(x^2-4)\dot{x} = \frac{d}{dt}\left(\dot{x} + k\left(\frac{1}{3}x^3 - 4x\right)\right)$, and thus we can introduce following variables.

$$\begin{cases} \omega = \dot{x} + k\left(\frac{1}{3}x^3 - 4x\right) \\ \dot{\omega} = 1 - x \end{cases} \Rightarrow \begin{cases} \dot{x} = \omega - k\left(\frac{x^3}{3} - 4x\right) \\ \dot{\omega} = 1 - x \end{cases}$$

Now it is reasonable to change variables in the following way; $y = \frac{\omega}{k} \Rightarrow \omega = ky$

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$$\begin{cases} \dot{x} = k\left(y - \frac{1}{3}x^3 + 4x\right) \\ \dot{y} = \frac{1}{k}(1-x) \end{cases}$$ Since $k \gg 1$

x is fast: $\dot{x} \sim O(k) \gg 1$ - changes fast

y is slow: $\dot{y} \sim O(\frac{1}{k}) \ll 1$ - chages slow

Phase plane

* Fixed points $\dot{x} = 0$, $\dot{y} = 0 \Rightarrow x^* = 1$;

$y^* - \frac{1}{3} + 4 = 0$ $\quad y^* = -\frac{11}{3}$; $\quad (x^*, y^*) = (1; -\frac{11}{3})$;

$$J(x,y) = \begin{bmatrix} -kx^2 + 4k & k \\ -\frac{1}{k} & 0 \end{bmatrix};$$ $\Delta = 1$

$\tau = k(4 - x^2)$ for $x = 1$

$\tau = 3k > 0$

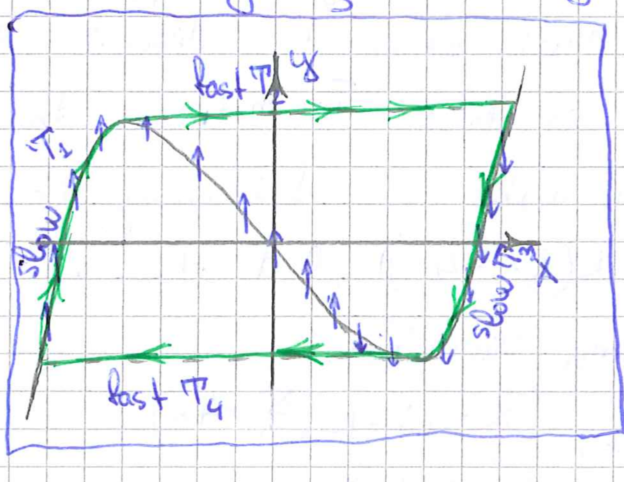
* Nullclines

* vertical $\dot{x} = 0$; $y = -4x + \frac{1}{3}x^3$; $\dot{y} = \frac{1}{k}(1-x)$ - slow drift

* horizontal $\dot{y} = 0$; $x = 1$; $\dot{x} = k(y + \frac{11}{3}) \to$ fast drop

$\dot{x} \gg \dot{y}$ if $\dot{x} \neq 0$;

$\dot{x} > 0$ if $y > -\frac{11}{3}$; and $y_{min} = -\frac{16}{3} < -\frac{11}{3}$;



Behaviour of trajectories

* The trajectories zap horizontally onto the cubic nullclines $y = \frac{1}{3}x^3 - 4x$

* Since the velocity is enormous in the horizontal direction trajectories move practically horizontally (if $y - (\frac{1}{3}x^3 - 4x) \sim O(1)$ i.e. initial condition is not too close to nullclines.)

* if the initial condition is above the nullcline

(15) $y - (\frac{1}{3}x^3 - 4x) > 0$ then $\dot{x} > 0$ (since $\dot{x} = k(y - \frac{1}{3}x^3 + 4x)$)
the trajectory moves sideways forward the nullcline.

* once trajectory gets so close that $y - (\frac{1}{3}x^3 - 4x) \sim O(\frac{1}{k^2})$
than $\dot{y}$ and $\dot{x}$ are comparable:

$$\dot{x} = k(y - \frac{x^3}{3} + 4x) \sim O(\tfrac{1}{k})$$

$$\dot{y} = \frac{1}{k}(x-1) \sim O(\tfrac{1}{k})$$

The trajectory crosses the nullclines vertically and then slides slowly along the backside of the branch with a velocity $\sim O(\frac{1}{k})$;

* the trajectory reaches the knee and it can jump sideways again.
* the trajectory reaches the next jumping-off point and the motion continues periodically after that.

Period of the trajectory

We have 2 time scales

the crawls require $\Delta t \sim O(k)$ (slow) $\dot{x} \sim O(\frac{1}{k})$; $\dot{y} \sim O(\frac{1}{k})$;

the jumps require $\Delta t \sim O(k^{-1})$ (fast) $\dot{x} \sim O(k)$; $\dot{y} \sim 0$;

$$T = T_1 + T_2 + T_3 + T_4 = T_1 + T_3 + O(\tfrac{1}{k})$$

$T = \int dt = \int \frac{dx}{\dot{x}}$ the period T is essentially the time required to travel along the two slow branches.

$T_1 = \int_{x_1}^{x_2} dt$; $T_2 = \int_{x_4}^{x_3} dt$; $y = \frac{1}{3}x^3 - 4x$; $\dot{y} = \dot{x}(x^2-4) = \frac{1}{k}(1-x)$

thus $\dot{x} = \frac{1-x}{k(x^2-4)}$ now $dt = \frac{dx}{|\dot{x}|} = dx \cdot k\left|\frac{x^2-4}{1-x}\right|$

$$T_1 = \int_{x_1}^{x_2} \frac{dx}{|\dot{x}|} = k\int_{x_1}^{x_2} \left|\frac{x^2-4}{1-x}\right| dx \,; \quad T_3 = k\int_{x_3}^{x_4} \left|\frac{x^2-4}{1-x}\right| dx \,;$$

by symmetry $T_1 = T_3$;

Now let's find points x_1, x_2, x_3, x_4;

(16) Isoclines ar given by function

$F(x) = \frac{1}{3}x^3 - 4x$ x_2 and x_4 are points of maximum and minimum of $F(x)$ given by equation

$\frac{\partial F}{\partial x} = x^2 - 4 = 0$ $x = \pm 2$ in fact $x_2 = -2$; $x_4 = 2$;

$F(x_2) = \frac{16}{3} = F(x_3) \Rightarrow x_3^3 - 12x_3 = 16$ as $x_2 = -2$ is solution of the same equation let's rewrite

$(x_3+2)(x_3^2 - 2x - 8) = 0$ solving quadratic equation

$x_3^2 - 2x_3 - 8 = 0$ gives us $x_3 = 1 \pm 3 = 4; -2.$

root we are interested in is $\boxed{x_3 = 4;}$

in the same manner due to the symmetry we can find $\boxed{x_1 = -4;}$

Finally we can obtain period

$$T_1 = k\int_2^4 \frac{x^2-4}{x-1}dx = k\int_2^4 \frac{x^2-1}{x-1}dx - 3k\int_2^4 \frac{1}{x-1}dx =$$

$$= k\left(\tfrac{1}{2}x^2 + x\right)\Big|_2^4 - 3k\ln|x-1|\Big|_2^4 = k(8 - 3\ln 3)$$

$\boxed{T_1 = k(8 - 3\ln 3)}$; now $T_3 = k\int_{-4}^{-2} \frac{x^2-4}{1-x}dx = k\left(-\frac{1}{2}x^2 - x + 3\ln|1-x|\right)\Big|_{-4}^{-2} =$

$= k(+6 - 2 + 3\ln 3 - 3\log 5) = k\left(4 + 3\log\frac{3}{5}\right)$

$\boxed{T_3 = k\left(4 + 3\log\frac{3}{5}\right)}$; then whole period is given by

$T = T_1 + T_3 = k(12 - 3\log 5)$; $\boxed{T \approx k(12 - 3\log 5) + O(\tfrac{1}{k})}$

(17) Weakly nonlinear oscillators

Weakle nonlinear oscillators are described by equation:

$\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0$, $0 \leq \varepsilon \ll 1$. Here in the same manner with previous example we have 2 time scales. But before they were relevant sequently, now they operate at the same time

Two time scales: fast time $\tau = t$, $O(1) \to \partial_\tau$

slow time $T = \varepsilon t$ $\partial_t \to \varepsilon \partial_T$

τ and T are independent variables and functions of T are constants on τ-scale.

Perturbation theory:

$$x(t, \varepsilon) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + \frac{1}{2}\varepsilon^2 x_2(\tau, T) + \dots$$

then $\dot{x}(t, \varepsilon) = \partial_\tau x_0 + \frac{\partial T}{\partial t}\partial_T x_0 + \varepsilon \partial_\tau x_1 + \varepsilon \frac{\partial T}{\partial t}\partial_T x_1 + O(\varepsilon^2) =$

$= \partial_\tau x_0 + \varepsilon \partial_T x_0 + \varepsilon \partial_\tau x_1 + O(\varepsilon^2)$;

$\ddot{x}(t, \varepsilon) = \partial^2_\tau x_0 + 2\varepsilon \partial_\tau \partial_T x_0 + \varepsilon \partial_\tau \partial_\tau x_1 + O(\varepsilon^2)$

now we gather all terms of the same order.

$O(1)$: $\partial^2_{\tau\tau} x_0 + x_0 = 0$;

$O(\varepsilon)$: $\partial^2_{\tau\tau} x_1 + 2\partial_\tau \partial_T x_0 + x_1 + h(x, \dot{x}) = 0$;

Solution of first equation is just usual harmonic oscillator $x_0 = A\sin\tau + B\cos\tau$, now let's assume A and B be the functions of T rather then constant terms. To determine $A(T)$ and $B(T)$ let's substitute solution of first equation into second and set the coefficients of resonant terms to 0.

$$\partial^2_{\tau\tau} x_1 + x_1 = -2(\partial_\tau \partial_T x_0 + \partial_\tau x_0) = -2(A' + A)\cos\tau + 2(B' + B)\sin\tau$$

$A' + A = 0$ $\boxed{A(T) = A(0)e^{-T}}$

$B' + B = 0$ $\boxed{B(T) = B(0)e^{-T}}$

Then we should use initial conditions for x and $\dot{x}$ to determine $A(0)$ and $B(0)$;

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Ex 7.6.5

Calculate the averaged equations and analyze the long-term behaviour of the system. Find the amplitude and frequency of any limit cycles for the original system. If possible, solve the averaged equations explicitely for $x(t,\varepsilon)$ given the initial conditions $x(0)=a$; $\dot{x}(0)=0$;

$h(x,\dot{x}) = x\dot{x}^2$

$\ddot{x} + x + \varepsilon x\dot{x}^2 = 0$, $0<\varepsilon \ll 1$;

$$\begin{cases} \partial^2_{\tau\tau} x_0 + x_0 = 0; & O(1) \\ 2\partial^2_{\tau T} x_0 + \partial^2_{\tau\tau} x_1 + x_1 + (\partial_\tau x_0)^2 x_0 = 0; & O(\varepsilon) \end{cases}$$

Now we substitute ansatz $x_0 = r(T)\cos(\tau + \varphi(T))$

$r(T)$ and $\varphi(T)$ are slowly-varying amplitude and phase of x_0

$\partial_T x_0 = r'\cos(\tau+\varphi(T)) - \varphi'\sin(\tau+\varphi(T))$

$\partial_\tau \partial_T x_0 = -r'\sin(\tau+\varphi(T)) - \varphi'\cos(\tau+\varphi(T))$

$(\partial_\tau x_0)^2 = r^2(T) \quad (\tau+\varphi(T))$

$x_0(\partial_\tau x_0)^2 = r^3(T)\sin^2(\tau+\varphi(T))\cos(\tau+\varphi(T))$

$$x_1 + \partial^2_{\tau\tau} x_1 = \underbrace{2r'\sin(\tau+\varphi) + 2r\cos(\tau+\varphi)\cdot\varphi'}_{\text{resonance force}} - r^3\cos(\tau+\varphi)\sin^2(\tau+\varphi)$$

resonance force, producing terms like $\tau\sin\tau$, i.e. growing with time amplitude of oscillations for x_1, which is not good for pertubation theory we are using.

To understand what is resonant component of the last force we shoul usually expand it in fourier series

$f(\theta) = \sum_{k=0}^{\infty} a_k \cos\theta + \sum_{k=1}^{\infty} b_k \sin(k\theta)$, but in this example there is no need to do even this.

(19) we should just use some trigonometric formulas

$\cos x \cdot \sin^2 x = \frac{1}{2}\sin 2x \cdot \sin x = \frac{1}{4}(-\cos 3x + \cos x)$, $x = \tau + \varphi$

in our case only $\cos(\varphi + \tau)$ is resonant term, thus

$$\partial^2_{\tau\tau} X_1 + X_1 = 2r' \sin(\tau + \varphi) + 2r\varphi' \cos(\tau + \varphi) - \frac{1}{4} r^3 \cos(\tau + \varphi) + \\ - \frac{1}{4} r^3 \cos(3\tau + 3\varphi)$$

thus resonat force is given by the following expression

$2r' \sin(\tau + \varphi) + 2r\varphi' \cos(\tau + \varphi) - \frac{1}{4} r^3 \cos(\tau + \varphi) = 0$ thus we get

$$\begin{cases} r' = 0 \\ \varphi' = \frac{1}{8} r^2 \end{cases} \qquad r(T) = \text{const} = r_0 \qquad \frac{d\varphi}{dT} = \frac{r_0^2}{8} \Rightarrow \varphi(T) = \frac{r_0^2}{8} T + \varphi_0$$

thus $\boxed{X(\tau, T) = r_0 \cos\left(\tau + \frac{r_0^2}{8} T + \varphi_0\right) \approx r_0 \cos\left(t + \frac{\varepsilon r_0^2}{8} t + \varphi_0\right)}$

Now r_0 and φ_0 are determined from initial conditions

$X(0) = a = r_0 \cos\varphi_0$

$\dot{x}(0) = 0 = r_0 \sin\left(t + \frac{\varepsilon r_0^2}{8} + \varphi_0\right)\left(1 + \frac{\varepsilon r_0^2}{8}\right) = -r_0 \sin\varphi_0 \cdot \left(1 + \frac{\varepsilon r_0^2}{8}\right)$

thus $\sin\varphi_0 = 0$, $\boxed{\varphi_0 = 0}$ and $\boxed{r_0 = a}$

And we finally get the answer:

$$\boxed{X(\tau, T) = r_0 \cos\left(\left(1 + \frac{\varepsilon a^2}{8}\right) t\right)}$$

1

Bifurcations in phase plane (Seminars 10, 11)

Def If phase portrait changes it's topological structure as parameter is varied, we say that bifurcation has occured.

By topologically equivalent we mean that there is homeomorphism (i.e. continious deformation with continious inverse) that maps one local phase portrait onto another such that all trajectories map onto trajectories and the sense of time is preserved.

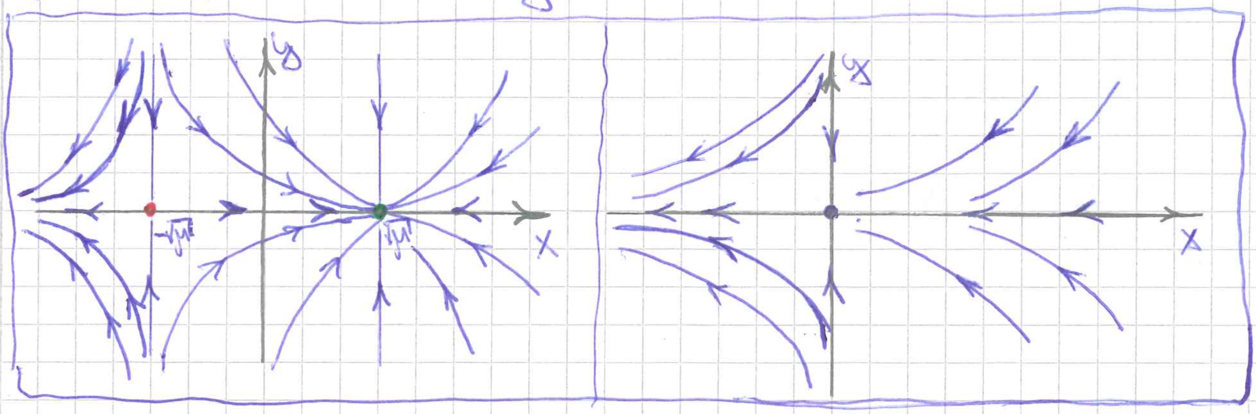
Types of bifurcations similar to 1d systems

Adding dimensions doesn't change very much for fixed points because bifurcations that occure, occure along some 1d manifolds while dynamics of the system along directions transverse to this 1d manifolds is not very interesting because it is simply attracting or repelling. Let's refresh information about such kind of bifurcations

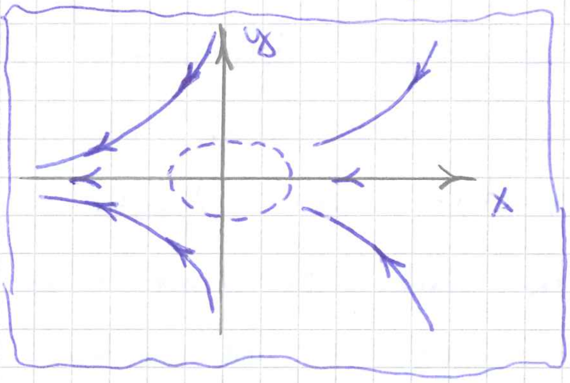
1. Saddle-Node bifurcation is creation/or destruction of pairs of fixed points

$$\begin{cases} \dot{x} = \mu - x^2 \\ \dot{y} = -y \end{cases}$$

this system is splitted in 2 1-dim systems. In x-direction we get usual saddle-node bifurcation and in y-direction we have just exponential decay.

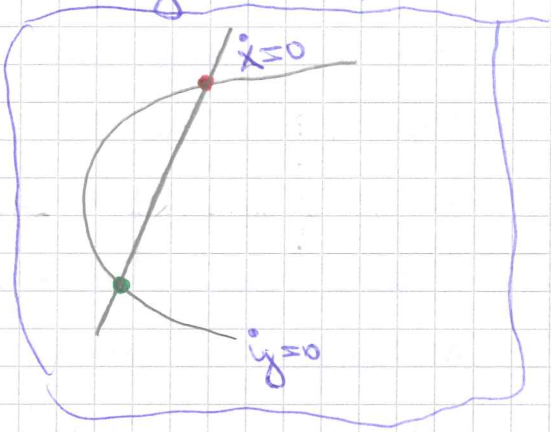


② What happans is that saddle point collides with node and eventually they annihilate, but bottle-neck (or ghost) is left after such bifurcation (i.e. the interval of coordinates where movement is slow).



In general $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$ depending on μ parameter. We can draw the nullclines $\dot{x} = 0$ and $\dot{y} = 0$; intersections of nullclines are fixed points by definition

While we change μ this nullclines move and eventually at some μ_c they become tangent. At this moment bifurcation occure.

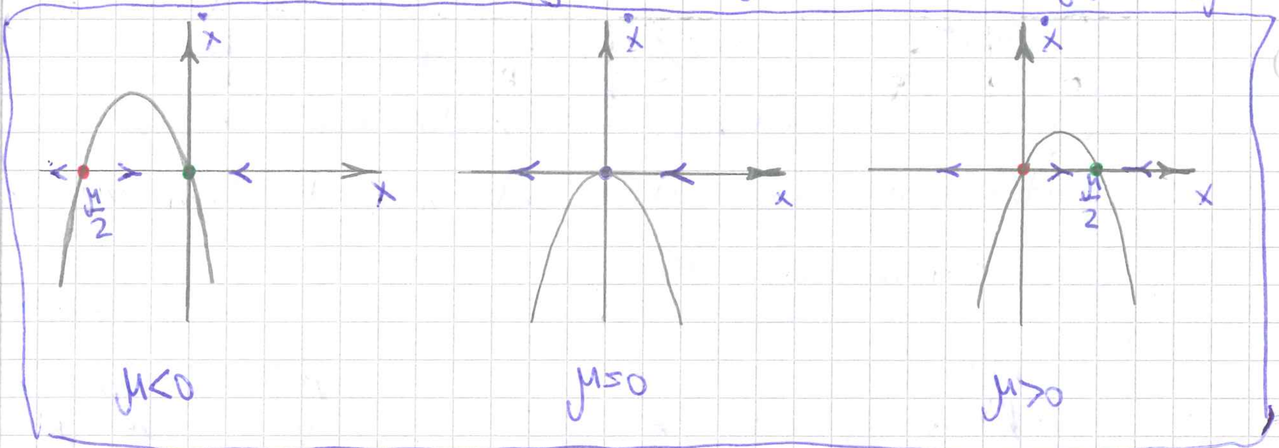


② Transcritical Bifurcation

$$\begin{cases} \dot{x} = \mu x - x^2 \\ \dot{y} = -y \end{cases}$$

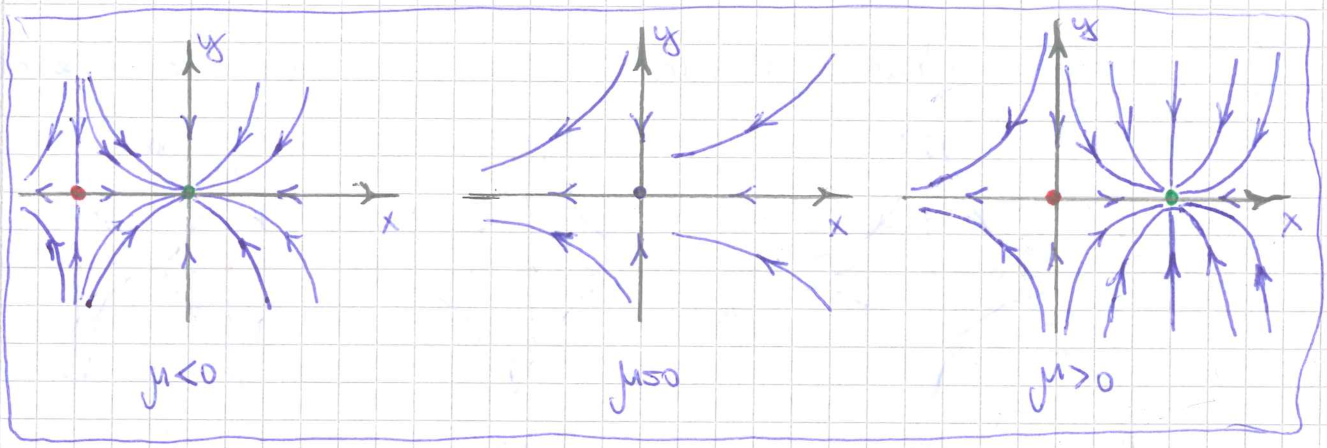
In y direction we get simple decay $y = y_0 e^{-t}$

in x direction we get: $\dot{x} = \mu x - x^2 = -(x - \frac{1}{2}\mu)^2 + \frac{1}{4}\mu^2$



Now if we draw 2d flow we get:

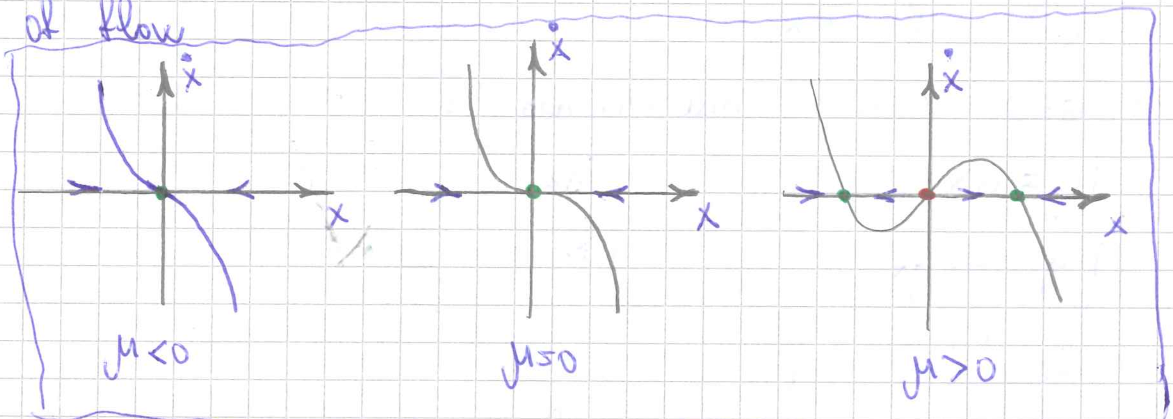
③



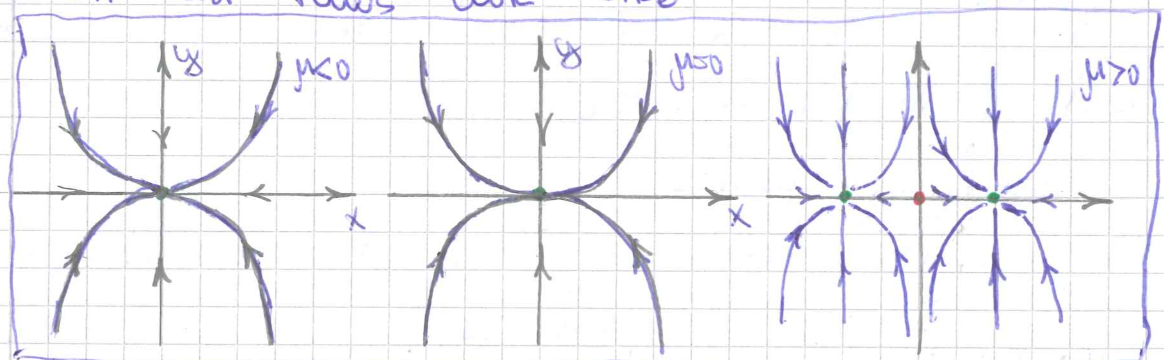
③ supercritical pitchfork bifurcation.

$$\begin{cases} \dot{x} = \mu x - x^3 \\ \dot{y} = -y \end{cases}$$

again we get simple decay along y-direction $y = y_0 e^{-t}$ and along x direction we get the following picture of flow



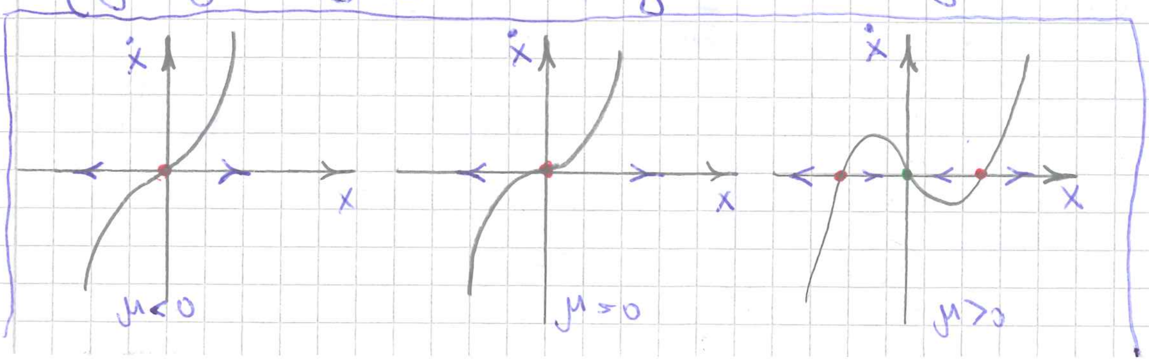
In 2d flows look like



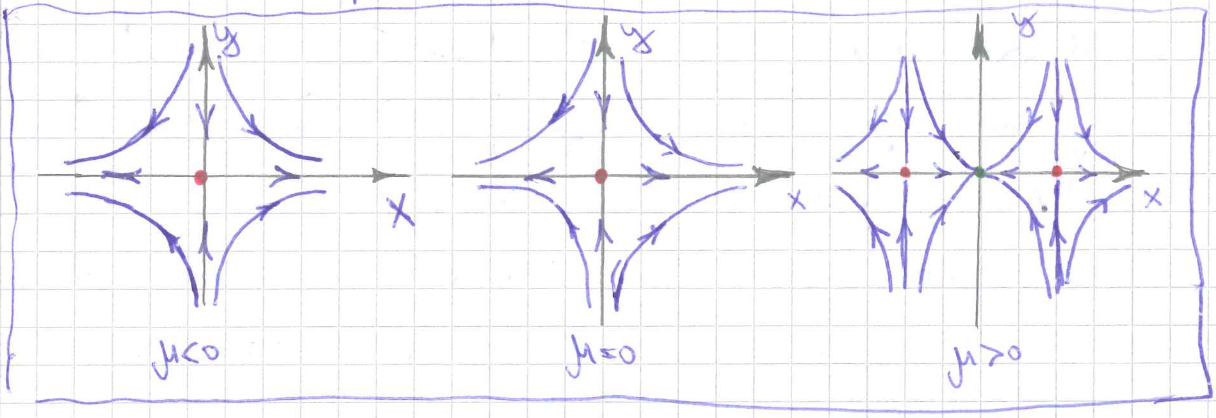
subcritical pitchfork bifurcation

$$\begin{cases} \dot{x} = \mu x + x^3 \\ \dot{y} = -y \end{cases}$$

once again we get decay along y direction $y = y_0 e^{-t}$. Along x-dir we get:



④ 2d flows picture in this case look like.



As example let's consider following exercise

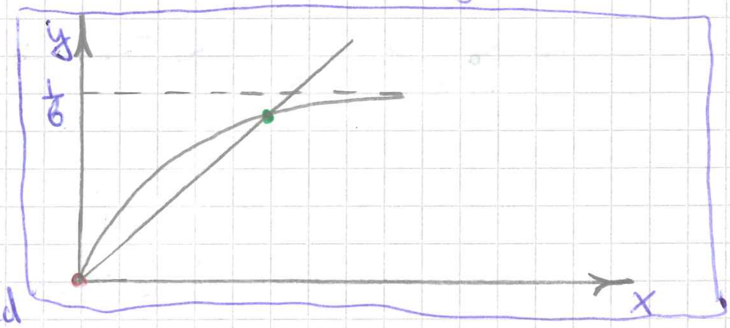
Ex. 8.1.7

$$\begin{cases} \dot{x} = y - ax \\ \dot{y} = -by + \frac{x}{1+x} \end{cases}$$

Find and classify all bifurcations for the system

* Let's consider null clines of this system

$$\begin{cases} y = ax \\ y = \frac{x}{b(1+x)} \end{cases}$$



intersections of nullclines give us fixed points.

The next question we should answer is what kind of fixed points are this?

* classifying points.

As usually let's write down the Jacobi matrix

$$J(x,y) = \begin{bmatrix} -a & 1 \\ \frac{1}{1+x} - \frac{x}{(1+x)^2} & -b \end{bmatrix} = \begin{bmatrix} -a & 1 \\ \frac{1}{(1+x)^2} & -b \end{bmatrix}$$

fixed points are given by: $(x^*, y^*) = (0,0)$;

$ab = \frac{1}{1+x} \Rightarrow \boxed{x^* = \frac{1}{ab} - 1 \;;\; y^* = \frac{1}{b} - a;}$

Now we substitute this values into Jacobian.

$$J(x^*, y^*) = \begin{bmatrix} -a & 1 \\ a^2b^2 & -b \end{bmatrix} \qquad \begin{aligned} \Delta &= +ab(1-ab) \\ \tau &= -(a+b) < 0 \end{aligned}$$

(5) As we can see $\Delta < 0$ for $ab > 1$ and this is saddle point, now if $ab < 1$ $\Delta > 0$ and $\tau^2 - 4\Delta = (a+b)^2 - 4ab + 4a^2b^2 = (a-b)^2 + 4a^2b^2 > 0$ always accept $a = b = 0$ point.

Now let's exemine fixed point at the origin

$$(x^*, y^*) = (0,0) \quad ; \quad J(0,0) = \begin{bmatrix} -a & 1 \\ 1 & -b \end{bmatrix} \quad ; \quad \begin{matrix} \Delta = ab - 1 \\ \tau = -(a+b) \end{matrix}$$

$$\tau^2 - 4\Delta = (a+b)^2 - 4ab + 4 = (a-b)^2 + 4 > 0$$

but $\Delta = ab - 1$ is $\Delta > 0$ for $ab > 1$ leading to the stable node in the origin and is negative $(\Delta < 0)$ for $ab < 1$. So we get:

$a > \frac{1}{b}$ $\quad (x^*, y^*) = (0,0)$ stable node

$(x^*, y^*) = \left(\frac{1}{ab} - 1; \ \frac{1}{b} - a\right)$ - negative (thus nonphysical) sadde point.

$a < \frac{1}{b}$ $\quad (x^*, y^*) = (0,0)$ saddle point.

$(x^*, y^*) = \left(\frac{1}{ab} - 1; \frac{1}{b} - a\right)$ positive stable node.

This type of bifurcation when saddle point and stable node go through each other and interchange. is subcritical bifurcation.

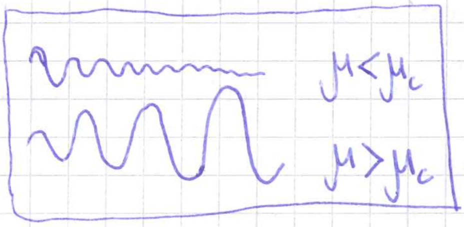
In all bifurcations considered (saddle-node, transcritical and pitchfork) at the bifurcation point we got $\Delta = 0$. Such kind of bifurcations are called zero-eigenvalue bifurcation., and it involve interaction of fixed points, but 2d systems can have more then just fixed points.

⑥ ## Hopf Bifurcations

Hopf bifurcation occurs when 2 complex conjugate eigenvalues simultaneously cross the imginary axis and go from one half-plane to another. Just as pitchfork bifurcations Hopf bifurcations can be supercritical or subcritical.

Supercritical Hopf bifurcations

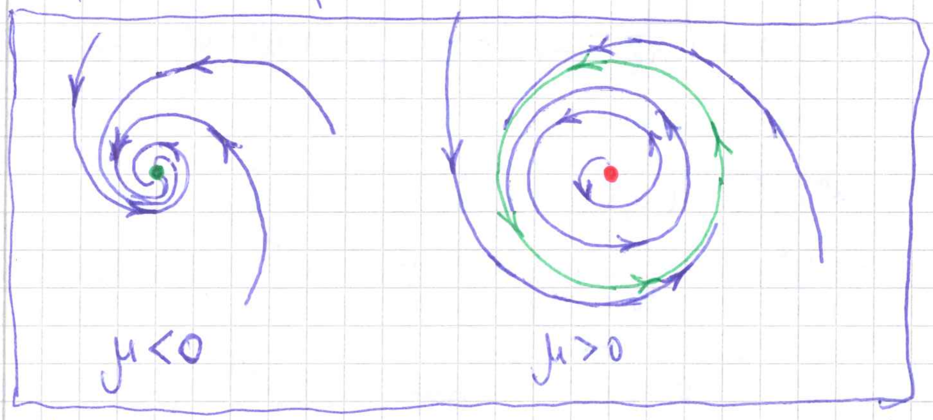
This occurs when stable spiral changes to unstable



simple example is given by

$$\begin{cases} \dot{r} = \mu r - r^3 \\ \dot{\theta} = \omega + b r^2 \end{cases}$$

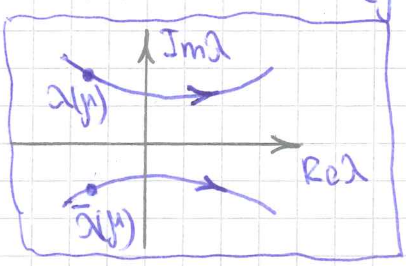
- in radial direction we just get supercritical pitchfork bifurcation.



Some features:

① The size of the limit cycle grows continiously from zero, and increases proportional to $\sqrt{\mu - \mu_c}$, for μ close to μ_c

② The frequency of the limit cycle is given approximately by $\omega = \operatorname{Im} \lambda$, evaluated at $\mu = \mu_c$. This formula is exact at the birth of the limit cycle, and correct within $O(\mu - \mu_c)$ for μ close to μ_c.



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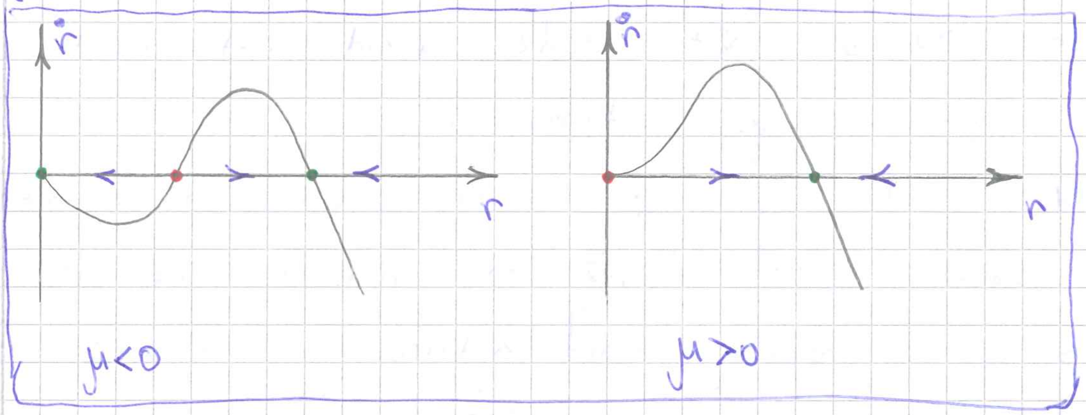
Subcritical Hopf bifurcations

This kind of bifurcations are more dengreous from engineering applications, because jump to a distant attractor happens.

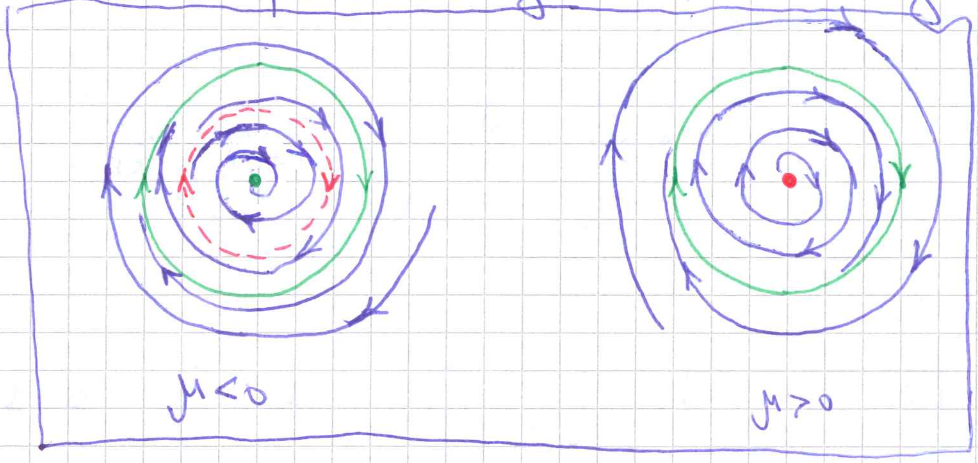
Simple example is given by the following system:

$$\begin{cases} \dot{r} = \mu r + r^3 - r^5 \\ \dot{\theta} = \omega + b r^2 \end{cases}$$



To understand this look at the position of fixed points $(\mu + r^2 - r^4) r = 0 \Rightarrow r^* = 0$ and $r^4 - r^2 - \mu = 0$

$r^2 = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\mu}$ if $\mu>0$ one of the roots is negative which is unreasonable for r^2. If $\mu<0$ both roots are ok. On the plane we get the following picture.



stable limit cycle stable f.p. at the origin + unstable limit cycle in between $\xrightarrow{\text{bifurc.}}$ Large amplitude limit cycle is the only attractor. At $\mu=0$ unstably cycle shrinks to radius 0 and thurn stable fixed point to unstable.

8 In this kind of systems we observe hysteresis. Solutions used to remain near the origin, now are forced to grow into large-amplitude oscillations

Degenerate Hopf bifurcation

Let's consider already familiar to us damped pendulum, described by:

$\ddot{x} + \mu\dot{x} + \sin x = 0$

For $\mu > 0$ we get stable fixed point at the origin.
For $\mu < 0$ we get unstable f.p.
Thus when we go from positive to negative μ stable spiral changes to unstable one. But this is not really Hopf Bifurcation because there is no limit cycle on either side and at $\mu = 0$ we get continious bunch of closed orbits which by definition are not limit cycles.

Exercise 8.2.1

Consider the biased van der Pol oscillator
$\ddot{x} + \mu(x^2-1)\dot{x} + x = a$; Find curves in (μ, a) space at which Hopf bifurcations occure.

$\begin{cases} \dot{x} = y \\ \dot{y} = a - x - \mu(x^2-1)y \end{cases}$ fixed points are given by $y=0$; $x=a$ $(x^*, y^*) = (a, 0)$

Jacobian is

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -1-2\mu xy & -\mu(x^2-1) \end{bmatrix}; \quad J(a,0) = \begin{bmatrix} 0 & 1 \\ -1 & -\mu(a^2-1) \end{bmatrix} \Rightarrow \begin{matrix} \Delta = 1 > 0 \\ \tau = -\mu(a^2-1) \end{matrix}$$

$$\lambda_{1,2} = \frac{1}{2}\tau \pm \frac{1}{2}\sqrt{\tau^2 - 4\Delta}; \quad \boxed{\lambda_{1,2} = -\frac{1}{2}\mu(a^2-1) \pm \frac{1}{2}\sqrt{\mu^2(a^2-1)^2 - 4};}$$

There are several cases that should be considered here:

⑨ ① $|\tau|>2$

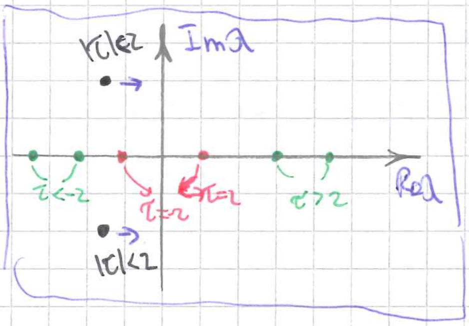
ⓐ $\tau>2$ $\lambda_{1,2}=\frac{\tau}{2}\pm\frac{1}{2}\sqrt{\tau^2-4}$ – Both values are real and have thesame (positive) sign – we get unstable node

ⓑ $\tau<-2$ now signs are opposite and we get stable node

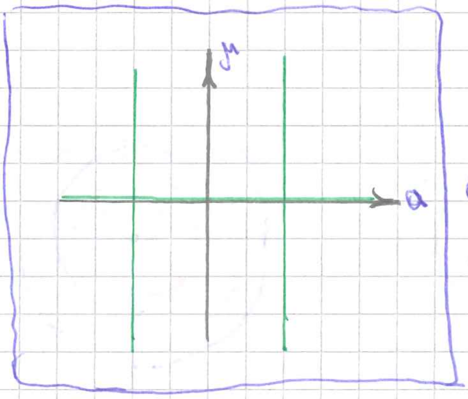
② $|\tau|=2$; $\lambda_1=\lambda_2=\frac{\tau}{2}=1$ or -1

③ $|\tau|<2$ eigenvalues are complex conjugated.

$\lambda_{1,2}=\frac{\tau}{2}\pm\frac{i}{2}\sqrt{4-\tau^2}$;



Hopf bifurcation occure where $Im\lambda\neq0$; $Re\lambda=0$; $Re\lambda=\frac{\tau}{2}=0$; $\tau=0$ $\mu(a^2-1)=0$; $Im\lambda=\pm2$. In $\mu-a$ plane we have lines $a=\pm1$ and $\mu=0$. This are manifold corresponding to bifurcation.



Global bifurcations of cycles

Limit cycles can be created or destroyed in 2 ways:

① Hopf bifurcations : occure locally i.e. limit cycle is born in fixed point and grows as $\sqrt{\mu-\mu_c}$

② Global bifurcations This kind of bifurcations happen when limit cycles coalesce or being born in pairs.

① Saddle-node bifurcations of cycles.

In case of this type of bifurcation 2 limit cycles coalesce and annihilate.

$$\begin{cases}\dot{r}=\mu r+r^3-r^5\\ \dot{\theta}=\omega+br^2\end{cases}$$

We have considered this system before. Let's consider it one more

(10) time. Before we have figured out that at $\mu=0$ point hopf bifurcation occurs. But there is one more point of bifurcation.

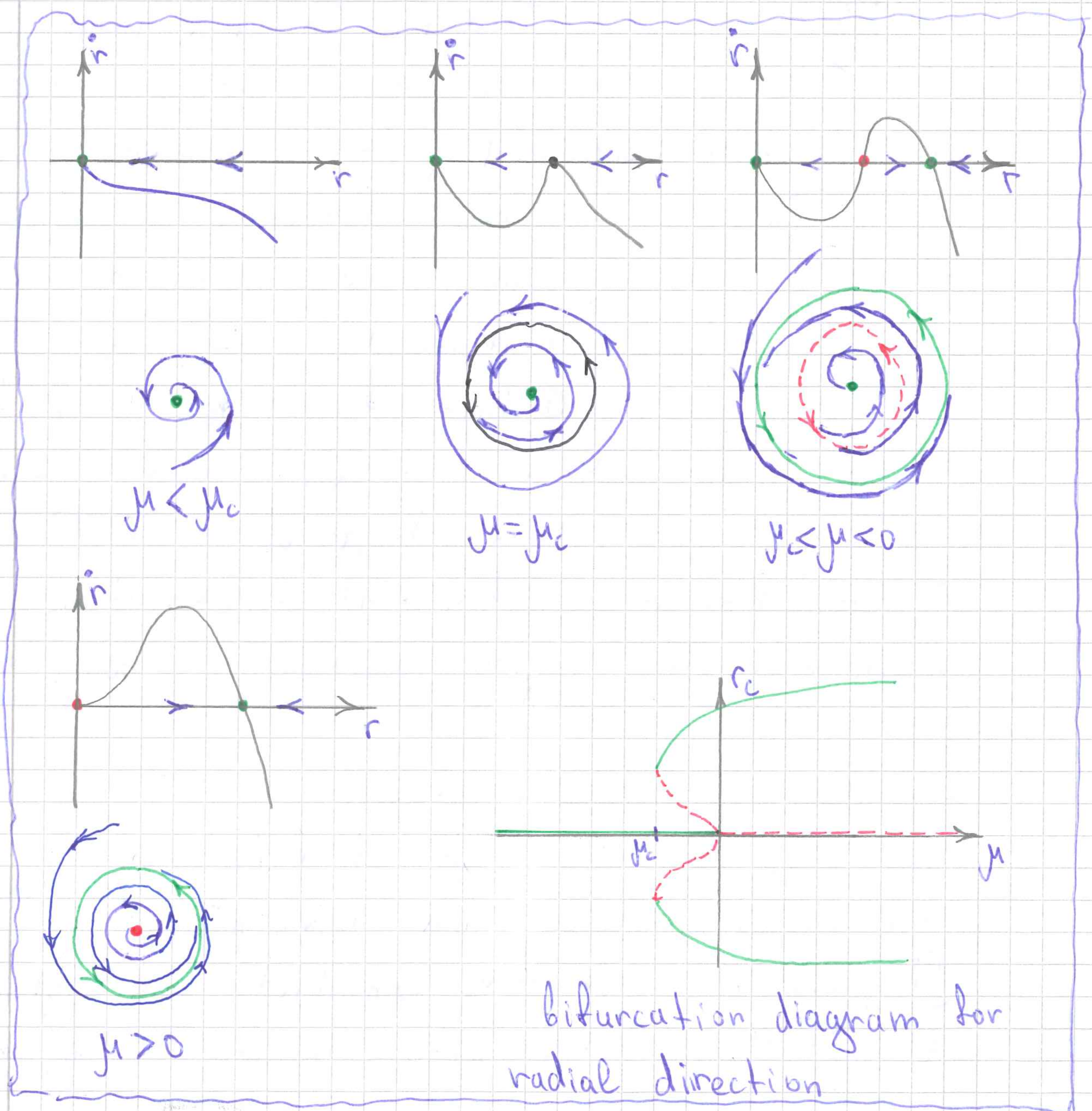
Let's look closer on dynamics in radial direction

$\dot{r} = (\mu + r^2 - r^4) r$ * fixed points: $r=0$;

$$r^2 = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\mu}$$

So we get the following picture

* $\mu < \mu_c = -\frac{1}{4}$ only on stable fixed point at $r=0$
* $\mu_c < \mu < 0$ 2 fixed points
* $\mu > 0$ unstable fixed point in the origin and stable limit cycle.



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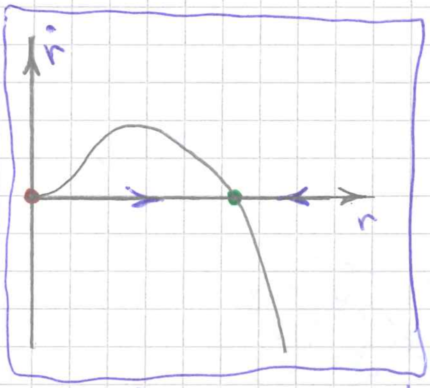
Infinite period bifurcation

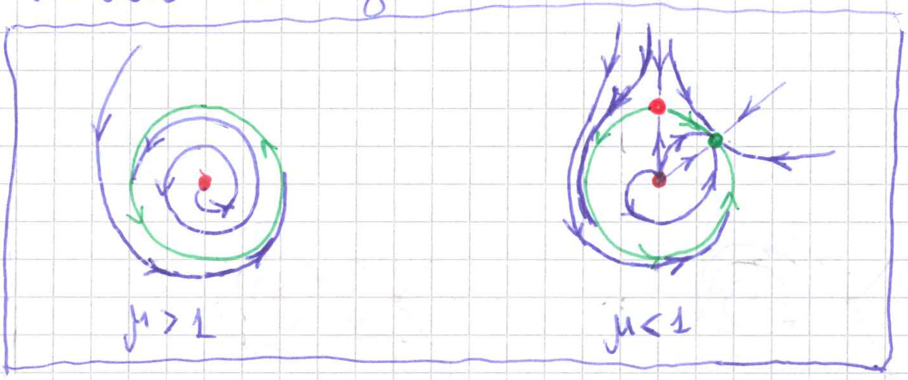
Let's consider the following system:

$$\begin{cases} \dot{r} = r(1-r^2) & \mu \geq 0 \\ \dot{\theta} = \mu - \sin\theta \end{cases}$$

Dynamics in radial direction is very simple



This mean that in "r" direction we always get unstable fixed point at the origin and stable limit cycle at some finite radius. As for dynamics of angle variable we get



* When $\mu > 1$ $\dot{\theta} > 0$ always and all trajectories approach limit cycle $r = 1$ counterclockwise
* When $\mu < 1$ $\dot{\theta} = \mu - \sin\theta$ can appear to be 0: there will be 2 invariant rays. When $\mu \to 1$ the limit cycle $r = 1$ develops a bottleneck at $\theta = \frac{\pi}{2}$ that becomes increasingly severe as $\mu \to 1^+$. The oscillation period increases and becomes infinite at $\mu = 1$ when half-stable fixed point appears on the limit cycle. Eventually we get two fixed points on the limit cycle: one saddle point and one stable node

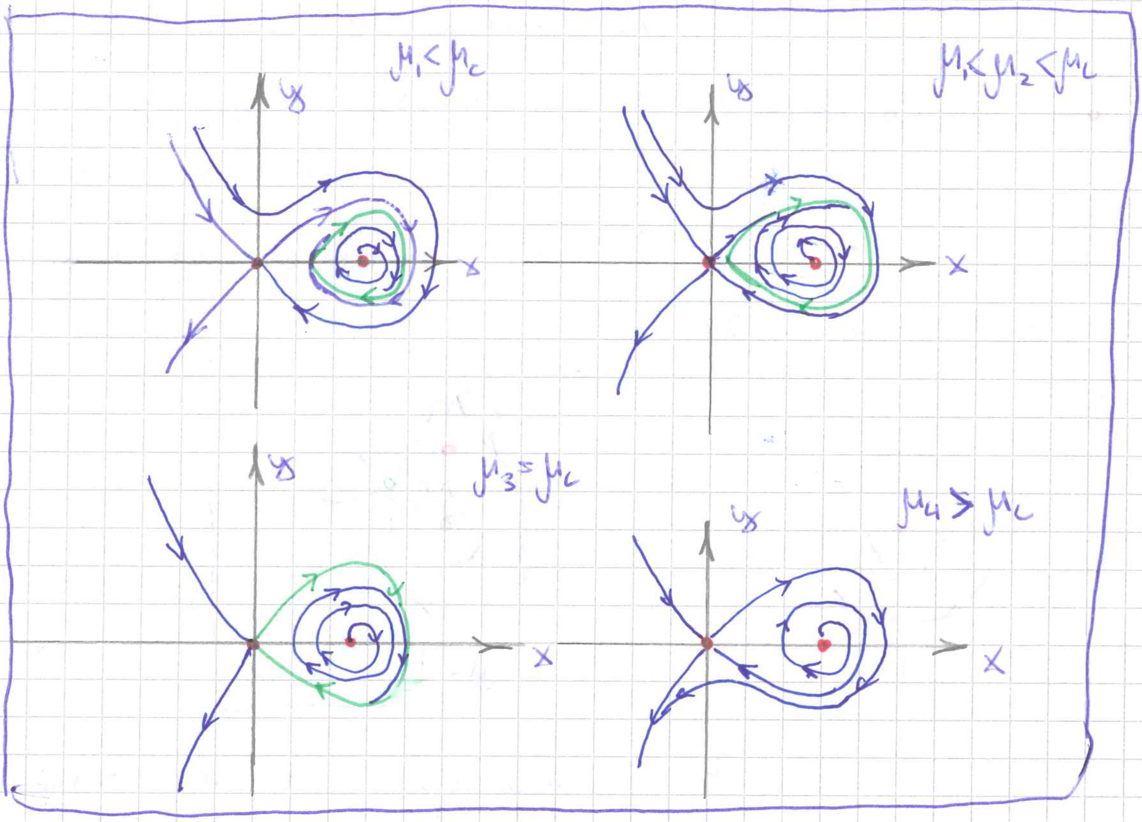
Homoclinic bifurcation

Qualitatively this bifurcation occure when part of limit cycle is coming closer and closer to saddle

(12) point and eventually turns into homclinic orbit (i.e. the one starting and ending on the same point) Example of such system is

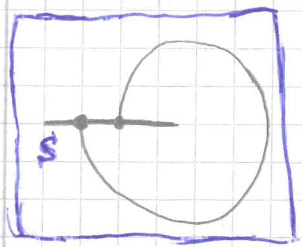
$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu y + x - x^2 + xy \end{cases}$$

This system is difficult to analize by hands so we just belive numerical results.



Poincare maps

To study the properties of swirling flows it is useful to introduce so called Poincare maps. Assume we have n-dimensional system $\dot{\bar{x}} = \bar{f}(\bar{x})$; and S is $(n-1)$ dimensional surface of section, transverse to the flow (all trajectories starting on S flow through it and not parallel) Simpliest example is 2d system and 1d surface.



Poincare map $P: S \to S$ i.e. this is mapping of S onto itself, so that $\bar{x}_k \to \bar{x}_{k+1} = P(\bar{x}_k)$. If $\bar{x}^*$ is fixed point

(13) of Poincare map P: $P(\bar{x}^*) = \bar{x}^*$, i.e. trajectory starting on $\bar{x}^*$ return to $\bar{x}^*$ after some finite T, then it is closed orbit.

Characteristic multiplyers.

Now let's try to understand if given fixed point of Poincare map is stable or not. Assume we are given system $\dot{\bar{x}} = \bar{f}(\bar{x})$ and fixed point $\bar{x}^*$ of it. Now let's make infinitesemal perturbation of fixed point $\bar{x}^* \to \bar{x}^* + \bar{v}_0$. Now let's observe how this perturbation behaves in time. On the second intersection

$$\bar{x}^* + \bar{v}_1 = P(\bar{x}^* + \bar{v}_0) = P(\bar{x}^*) + [DP(\bar{x}^*)]\bar{v}_0 + O(\|\bar{v}_0\|^2);$$

where $DP(\bar{x}^*)$ is an $(n-1)\times(n-1)$ matrix called linearized Poincare map. at $\bar{x}^*$ so we get finally

$\bar{v}_1 = [DP(\bar{x}^*)]\bar{v}_0$; assuming we neglect the smale $O(\|v_0\|^2)$ terms.

Stability criterion Assume λ_j are eigenvalues of $DP(\bar{x}^*)$. Then closed orbit is linearly stable if and only if $|\lambda_j| < 1$ for all $j = 1, \dots, n-1$;

To understand this criterion, let's look on time evolution of perturbation. Let expand $\bar{v}_0$ over the basis of $DP(\bar{x}^*)$ matrix eigenvectors: $\bar{v}_0 = \sum_{j=1}^{n-1} \nu_j \bar{e}_j$;

$\bar{v}_1 = DP(\bar{x}^*) \sum_{j=1}^{n-1} \nu_j \bar{e}_j = \sum_{j=1}^{n-1} \nu_j \lambda_j$, iterating this map more and more (eventually k times)

$\bar{v}_k = \sum_{j=1}^{n-1} \nu_j (\lambda_j)^k \bar{e}_j$; now we see that if $|\lambda_j| > 1$ perturbations along $\bar{e}_j$ grow with time and $\bar{x}^*$ appears to be unstable. Now if $|\lambda_j| < 1$ than

(14) perturbations along $\bar{e}_j$ decay geometrically. ($\bar{x}^*$ is stable)
At borderline $|\lambda_m|=1$ nonlinear stability analysis is required.

Def λ_j are called characteristic or Floquet multipliers of the periodic orbit.

Exercise 8.5.3

Logistic equation with periodically varying carrying capacity. Consider the logistic equation $\dot{N} = rN(1 - N/K(t))$, where the carrying capacity is positive, smooth, and T-periodic in t.

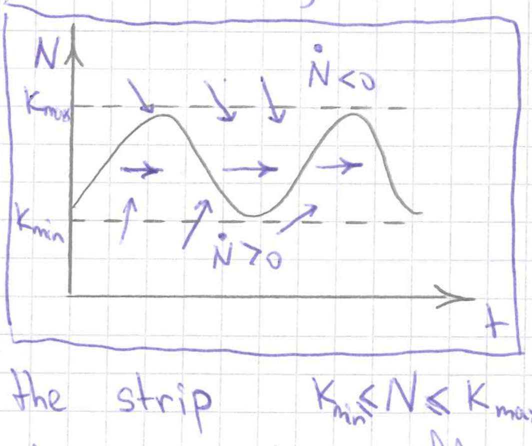
(a) using a Poincare map argument like that in the text, show that the system has at least one stable limit cycle of period T, contained in the strip $K_{min} \leq N \leq K_{max}$

This system is nonautonomous because K depends on time variable t. In such kind of systems we usually introduce time-variable as independent and add one more equation to the system.

$$\begin{cases} \dot{t} = 1 \\ \dot{N} = rN\left(1 - \frac{N}{K(t)}\right); \end{cases}$$

Nullclines $\dot{N}=0$ is where $N=0$ or $N=K(t)$

$K_{min} \leq K(t) \leq K_{max}$ due to periodicity (+ K(t) should be continious)



$\dot{N}>0$ when $N \leq K_{min}$ since $K(t)-N>0 \ \forall t$

$\dot{N}<0$ when $N > K_{max}$ since $K(t)-N<0 \ \forall t$

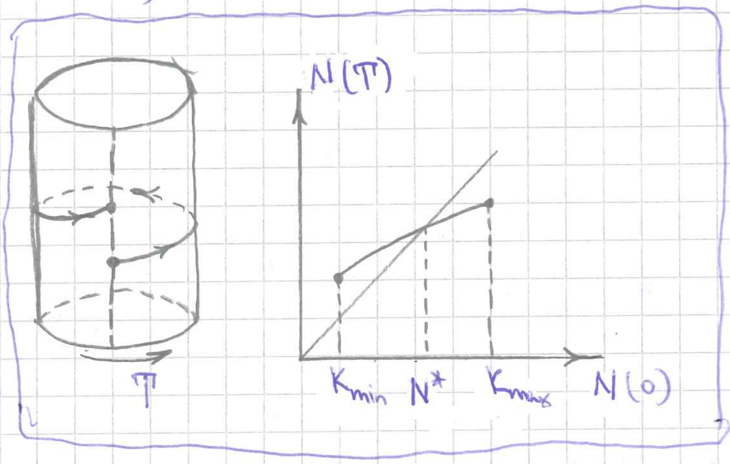
All trajectories eventually enter the strip $K_{min} \leq N \leq K_{max}$ and stay there forever

Inside strip flow is to the right since $\dot{t}=1>0$

Let's consider Poincare map.

(15) $P(N(0)) = N(T)$ How does it behave?

* We can say that $P(k_{min}) > k_{min}$ because the flow is strictly upward in the beginning and can't come back as flow is upward everywhere on the $k = k_{min}$ line.
* now we are able to observe that $P(k_{max}) < k_{max}$
* $P(N)$ is continious (solutions of different equations depend continiously on initial conditions)
* $P(N)$ is monotonic function (2 trajectories cannot cross)



(b) We have proven existence, now let's prove uniqueness

We define $E(N,t)$ T periodic in t:

$E(N, t=0) = E(N, t=T)$

After one period $\quad 0 = \Delta E = \int_{t=0}^{t=T} dE = \int_0^T \frac{dE}{dt}\, dt;$

Let's choose $E(N) = \log N$, because in this case $\frac{dE}{dt} = \frac{\dot N}{N} = r\left(1 - \frac{N}{K}\right)$ is simple enough. We get

$$\int_0^T dt - \int_0^T \frac{N(t)}{K(t)}\, dt = 0 \Rightarrow T = \int_0^T \frac{N(t)}{K(t)}\, dt$$ for all closed trajectories

But if solution is not unique (i.e. we have banch of solutions) since $N_1(t) > N_2(t)$ for 2 of them (as $P(N)$ is monotonic). Then

$$\int_0^T \frac{N_1(t)}{K(t)}\, dt > \int_0^T \frac{N_2(t)}{K(t)}\, dt$$ and this contradiction prove uniquenes.

(16)

Ex 8.7.2

Consider the vector field on the cylinder given by $\dot\theta = 1$; $\dot y = ay$. Define an appropriate Poincare map and find a formula for it. Show that the system has a periodic orbit. Classify it's stability for all real values of a.

$$\begin{cases} \dot\theta = 1 \\ \dot y = ay \end{cases} \qquad T = 2\pi \text{ is period} \quad T = \int_0^T dt = \int_0^{2\pi} \frac{1}{\dot\theta} d\theta = 2\pi;$$

$P(y(0)) = y(2\pi)$ - definition of Poincare map

$y(0) = y_0$; $\frac{dy}{dt} = ay \rightarrow y(t) = y(0)e^{at}$

$P(y_0) = y(T) = y(2\pi) = y_0 e^{2\pi a}$;

thus map is given by $\boxed{P(y) = ye^{2\pi a}}$

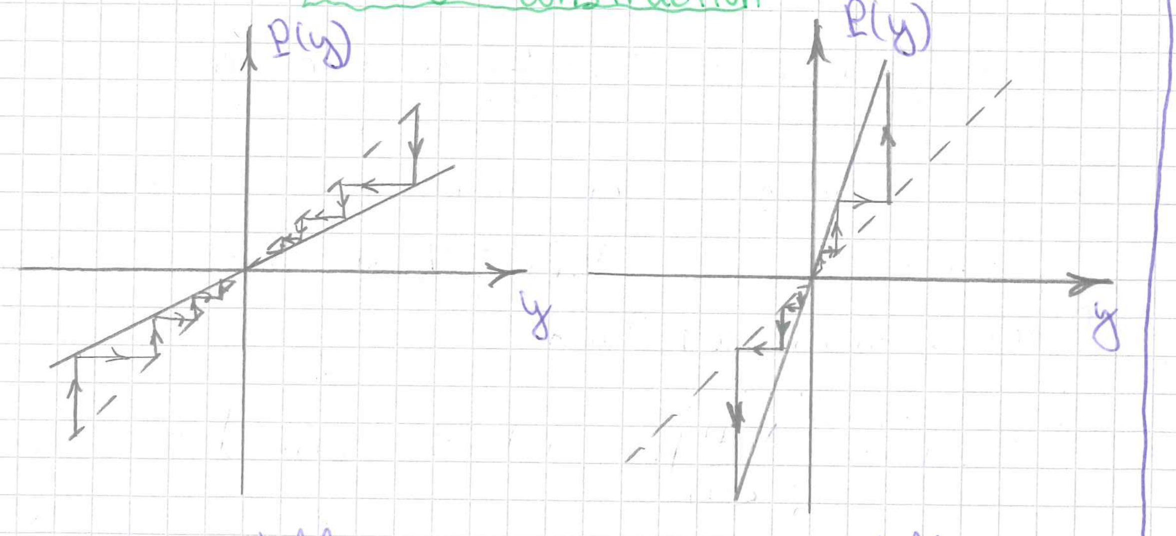
* fixed point of this map is given by:

$y^* = P(y^*) \Rightarrow y^*(1 - e^{2\pi a}) = 0$ the only fixed point is $y^* = 0$

* stability of fixed point

$\left|\frac{dP(y)}{dy}\right| = e^{2\pi a}$ if $a<0$ we get $\left|\frac{dP}{dy}\right| < 1$ and fixed point is stable. If $a>0$ vice versa fixed point is unstable

Cobweb construction



$a<0$ stable $\qquad$ $a>0$ unstable

1

Lorenz equations (seminar 12)

Simple properties of the Lorentz equations

Lorenz equations is the following system of equations:

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

$\sigma, r, b > 0$

σ - Prandtl number

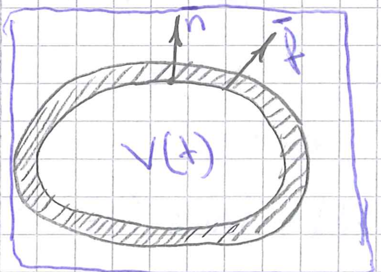
r - Rayleig number

(1) This is 3 dimensional nonlinear system containing chaotic behaviour

(2) symmetry $(x,y) \to (-x;-y)$ (all solutions are either symmetric themselves or have a symmetric partner)

(3) Volume contraction

The Lorenz system is dissipative, i.e. volumes in phase space contract under the flow. Let's consider system $\dot{\bar{x}} = \bar{f}(\bar{x})$. $dV = \bar{f} \cdot \bar{n}\, dt\, dA$



$$\frac{\Delta V}{\Delta t} = \int_S (\bar{f} \cdot \bar{n})\, ds = \int_V \bar{\nabla} \cdot \bar{f}\, dV$$

Thats general behaviour of phase volume with time

$$\bar{\nabla} \cdot \bar{f} = \frac{\partial}{\partial x}\sigma(y-x) + \frac{\partial}{\partial y}(rx - y - xz) + \frac{\partial}{\partial z}(xy - bz) =$$

$= -\sigma - 1 - b$, thus $\frac{dV}{dt} = -(1+\sigma+b)V \Rightarrow$

$\Rightarrow V = V_0 \exp(-\sigma-1-b)$ thus volume exponentally decreases with time.

There are 2 important consequencies

* There are no quasiperiodic solutions of Lorenz equations if there are some then they should lie on torus $\Rightarrow$ than this torus is invariant

② under the flow ⇒ volume inside torus is constant, and we came to contradiction.

* it's impossible for Lorenz system to have either repelling fixed points or repelling closed orbits (repellers are sources of volume)

Fixed point

$(x^*, y^*, z^*) = (0,0,0)$ is universal fixed point valid for all values of parameters.

another fixed point is given by:

$y^* = x^*$; $z^* = r-1$; $(x^*)^2 = bz^* = b(r-1)$

thus we get $\boxed{(x^*; y^*; z^*) = (\pm\sqrt{b(r-1)}; \pm\sqrt{b(r-1)}; r-1)}$

this solution exist only for $r>1$, as $r \to 1$; C_+ and C_- coalesce in pitchfork bifurcation

Let's linearize the system near origin:

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = rx - y \\ \dot{z} = -bz \end{cases}$$

as we see dynamics in z-direction is decoupled and we have exponential decay in this direction

$z = z_0 e^{-bt}$; and in xy-plane we get:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma \\ r & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

determinant and trace of this equations are respectively

$\boxed{\Delta = \sigma(1-r); \; \tau = -1-\sigma}$

$\tau < 0$ always

$\Delta < 0$ when $r > 1$ - saddle point.

$\Delta > 0$ when $r < 1$ - stable node

Real state of affairs is such that for $r<1$ origin is globally stable, i.e. all trajectories approaches the origin as $t \to \infty$; Hence there can be no limit cycles or chaotic behaviour for $r<1$

③ To proof global stability of origin we will use Liapunov function. In fact we will use $V(x,y,z) = \frac{1}{\sigma}x^2 + y^2 + z^2$ (surfaces of constant V are concentric ellipsoids about the origin). Now let's calculate $\dot{V}(x,y,z)$:

$$\frac{1}{2}\dot{V}(x,y,z) = \frac{1}{\sigma}\dot{x}x + y\dot{y} + z\dot{z} = (y-x)x + y(rx-y-xz) +$$
$$+ z(xy-bz) = xy - x^2 + rxy - y^2 - xyz + xyz - bz^2 =$$
$$= (r+1)xy - x^2 - y^2 - bz^2 = -\left(x + \frac{1}{2}(r+1)y\right)^2 - bz^2 -$$
$$-y^2\left(1 - \frac{1}{4}(r+1)^2\right)$$ if $r<1$ we get $1 - \frac{1}{4}(r+1)^2 \geq 0$

and thus $\dot{V} \leq 0$ $\dot{V}=0$ only if $x=y=z=0$ and thus $V(x,y,z)$ is appropriate Lyapunov function

Exercise 9.2.1 (Parameter where Hopf bifurcation occurs)

ⓐ For the Lorenz equations show that the characteristic equation for the eigenvalues of the Jacobian matrix at C^+, C^- is

$$\lambda^3 + (\sigma+b+1)\lambda^2 + (r+\sigma)b\lambda + 2b\sigma(r-1) = 0;$$

Let's consider our system

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = rx - xz - y \\ \dot{z} = xy - bz \end{cases}$$

* fixed point's $(x^*, y^*, z^*) = (0,0,0)$

$C^{\pm}$: $(x^*, y^*, z^*) = (\pm c; \pm c; r-1)$;

$c = \sqrt{b(r-1)}$

$$J(x,y,z) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{bmatrix};$$ at the origin we have

already considered linearization near origin

now let's consider $c^{\pm}$ fixed points.

$$J(c^{\pm}) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp c \\ \pm c & \pm c & -b \end{bmatrix};$$ now let's find characteristic equation for eigenvalues

(4)

$$\det[J(C^{\pm})-\lambda I]=\begin{vmatrix}-\sigma-\lambda & \sigma & 0\\ 1 & -1-\lambda & \mp C\\ \pm C & \pm C & -b-\lambda\end{vmatrix}=$$

$$=-(\sigma+\lambda)(1+\lambda)(b+\lambda)-\sigma C^2+\sigma(b+\lambda)+(\sigma+\lambda)C^2=0$$

$$-\lambda^3-\lambda^2(\sigma+b+1)-\lambda(b+\sigma b+\sigma)-\sigma b-\sigma C^2+\sigma b+\sigma\lambda+$$
$$-\sigma C^2-\lambda C^2=0$$

$$-\lambda^3-\lambda^2(\sigma+b+1)-\lambda(b+\sigma b+\sigma-\sigma+C^2)-\sigma(C^2+C^2)=0$$

Now substituting $C=\sqrt{b(r-1)}$ we get

$$\boxed{\lambda^3+\lambda^2(1+\sigma+b)+\lambda b(r+\sigma)+2\sigma b(r-1)=0} \quad (*)$$

(b) By seeking solutions of the form $\lambda=i\omega$, where ω is real, show that there is a pair of pure imaginary eigenvalues when $r=r_H=\sigma\left(\frac{\sigma+b+3}{\sigma-b-1}\right)$. Explain why we need to assume $\sigma>b+1$.

Let's substitute $\lambda=i\omega$ into $(*)$ and get:

$$-i\omega^3-\omega^2(1+\sigma+b)+i\omega b(r+\sigma)+2\sigma b(r-1)=0$$

$$\omega(\omega^2-b(r+\sigma))=0;\quad \omega^2=\frac{2\sigma b(r-1)}{1+\sigma+b}$$

thus if $\omega\neq 0$ equations

$\omega^2=b(r+\sigma)$ and $\omega^2=\frac{2\sigma b(r-1)}{1+\sigma+b}$ should be satisfied simultaneously

$$(r+\sigma)(1+\sigma+b)=2\sigma(r-1)$$

$$\sigma^2+\sigma(r+1+b-2r+2)+r(b+1)=0$$

$$r_H(1+\sigma+b-2\sigma)=-2\sigma-\sigma(1+\sigma+b) \quad \text{thus}$$

$\boxed{r_H=\frac{\sigma(3+b+\sigma)}{\sigma-b-1}};$ since $r>0$ we should have $\sigma>b+1$ for this equation to be valid.

(c) Find third eigenvalue.

(5) $\lambda^3 + (\sigma+b+1)\lambda^2 + b(r+\sigma)\lambda + 2b\sigma(r-1) = 0$

This equation should be written in the following form:

$(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) = \left[\lambda_{1,2} = \pm i\omega\right] = (\lambda^2+\omega^2)(\lambda-\lambda_3) =$

$= \lambda^3 - \lambda_3\lambda^2 + \omega^2\lambda - \omega^2\lambda_3 = 0$, thus

$\boxed{\lambda_3 = -(1+b+\sigma)}$; $\omega^2 = b(r+\sigma)$ (which is consistent with previous observations), and $-\omega^2\lambda_3 = 2b\sigma(r-1)$. Let's check if this equation is satisfied. If first 2 equations are satisfied we get:

$+(1+b+\sigma)b(r+\sigma) = 2b\sigma(r-1) \Rightarrow 2\sigma(r-1) = (1+b+\sigma)(r+\sigma) \Rightarrow$

$\Rightarrow 2\sigma r - 2\sigma = r + \sigma + br + b\sigma + \sigma r + \sigma^2$ thus

$r(1+b-\sigma) = -\sigma(3+b+\sigma) \Rightarrow \boxed{r = \frac{3+b+\sigma}{\sigma-b-1} = r_H}$ i.e.

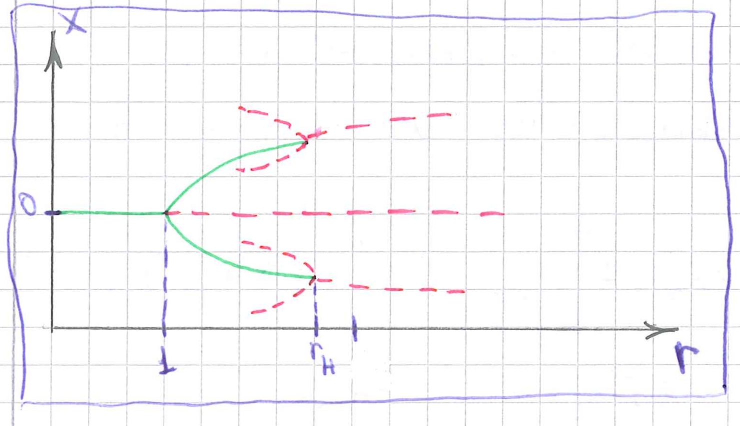
that condition is satisfied when $r = r_H$

Conclusions what appears is that points C^+ and C^- are linearly stable if $1 < r < r_H = \frac{3+b+\sigma}{\sigma-b-1}$ after, this stable spirals become unstable.

What we should expect in principle is supercritical bifurcation at $r = r_H$ (i.e. after bifurcation we get stable limit cycle and unstable spiral at $C^\pm$ points but really bifurcation is subcritical, i.e. limit cycles are unstable and exist for $r < r_H$

When $r \to r_H$ cycle shrinks down around fixed point, when $r = r_H$ fixed point adsorbs the saddle cycle and become saddle point., and for $r > r_H$ trajectories fly away to distinct attractor.

(6)



Long term behaviour of trajectories:

For $r > r_H$ trajectories are repelled from one unstable manifold to another and they are confined to a bounded set of volume 0, they move in this set forever without intersecting themselves or others.

Numerical example of Lorenz was $r_H = 24,74$; $\sigma = 10$; $b = \frac{8}{3}$; $r = 28$. What he has found was strange: after an initial transient, the solution settles into an irregular oscillation that persists as $t \to \infty$, but never repeats exactly, i.e. movement is aperiodic.

Peribdic motion (butterfly wings)

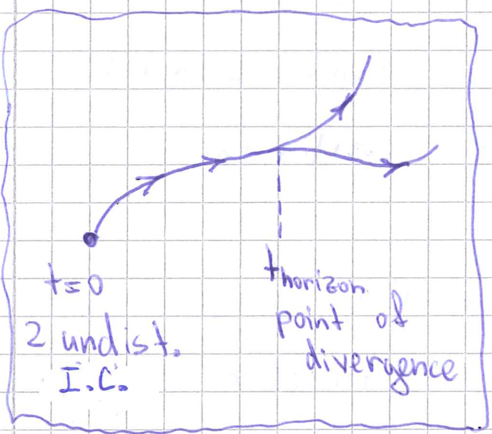
Strange attractor

* geometrical structure: fractal. It is a set of points with 0 volume but infinite surface area (dimension 2.05)

* sensetive dependence on initial condition (long term prediction become impossible, small uncertainties amplified enormously fast)

What does this sensitivity mean. Let's take some point $\bar{x}(t)$ on the attractor. Let's disturb it $\bar{x}(t) + \delta(t)$ and observe how $\delta(t)$ behaves with time. Usually this law looks like $\|\delta(t)\| \sim \|\delta_0\| e^{\lambda t}$, and λ is called Lyapunov exponent.

(7) If $\lambda>0$ disturbance grow exponentally with time and we can introduce so called time horizon, i.e. time when two almost undistinguishable initial conditions diverge.



$t_{horizon} \sim O(\frac{1}{\lambda} \log \frac{a}{\|\delta_0\|})$ where

a is measure of our tolerance.

Chaos

Def Aperiodic long-term behaviour in a determenistic system that exhibits sensitive dependence on initial conditions. What does it mean?:

* Aperiodic long-term behaviour: trajectories do not settle down to fixed points, periodic or quasiperiodic orbits as $t\to\infty$
* Determenistic System has no random or noisy inputs or parameter (irregular behaviour arises from system's nonlinearity)
* Sensetive dependence on I.C. nearby trajectories separate exponentally fast (see Lyapunov exponent)

Attractor (Definition)

Attractor is closed set A with the following properties.

* A is invariant set any trajectory $\bar{x}(t)$ starting in A stays in A for all time.
* A attracts an open set of initial conditions.

There is open set U containing A such that if $\bar{x}(0)\in U$, then the distance from $\bar{x}(t)$ to A tends

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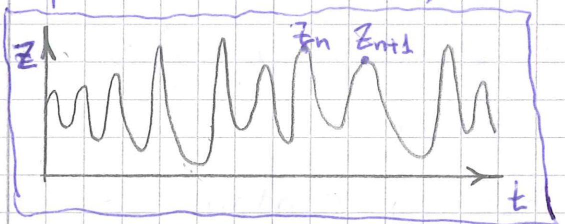
to zero as $t \to \infty$; i.e. A attracts all trajectories that start sufficiently close to it. The larges such U is called basin of attraction.

* A is minimal: there are no subsets satisfying first 2 conditions

Strange attractor is attractor that exhibits sensitive dependence on initial conditions.

Quite effective and at the same time simple way to analyze strange attractor is Lorenz map. Assume we take one of coordinates (z for example) as attractor has nearly periodic behaviour it will look approxametely as on the picture below, so we can write down sequence



of maximums of z and draw function $z_{n+1} = f(z_n)$. Data from chaotic time series appear to fall nearly on a curve

Difference with Poincare map is that for Lorenz system Poincare map should be 2-dimensional but here we get just one parameter, but stability criterion here is just the same as for Poincare maps: if $|f'(z)| > 1$ for any z then all closed orbit are unstable.

(9) Exercise 9.4.2 Tent map as model of Lorenz map.

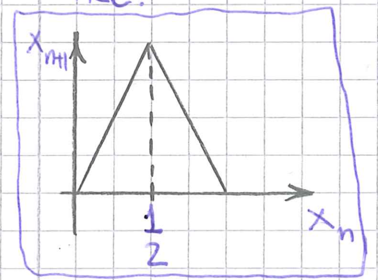
Consider the map:

$$x_{n+1} = \begin{cases} 2x_n ; & 0 \leq x_n \leq \frac{1}{2} \\ 2-2x_n ; & \frac{1}{2} \leq x_n \leq 1 \end{cases}$$

as a simple model of the Lorenz map

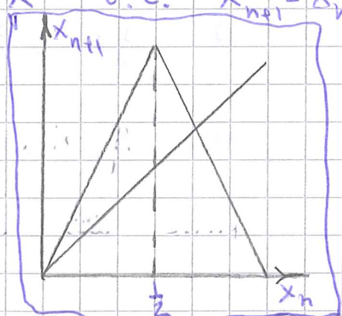
(a) Why is it called the "tent map"?

This is the simpliest question :) To undestand the answer it is enough just to draw how it looks like.



(b) Find all the fixed points and classify their stability.

* fixed points. x^* is stable point of map if $f(x^*) = x^*$ i.e. $x^*_{n+1} = x^*_n$



for our map this happens

$\boxed{x^* = 0; \quad x^* = \frac{2}{3}}$

* stability of this fixed points:

$$|f'(x^*)| = \begin{cases} 2 & 0 \leq x \leq \frac{1}{2} \\ |-2| = 2 & 0 \leq x \leq 1 \end{cases}$$

we see that $|f'(x^*)| > 1$ alwas and thus all fixed points we have are unstable

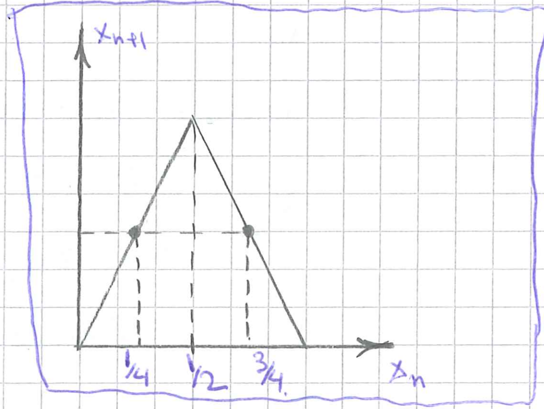
(c) Show that the map has a period-2 orbit. Is it stable or unstable?

Def: assume we have map $x_{n+1} = f(x_n)$. Now if $x^*_{n+p} = x^*_n$ we say that map has p-periodic orbit. x_n is called period-p point and p is period of sequence, since the sequence repeats after p iterations.

In our case we are looking for $p = 2$.

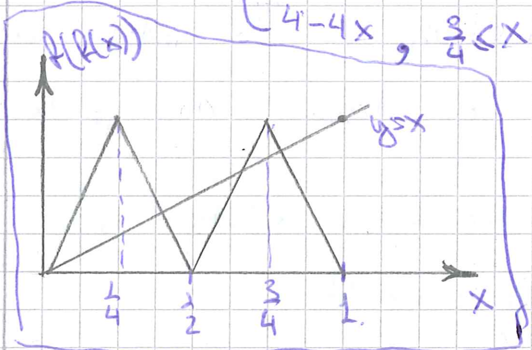
⑩ if we want $x_{n+2} = x_n$ we get $x_n = f(f(x_n))$

Let's build $f(f(x_n))$ function



$$f(f(x)) = \begin{cases} 2f(x) & ; \quad 0 \le f(x) \le \frac{1}{2} \\ 2-2f(x) & ; \quad \frac{1}{2} \le f(x) \le 1 \end{cases}$$

$$f(f(x)) = \begin{cases} 4x, & 0 \le x \le \frac{1}{4} \\ 2-4x, & \frac{1}{4} \le x \le \frac{1}{2} \\ 4x-2, & \frac{1}{2} \le x \le \frac{3}{4} \\ 4-4x, & \frac{3}{4} \le x \le 1 \end{cases}$$



* fixed points are given by $x^* = f(f(x^*))$; there are 4 intersections.

$x^* = 0$

if $\frac{1}{4} \le x \le \frac{1}{2}$ $\quad 2-4x^* = x^*; \quad x^* = \frac{2}{5}$

if $\frac{1}{2} \le x \le \frac{3}{4}$ $\quad 4x^* - 2 = x^* \Rightarrow x^* = \frac{2}{3}$

if $\frac{3}{4} \le x \le 1$ $\quad 4-4x^* = x^* \Rightarrow x^* = \frac{4}{5}$

thus we get 4 fixed points: $\boxed{x^* = 0; \ x^* = \frac{2}{5}; \ x^* = \frac{2}{3}; \ x^* = \frac{4}{5};}$

* What can we say? as usually we should take a look on derivative of map :

$\left| \frac{d}{dx} f(f(x)) \right| = 4 > 1$ alway thus all fixed points are unstable. Another thing we should notice here that 1-periodic orbits are 2-periodic at the same time., thus only really 2-periodic solutions correspond to fixed points $\boxed{x^* = \frac{2}{5}; \ x^* = \frac{4}{5};}$

ⓒ 3-periodic points.

Just in the same way as in previous part we get.

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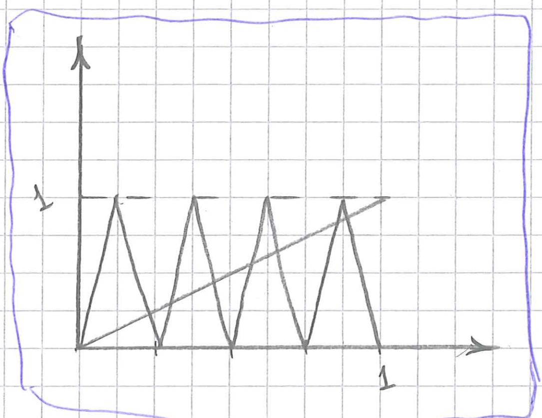
$$f^{(3)}(x) = f(f(f(x))) = \begin{cases} 2f(f(x)) & 0 \leq f^{(2)} \leq \frac{1}{2} \\ 2 - 2f^{(2)}(x) & \frac{1}{2} \leq f^{(2)} \leq 1 \end{cases}$$



$$f^{(3)}(x) = \begin{cases} 8x; & 0 \leq x \leq \frac{1}{8}; & x^* = 0 & 1\text{-p} \\ 2-8x; & \frac{1}{8} \leq x \leq \frac{1}{4}; & x^* = \frac{2}{9} & 3\text{-p} \\ -2+8x; & \frac{1}{4} \leq x \leq \frac{3}{8}; & x^* = \frac{2}{7} & 3\text{-p} \\ 4-8x; & \frac{3}{8} \leq x \leq \frac{1}{2}; & x^* = \frac{4}{9} & 3\text{-p} \\ -4+8x; & \frac{1}{2} \leq x \leq \frac{5}{8}; & x^* = \frac{4}{7} & 3\text{-p} \\ 6-8x; & \frac{5}{8} \leq x \leq \frac{3}{4}; & x^* = \frac{2}{3}, & 1\text{-p} \\ -6+8x; & \frac{3}{4} \leq x \leq \frac{7}{8}; & x^* = \frac{6}{7}; & 3\text{-p} \\ 8-8x; & \frac{7}{8} \leq x \leq 1; & x^* = \frac{8}{9}; & 3\text{-p} \end{cases}$$

So to reflect picture with respect to $y = \frac{1}{2}$ and stretch it twice to obtain $f^{(n)}(x)$ out of $f^{(n-1)}(x)$

Here we get following 3-periodic fixed points $x^* = \frac{2}{9}; \frac{2}{7}; \frac{4}{9}; \frac{4}{7}; \frac{6}{7}; \frac{8}{9}$. All points are unstable as $\left|\frac{d}{dx} f^{(3)}(x)\right| = 8 > 1$ always.

①

One-dimensional maps (seminar 13)

Def The rule $x_{n+1} = f(x_n)$ is called one-dimensional map, because points x_n belong to the one-dimensional space of real numbers. The sequence $x_0, x_1, x_2, \ldots$ is called the orbit starting from x_0. One-dimensional maps are used widely as:

* Tools for analyzing diff. equations (Poincare map → periodic orbit; Lorenz map → strange attractor)
* Models of nature (some phenomenon contain descrete time, such as electronics, finance or problems of populaton)
* Simple examples of chaos many maps, even abstract mathematical maps, provide us with some facts about chaotic systems, so 1d maps can be considered as computer laboratories of chaos.

Tools for analyzing 1d maps

Let's consider 1d map

$x_{n+1} = f(x_n)$

* First thing we should explore are fixed points of our map which by definition are given by $x^* = f(x^*)$, i.e. after coming to x^* (exactly) orbit remains there forever.

$x_n = x^* + \eta_n$ nearby orbit

$x^* + \eta_{k+1} = x_{n+1} = f(x_n) = f(x^* + \eta_n) = f(x^*) + f'(x^*)\eta_n + O(\eta_n^2)$

thus we conclude that

$\eta_{n+1} = f'(x^*)\eta_n + O(\eta_n^2)$. If we now go on with this itterations

$\eta_{n+k} = (f'(x^*))^k \eta_n$, $f'(x^*)$ is called linearized map

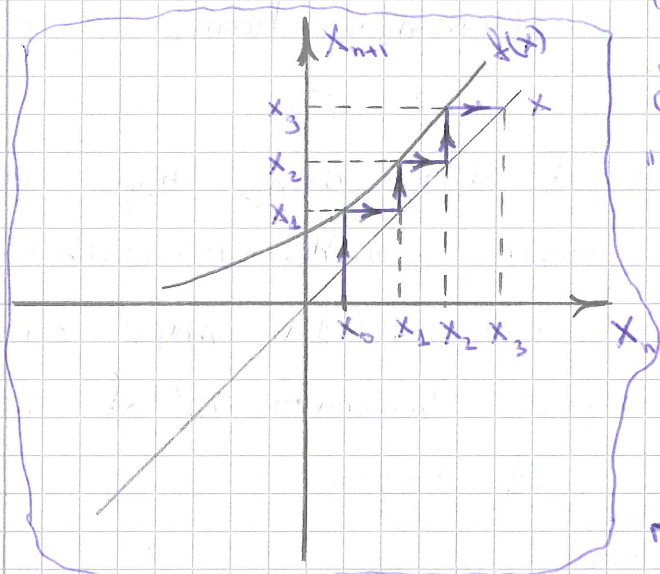
②

just as in the case of Poincare map.
we can note $f'(x^*) = \lambda$ and than perturbation after n^{th} itteration becomes:

$\eta_n = \lambda^n \eta_0$, now

* if $|\lambda| = |f'(x^*)| < 1$ then $\eta_n \to 0$ as $n \to \infty$, then fixed point is linearly stable.
* if $|\lambda| = |f'(x^*)| > 1$ then $\eta_n \to \infty$ as $n \to \infty$ and point appears to be linearly unstable

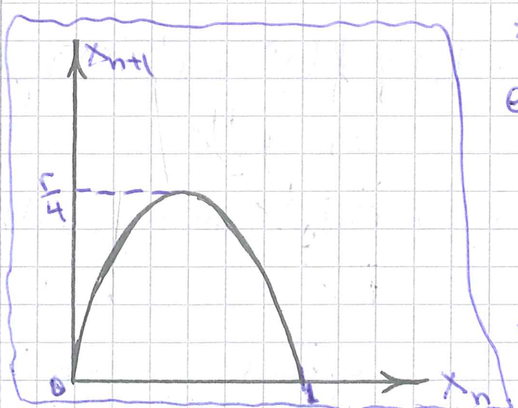
Another way to understand whethe map is stable or not is using cob web construction.



To construct this we should draw graph of "$f(x)$" and "x" streight line
Now we denote x_0 and then where it is mapped by $x_1 = f(x_0)$. For this we just draw vertical line. Then we should pict this x_1 on x-axis for this we start on the graph of $f(x)$ where we finished our previous step and draw horizontal line up to intersection with line $y = x$.

Logistic maps

$x_{n+1} = r x_n (1 - x_n)$ $\quad r \geq 0$ intrinsic growth rate $\quad x_n \geq 0$



* $r < 1$ population always goes extinct $x_n \xrightarrow[n \to \infty]{} 0$
* $r = 1$ transcritical bifurcation (origin loses stability)
* $1 < r < 3$ population grows and

(3) eventually reaches nonzero steady state

* $r=3$ flip bifurcation occurs (critical slope $f'(x^*)=-1$ at $r=3$ and $x^*=1-\frac{1}{r}$ loses stability.

* $3<r<4$ population builds up but now oscillates about former steady state

Alternating between a large population in one generation and smaller population. In next.

When we go on increasing r sometimes steady state is repeated, this periodicity of cycles is being doubled when we increase r.

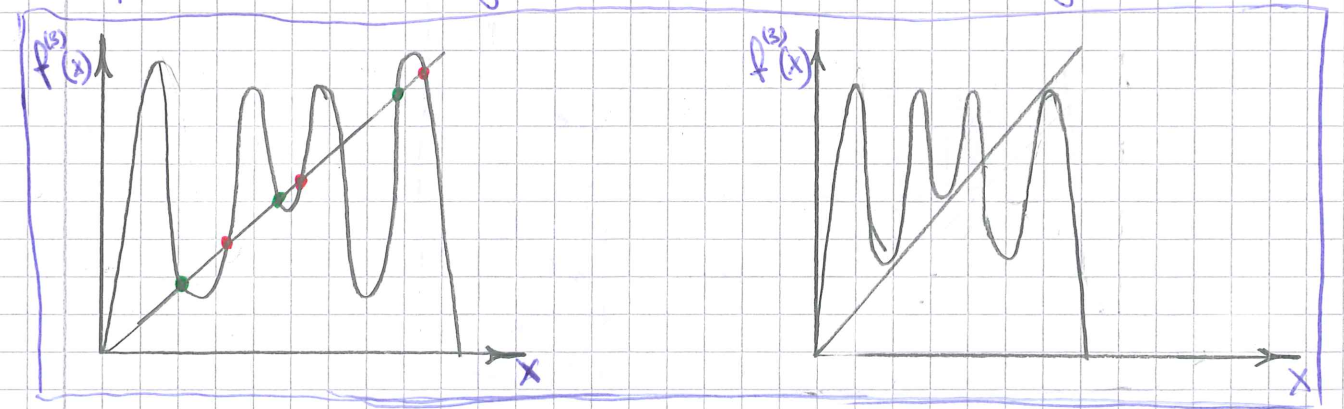
Let's say that at r_n 2^n cycle born.

than at some point (from computer simulations we know that this point is $r_\infty = 3,5699$ $n\to\infty$

when $r>r_\infty$ we get mix of chaotic and ordered behaviour. The convergence of r_n series is geometrical: in the limit of large n, the distance between successive transitions shrinks by a constant factor $\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1}-r_n} = 4,669$;

When $r>r_\infty$ sometimes periodic windows occure.

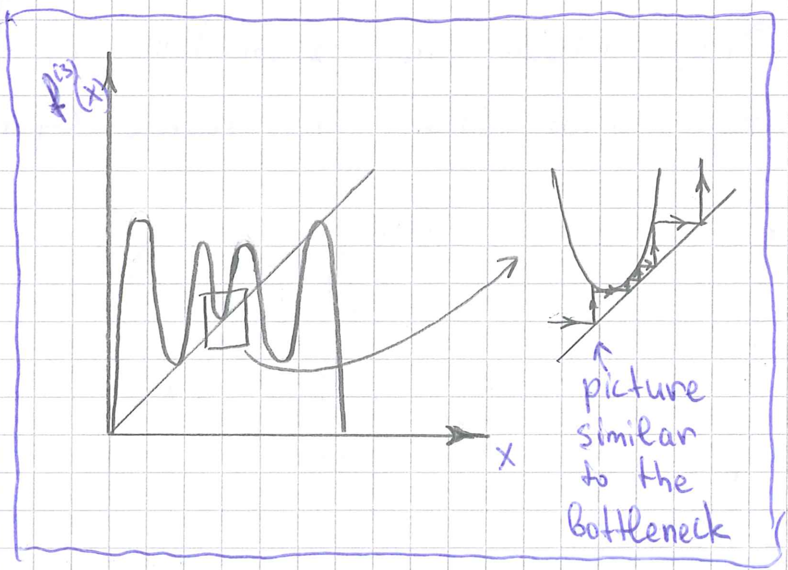
For example for $3,8284 \leq r \leq 3,8415$ there is period-3 window. Any p in period-3 cycle repeats every 3 iterations $p=f^{(3)}(p)$ (there are 8 solutions of this equation but only 6 of them are really 3-periodic)



④ While we decrease "r" hills move down and valleys rise up. Eventually tangent bifurcation occurs. that is this 3 period cycles disappear and we get chaotic behaviour. This is the begining of period-3 window. Tangent Bifurcation occurs at $r = 1 + \sqrt{8}$;

Intermittency

For r just below the 3 period window part of orbit looks like a stable 3-period cycle, and called ghost 3-cycle. Orbit returns to the ghosty 3-cycle repeatedely with intermittent beats of chaos between visits. Intermittency is common in the systems where transition from periodic to chaotic behaviour through a saddle-node bifurcation occurs. In experiments we can see often nearly periodic movement interrupted by occasional irregular bursts. Time between bursts disturbed statistically (it is random variable, even if system is determenistic). As parameter is taken away from the periodic window bursts became more frequent untill the system is fully chaotic - intermittency route to chaos



(5)

Lyapunov exponent

Now the question comes whether logistic map really reveals chaotic behaviour? We have already seen that there are aperiodic orbits for certain values of r-parameter. But chaotic behaviour means more than this - system should be sensetive to initial conditions.

Let's disturb initial condition x_0:

$x_0 \to x_0 + \delta_0$, $\delta_0 \ll 1$; and say that after n iterations we get $|\delta| = |\delta_0| e^{\lambda n}$ then λ is called Lyapunov exponent.

If we take now $\delta_n = f^n(x_0+\delta_0) - f^n(x_0)$; we get:

$$\lambda = \frac{1}{n} \ln \left|\frac{\delta_n}{\delta_0}\right| = \frac{1}{n} \ln \left| \frac{f^n(x_0+\delta_0) - f^n(x_0)}{\delta_0} \right| = \frac{1}{n} \ln |(f^n)'(x_0)|$$

Now we can find derivative standing in logarithm using chain rule:

$$f^n(x) = \underbrace{f(f(\dots(f(x_0))\dots)}_{n \text{ times}};$$

$$(f^n)'(x_0) = f'(f(\dots(f(x_0)))\dots) = \frac{df^{n-1}}{dx}\frac{df^n}{df^{n-1}} = \frac{df^{n-2}}{dx} \cdot \frac{df^{n-1}}{df^{n-2}} \cdot \frac{df^n}{df^{n-1}} =$$

$$= \prod_{i=0}^{n} f'(x_i) = f'(x_0) f'(x_1) f'(x_2) \dots$$

$$\lambda = \frac{1}{n} \log \left| \prod_{i=0}^{n} f'(x_i) \right| = \frac{1}{n} \sum_{i=0}^{n-1} \log |f'(x_i)|$$

And now we define Lyapunov exponent for orbit starting in x_0 as

$$\boxed{\lambda = \lim_{n\to\infty} \sum_{i=0}^{n-1} \log |f'(x_i)|}$$

criteria of stability is the following

* if $\lambda > 0$ then orbit is stable
* if $\lambda < 0$ orbit is unstable

6

Universality

Let's consider unimodal map (i.e. smooth, concave down, and having single maximum), and multiply it with parameter r.

$x_{n+1} = r f(x_n)$, $f(0) = f(1) = 0$;

example of unimodal map

Now we start varying r and observe that the order in which stable periodic solutions appear is independent of the unimodal map being iterated. Periodic attractors always occur in the same sequence which is called universal u-sequence

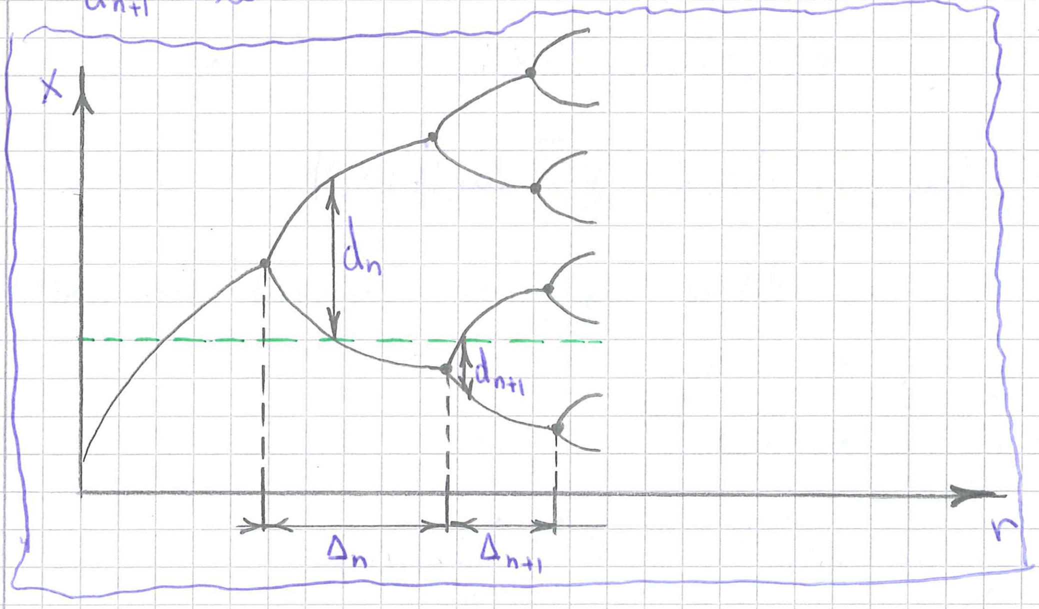
Quantative universality

* same convergence rate appears no matter what unimodal map is iterated, i.e

$\delta = \lim\limits_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4{,}669\ldots$ is universal number.

* d_n = distance from x_m (maximum of f) to the nearest point in a 2^n-cycle then

$\frac{d_n}{d_{n+1}} \xrightarrow[n\to\infty]{} -2{,}5$ - universal limit.



7

Exercise 10.3.7 (a chaotic map that can be analized completely)

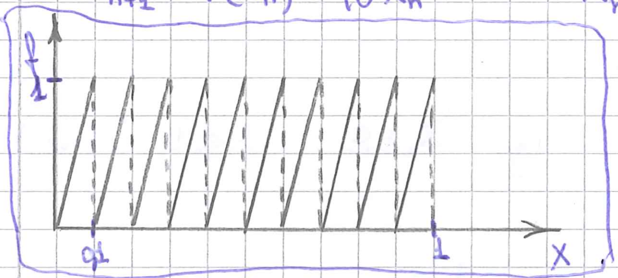
Decimal shift map on the unit interval.

This is map of form

$x_{n+1} = 10x_n \pmod 1$ "mod 1" means that we look only at noninteger part of x)

(a) Draw the graph of the map.

$x_{n+1} = f(x_n) = 10x_n \quad x_n \in [0, 1)$



(b) Find all the fixed points.

Let's take x being written in decimal form, i.e.

$x = 0, a_1 a_2 a_3 \ldots \quad 0 \leq x \leq 1$

this point is mapped onto $f(x) = 10x \pmod 1 =$

$= a_1, a_2 a_3 a_4 \ldots \pmod 1 = 0, a_2 a_3 a_4 \ldots$

so condition for this point to be fixed point is

$a_i = a_{i+1}$. $\forall i$, so $x^* = \{0; 0,(1); 0,(2); 0,(3); \ldots; 0,(9)\}$;

(c) Show that the map has periodic points of all periods, but that all of them are unstable

Let's take points:

$x = 0, a_1 a_2 a_3 a_4 \ldots$

$f(x) = 0, a_2 a_3 a_4;\quad f^{(2)}(x) = 0, a_3 a_4 a_5 \ldots;\quad f^{(3)}(x) = 0, a_4 a_5 a_6 \ldots; \ldots;$

$f^{(n)}(x) = 0, a_{n+1} a_{n+2} a_{n+3} \ldots$

Existence of p-periodic points:

$x_{n+p} = f^{(p)}(x_n)$, $a_{i+p} = a_i \ \forall i$ is condition for p-periodic orbits

$$\boxed{x^* = 0, \underbrace{a_1 a_2 a_3 \ldots a_p}_{p \text{ numbers}} \underbrace{a_1 a_2 \ldots a_p}_{p \text{ numbers}} \ldots}$$

(d) Show that the map has infinitely many aperiodic

(8) orbits.

Any orbit starting at an irrational number x_0 will be aperiodic, since the decimal expansion of an irrational number x_0 will be aperiodic, since the decimal expansion of an irrational number never repeats.

(e) By considering the rate of separation between two nearby orbits, show that the map has sensitive dependence on initial conditions.

Let's consider 2 orbits

$x_0; x_1; x_2; \dots$ where $y_0 = x_0 + \varepsilon$, $\varepsilon \ll 1$ $(\varepsilon > 0)$

$y_0; y_1; y_2; \dots$ thus

$$y_1 = f(x_0+\varepsilon) = 10x_0 + 10\varepsilon \pmod 1 = 10x_0 \pmod 1 + \varepsilon_1 = x_1 + \varepsilon_1$$

$$y_2 = f(y_1) = f(x_1+\varepsilon_1) = 10x_1 \pmod 1 + 10\varepsilon_1 \pmod 1 = x_2 + \varepsilon_2$$

We can calculate Lyapunov exponent:

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log|f'(x_i)| = n \cdot \frac{1}{n} \log 10 = \log 10 > 0$$

, here we have used $f'(x_i) = 10 \ \forall i$ as $\lambda > 0$ map is indeed sensitive.

Exercise 10.3.13

The orbit diagramm of logistic map exhibits some striking features that are rarely discussed in books

(a) There are several smooth, dark tracks of points running through the chaotic part of diagram. What are these curves?

Let's consider logistic map again $x_{n+1} = f(x_n) = r x_n(1-x_n)$

*f.p. is in $x^* = 0$ and $x^* = 1 - \frac{1}{r}$;

*stability: $f'(x^*) = r - 2rx^*$

$f'(0) = r$ stable for $r < 1$

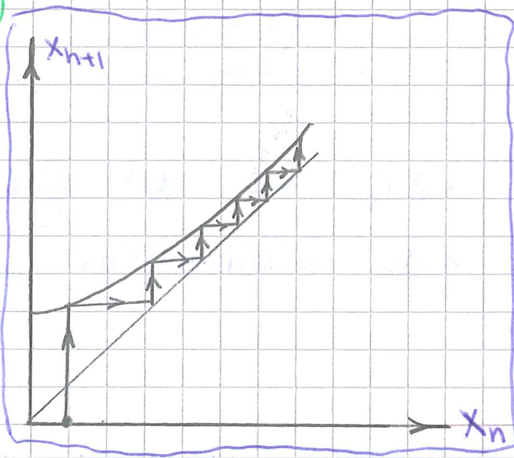
(9) $f'(1-\frac{1}{r}) = r - 2r + 2 = 2 - r$ stable for $1<r<3$
at some critical value $r_\infty = 3,57$ system reveals chaotic behaviour.
Let's consider point of maximum $x_m = \frac{1}{2}$;
$f = rx(1-x)$; $f(\frac{1}{2}) = \frac{r}{4}$; $f^{(2)}(\frac{1}{2}) = \frac{r^2}{4}(1-\frac{1}{4}r)$;
$f^{(3)}(\frac{1}{2}) = f(\frac{r^2}{4}(1-\frac{r}{4}))$, ... as $f'(x)=0$ in $x=\frac{1}{2}$ then points are mapped almost to the same point and density of maps for this point is high $\Rightarrow$ dark lines are just curves of $f^{(k)}(x_m, r)$.

(b) Can you find the exact value of r at the corner of the "Big wedge"

The corner occurs when $f^{(3)}(\frac{1}{2}) = f^{(4)}(\frac{1}{2})$ (by the graph)
$f^{(3)}(\frac{1}{2}) = x$, $x = f(x)$, $x = rx(1-x) \Rightarrow x(r-1-rx)=0$
thus we get $x=0$ or $x = \frac{r-1}{r} = 1 - \frac{1}{r}$;
$f^{(3)}(\frac{1}{2}) = 1 - \frac{1}{r} = f(\frac{r^2}{4}(1-\frac{r}{4})) = 1 - \frac{1}{r}$
Solution is $r = 3,67$.

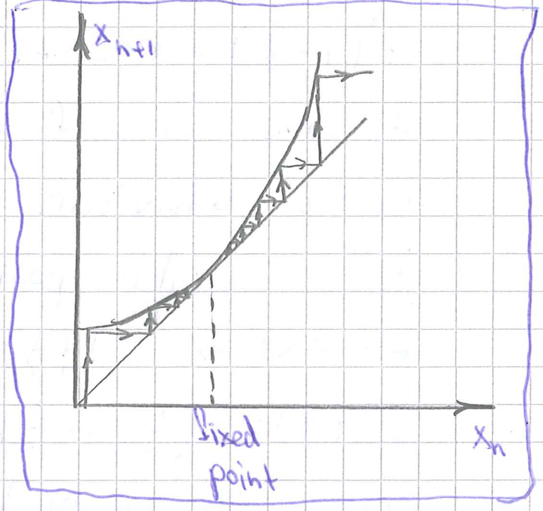
Exercise 10.4.1 (Periodic window)
Consider the map $x_{n+1} = r\exp(x_n)$ for $r>0$
(a) analyze map drawing cobweb
Here we should consider 3 cases
* $r > r_c$ (r_c is parameter when tangent bifurcation occur)

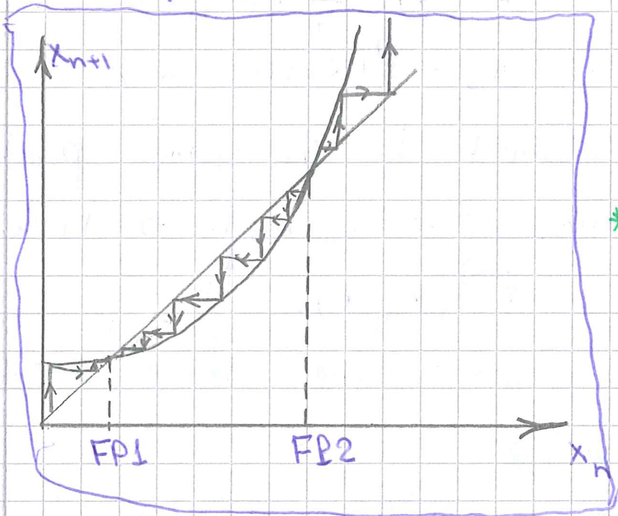
This case is picted on picture to the right and as we see there are no fixed points



(10)

* $r = r_c$ point where tangent bifurcation occurs. and one half-stable fixed point occurs





* $r < r_c$

fixed point 1 is stable

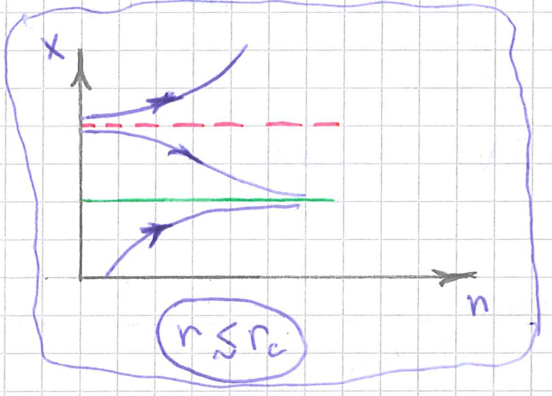
fixed point 2 is unstable.

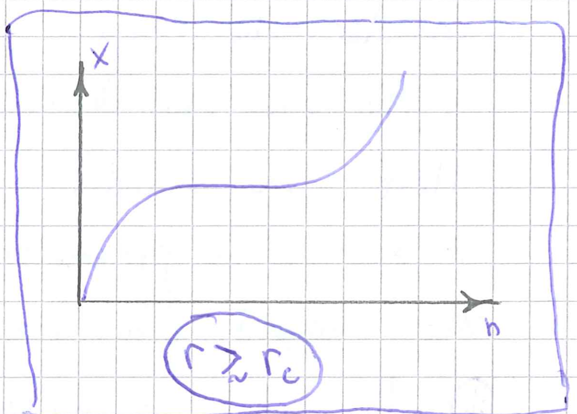
(b) Show that a tangent bifurcation occurs at $r = \frac{1}{e}$;

Bifurcation occurs where line $x_{n+1} = x_n$ is tangent to map wich is given by solution of the following system

$$\begin{cases} x = f_r(x) = re^x \\ \dfrac{df_r}{dx} = re^x = 1 \end{cases}$$ thus $x = 1$ and $r_c = \frac{1}{e}$

(c) Sketch the time series x_n vs. n for r just above and just below $r = \frac{1}{e}$;





$r \gtrsim r_c$ we get ghost wich always occurs near saddle-node bifurcations, namely tangent bifurcation.

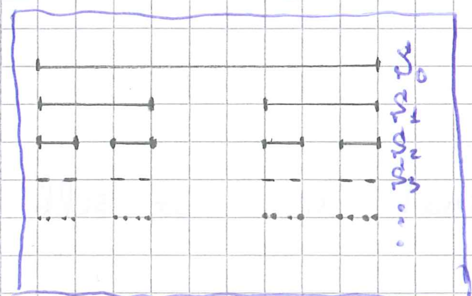
① Fractals (Seminar 14)

Def: Complex geometric shapes with fine structure usually with some degree of self-similarity.

Some definitions

* 2 set's X, Y have the same cardinality (or number of elements) if there exists invertible mapping that pairs each element $x \in X$ with precisely one $y \in Y$. This mapping is called one-to-one correspondence
* set is countable if X can be put into one-to-one correspondence with the natural numbers. Otherwise set is uncountable.

Example of fractal is Cantor set.



$S_0 = [0;1]$; $S_1 = [0;\frac{1}{3}] \cup [\frac{2}{3};1]$;

$S_2 = [0;\frac{1}{9}] \cup [\frac{2}{9};\frac{1}{3}] \cup [\frac{2}{3};\frac{7}{9}] \cup [\frac{8}{9};1]$;

$S_3 = \ldots\ldots$

$\ldots\ldots$

So we, on each iteration, remove middle thirds of intervals

Cantor set is $C \equiv S_\infty$ - infinite number of infinitely small pieces separated by gaps of various sizes.

Properties of fractal.

* C has a structure at arbitrarily small scales
* C is self similar: small copies of itself at all scales (left half of S_∞ looks like S_∞ scaled down by 3)
* The dimension of C is not an integer ($\frac{\log 2}{\log 3} = 0,63$)
* C has measure 0 and it consists of uncountably

② many points)

* C consists of all points $c \in [0,1]$ that have no 1 in their base-3 expansion

Dimensions of fractals

Dimension, by definition, is the minimum number of coordinates needed to describe every point in the set.

There are 3 ways to define dimension of fractals:

* Similarity dimension

Simple fractals are self-similar usually. So we can define dimension for such constructions in the following way

$$\boxed{d = \frac{\log m}{\log r}}$$

Here self-similar set is composed of m copies of itself scaled down by a factor r

* Box dimension

No we go to fractals that are not self similar

Let S be a subset of D-dimensional Euclidian space and $N(\varepsilon)$ is minimum number of D-dimensional cubes of side ε needed to cover S. usually $N(\varepsilon)$ scales as powers of ε:

$N(\varepsilon) \sim \frac{1}{\varepsilon^d}$, and we are able to define box dimension as

$$\boxed{d = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln (1/\varepsilon)}}$$

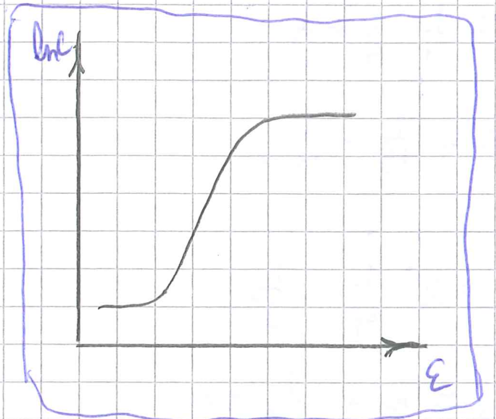
if the limit exists.

* Pointwise and correlation dimensions.

Let's consider chaotic system that settles down to a strange attractor in phase space $\bar{X} \in A$ is some point on the attractor. Let

③ $N_{\bar{x}}(\varepsilon)$ be number of points in the ball of radius ε. usually $N_{\bar{x}}(\varepsilon)$ is scaled as $N_{\bar{x}}(\varepsilon) \sim \varepsilon^d$ and d is then called pointwise dimension at $\bar{x}$. Other way is to average $N_{\bar{x}}(\varepsilon)$ over may $\bar{x}$ The resulting quantaty is scaled as $C(\varepsilon) \sim \varepsilon^d$ (this is empirical result) then d is called correlation dimension

to measure correlation dimension one usually draws $\ln C(\varepsilon)$ vs ε and get something like

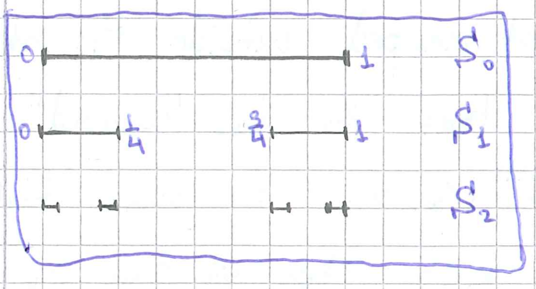


Slope should be measured in intermediate values of ε Because when ε becomes larger than size of attractor $N(\varepsilon)$ saturates and vice versa when ε is small enough only $\bar{x}$ point is covered by ball ε.

Ex 11.3.1 (middle-halves Cantor set)

Construct a new kind of Cantor set by removing the middle half of each sub-interval, rather than middle third.

ⓐ Find similarity dimension of the set.



$$d = \frac{\log m}{\log r} = \frac{\log 2}{\log 4} = \frac{1}{2};$$

m - number of copies $m=2$

r - scaling of copies $r=4$

Box dimension

$$d = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log\left(\frac{1}{\varepsilon}\right)} \quad \text{by definition}$$

④ * step 1 : $\varepsilon = \frac{1}{4}$, $N = 2$;

* step 2 : $\varepsilon = \frac{1}{16}$, $N = 4$;

....

* step n : $\varepsilon = \frac{1}{4^n}$, $N = 2^n$;

then box dimension $d = \lim_{n\to\infty} \frac{\log 2^n}{\log 4^n} = \frac{1}{2}$ which coincides with similarity dimension

ⓑ measure of the set.

length of S_0': $L_0 = 1$;

S_1': $L_1 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$;

S_2': $L_2 = 4 \cdot \frac{1}{16} = \frac{1}{4}$;

....

S_n': $L_n = 2^n \cdot \frac{1}{4^n} = \frac{1}{2^n}$;

S_∞': $L_\infty = \lim_{n\to\infty} \frac{1}{2^n} = 0$;

So the set has measure 0, it consists of uncountably many points, the dimension is not an integer, it is self-similar at arbitrarily small scales.

Exercise 11.4.6 (strange repeller of tent map)

Tent map on the interval $[0,1]$ is defined as

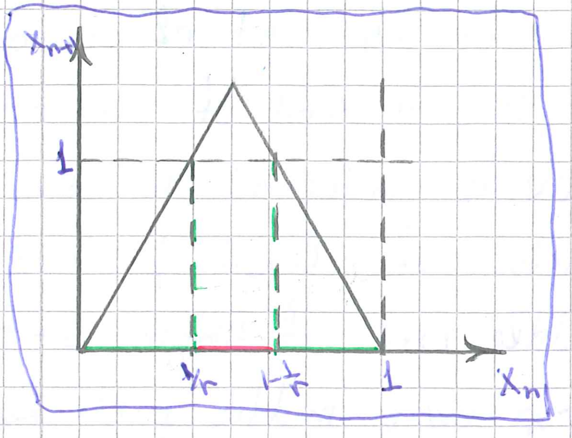
$$f(x) = \begin{cases} rx & ; \ 0 \le x \le \frac{1}{2} \\ r(1-x) & ; \ \frac{1}{2} \le x \le 1 \end{cases}$$

Assume in this exercise that $r > 2$ get mapped outside the interval $[0,1]$

if $f(x_0) > 1$, then we say that x_0 has "escaped" after one iteration. Similarly, if $f^{(n)}(x_0) > 1$ for some finite n, but $f^{(k)}(x_0) \in [0,1]$ for all $k < n$, then we say x_0 has escaped after n iterations.

ⓐ Find the set of initial conditions x_0 that escape after one or two iterations.

⑤ * first iteration



Let's find interval for which $f(x) > 1$;

$rx = 1 \Rightarrow x_1 = \frac{1}{r}$

$r - rx = 1 \Rightarrow x = 1 - \frac{1}{r}$;

for interval $\frac{1}{r} < x < 1 - \frac{1}{r}$ $f(x) > 1$ and thus this points escape after first iteration

$\boxed{x \in \left(\frac{1}{r}; 1 - \frac{1}{r}\right) \text{ escape after first iteration.}}$

* second iteration

$$f^{(2)}(x) = f(f(x)) = \begin{cases} r f(x); & 0 \leq f(x) \leq \frac{1}{2} \\ r(1 - f(x)); & \frac{1}{2} \leq f(x) \leq 1 \end{cases}$$

$f(x) = \frac{1}{2}$; $rx = \frac{1}{2}$, $x = \frac{1}{2r}$; $\Rightarrow 0 \leq f(x) \leq \frac{1}{2}$ when $0 \leq x \leq \frac{1}{2r}$

and $r - rx = \frac{1}{2}$, $x = 1 - \frac{1}{2r}$; $\Rightarrow 0 \leq f \leq \frac{1}{2}$ when $1 - \frac{1}{2r} \leq x \leq 1$

thus we get

$0 \leq f(x) \leq \frac{1}{2}$ when $0 \leq x \leq \frac{1}{2r}$ $(f(x) = rx)$ and $1 - \frac{1}{2r} \leq x \leq 1$ $(f(x) = r(1-x))$

$\frac{1}{2} \leq f(x) \leq 1$ when $\frac{1}{2r} \leq x \leq \frac{1}{r}$ $(f(x) = rx)$ and $1 - \frac{1}{2r} \leq x \leq 1 - \frac{1}{r}$ $(f(x) = r(1-x))$

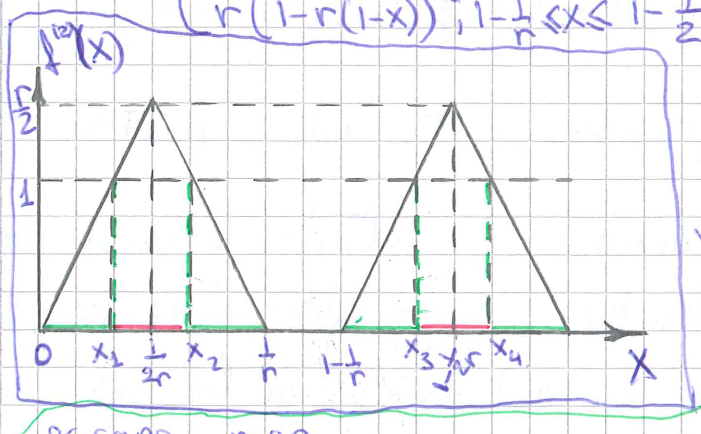
$$f^{(2)}(x) = \begin{cases} r^2 x; & 0 \leq x \leq \frac{1}{2r} \\ r^2(1-x); & 1 - \frac{1}{2r} \leq x \leq 1 \\ r(1 - rx); & \frac{1}{2r} \leq x \leq \frac{1}{r} \\ r(1 - r(1-x)); & 1 - \frac{1}{r} \leq x \leq 1 - \frac{1}{2r} \end{cases} \Rightarrow$$

point 1: $r^2 x = 1$, $x_1 = \frac{1}{r^2}$;

point 4: $r^2(1-x) = 1$; $x_4 = 1 - \frac{1}{r^2}$;

point 2: $r(1 - rx) = 1$; $x_2 = \frac{1}{r}\left(1 - \frac{1}{r}\right)$;

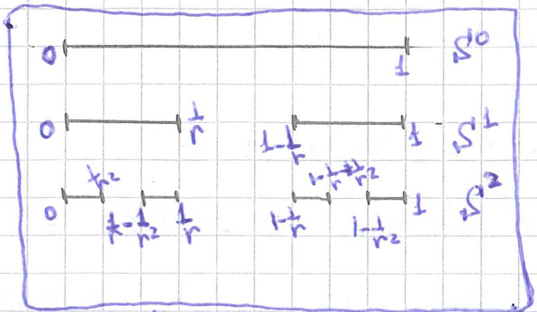
point 3: $r(1 - r(1-x)) = 1$; $x_3 = \frac{1}{r^2} + \frac{r-1}{r}$;



now if we take a look on the picture of $f^{(2)}(x)$ map we can see that

points in intervals $\left(\frac{1}{r^2}; \frac{1}{r}\left(1 - \frac{1}{r}\right)\right) \cup \left(\frac{1}{r^2} + 1 - \frac{1}{r}; 1 - \frac{1}{r^2}\right)$ escape map.

6 b) c) Find the box dimension of the set of x_0 that never escape and describe the set of points that never escape. This is a kind of Cantor set:



devide interval in thre parts: 2 of length $\frac{1}{r}$ and middle part of length $1-\frac{2}{r}$, then drop away middle part. Repeat procedure with 2 left parts and so on. The points of Cantor set we get after sufficiently large number of iterations will never escape.

Now let's find Box dimension:

S_0 : $n=1$ box $\varepsilon=1$;

S_1 : $n=2$ boxes $\varepsilon=\frac{1}{r}$;

S_2 : $n=4$ boxes $\varepsilon=\frac{1}{r^2}$;

. . . .

S_N $n=2^N$ boxes $\varepsilon=\frac{1}{r^N}$;

Box dimension is given by $d=\lim\limits_{\varepsilon\to0}\frac{\log N(\varepsilon)}{\log\left(\frac{1}{\varepsilon}\right)}=\lim\limits_{N\to\infty}\frac{\log 2^N}{\log(r^N)}=$

$=\log_r 2$; $\boxed{d=\log_r 2}$

d) Show that Liapunov exponent is positive at each point in the invariant set.

$$\lambda=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log|f'(x_i)|=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\log r^i=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n-1}i\log r=$$

$$=\lim_{n\to\infty}\frac{1}{n}\cdot\frac{n(n-1)}{2}\log r\to+\infty;$$ it's indeed positive.

The invariant structure is called strange repeller. It has fractal structure, it repells all nearby points that are not in the set and points in the set walk around chaotically under iterations of the tent map.

(7) **Exercise 11.2.6** Devils staircase.

Suppose that we pick a point at random from Cantor set. What is the probability that this point lies to the left of x, where $0 \leq x \leq 1$ is some fixed number

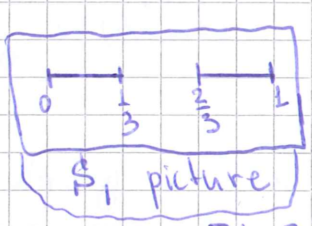
ⓐ Visualize $P(x)$ by building it up in stages.

First consider the set S_0. Let $P_0(x)$ be the probability that a randomly chosen point in S_0 lies to the left of x. Show that $P_0(x)=x$

We understand that

$P(0)=0$; $P(1)=1$. Points are equally distributed so $P(x)$ is linear in x and we get $P(x)=x$.

ⓑ Consider S_1 and $P_1(x)$.

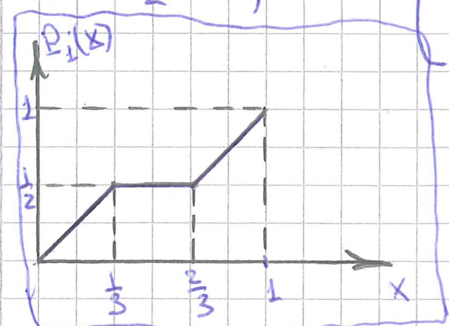


Again we say that $P_1(0)=0$; $P_1(1)=1$
$P(x)$ should be linear on both intervals $[0;\frac{1}{3}] \cup [\frac{2}{3},1]$ and constant on the interval $[\frac{1}{3};\frac{2}{3}]$

$$P = \begin{cases} \alpha x & ; 0 \leq x \leq \frac{1}{3} \\ \frac{1}{3}\alpha & , \frac{1}{3} \leq x \leq \frac{2}{3} \\ \frac{1}{3}\alpha + \alpha(x-\frac{2}{3}) & ; \frac{2}{3} \leq x \leq 1 \end{cases}$$ - this is appropriate ansatz

Now let's use boundary condition $\alpha(-\frac{1}{3}+x)\Big|_{x=1} = 1 \Rightarrow$

$\Rightarrow \alpha = \frac{3}{2}$, $P = \begin{cases} \frac{3}{2}x & , x \in [0;\frac{1}{3}] \\ \frac{1}{2} & x \in [\frac{1}{3};\frac{2}{3}] \\ \frac{3}{2}x - \frac{1}{2} & x \in [\frac{2}{3};1] \end{cases}$

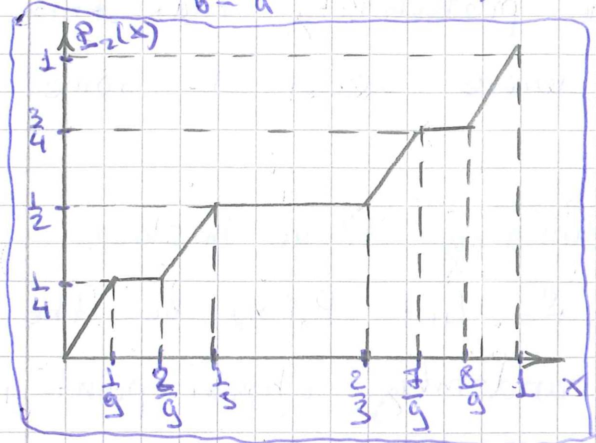


ⓒ Draw graphs of $P_2(x)$, $P_3(x)$, $P_4(x)$

For arbitrary n we have equal slope for 2^n intervals. Their summ gives 1 and difference $P(a)-P(b)$ is equal for each interval $[a,b]$; $P(b)-P(a) = \frac{1}{2^n}$. The length of intervals is given by: $L_n = \frac{1}{3^n}$;

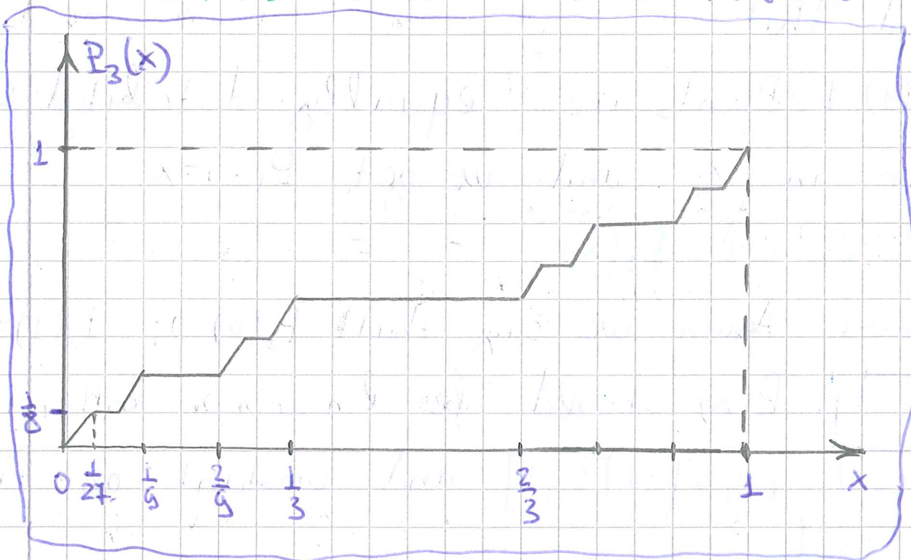
8 Then, finally we can find the slope:

$\alpha = \frac{P(b) - P(a)}{b - a} = \left(\frac{3}{2}\right)^n$ and graph looks like



$$P_n'(x) = \begin{cases} \left(\frac{3}{2}\right)^n, & x \in S_n \\ 0, & \text{everywhere else} \end{cases}$$

d) The limiting function $P_\infty(x)$ is the devils staircase. Is it continuous. How the graph of derivatives looks like? Derivatives themselves are given by expression above. In the limit $n \to \infty$



$$P_\infty'(x) = \begin{cases} +\infty & x \in S_\infty \\ 0 & \text{everywhere else} \end{cases}$$

i.e. graph looks like delta-functions over Cantor set.

1

Seminar 15 (exam of 30-05-2011)

№2

Consider equations of the form

$$\ddot{x} + f(\dot{x}) + g(x) = 0$$
$$f(x) = f(-x);$$

ⓐ Show that equation is invariant under time reversal symmetry.

Time reversal symmetry acts in the following way: $t \to -t$; $\dot{x} \to -\dot{x}$; $\ddot{x} \to \ddot{x}$

so equation changes in the following way:

$\ddot{x} + f(\dot{x}) + g(x) = 0 \to \ddot{x} + f(-\dot{x}) + g(x) = 0$ which, because of $f(x)$ being even is just the same equation. $\ddot{x} + f(\dot{x}) + g(x) = 0$.

ⓑ Show that equilibrium points can not be stable nodes or spirals.

Let's write down system in the following form

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(y) - g(x) \end{cases}$$

As system is reversible we may draw part of the trajectory and make mirror reflection of it. Fixed point should be somewhere on x-axis.

* if we assume that point is stable node we get.

(2) * now if we assume that we get spiral:

upper half-plane picture

after reflection we get centers rather than cycles

(N3) The system:

$$\begin{cases} \dot{x} = (r^2-6)x - y + xy^2 \\ \dot{y} = x + (r^2-6)y + y^3 \end{cases}$$

undergoes 2 different Hopf bifurcations as the parameter r is varied. Find the values of r where the Hopf bifurcation occurs.

Linearized system looks like

$$J(x,y) = \begin{bmatrix} r^2-6+y^2 & -1+2xy \\ 1 & (r^2-6)+3y^2 \end{bmatrix}$$

Let's consider fixed point at the origin $(x^*, y^*) = (0,0)$

$$J(0,0) = \begin{bmatrix} r^2-6 & -1 \\ 1 & r^2-6 \end{bmatrix} \qquad \Delta = (r^2-6)^2+1; \quad \tau = 2(r^2-6)$$

Hopf bifurcation occur when solutions are purely imaginary thus $\tau^2 - 4\Delta = 4((r^2-6)^2 - (r^2-6)^2 - 1) < 0$ always and $\tau = 0$ $\boxed{r = \pm\sqrt{6}}$ - this are points where Hopf bifurcations occur.

3

N4

Calculate Liapunov exponent for linear map.

By definition Liapunov exponent is given by

$$\lambda = \frac{1}{N}\sum_{i=0}^{N-1} \log |f'(x_i)|$$

$x_{n+1} = r x_n$

for this map $f(x) = rx$ and $f'(x_i) = r$, thus

$$\lambda = \frac{1}{N}\sum_{i=0}^{N-1} \log r = \log r \quad ; \quad \boxed{\lambda = \log r}$$

N5

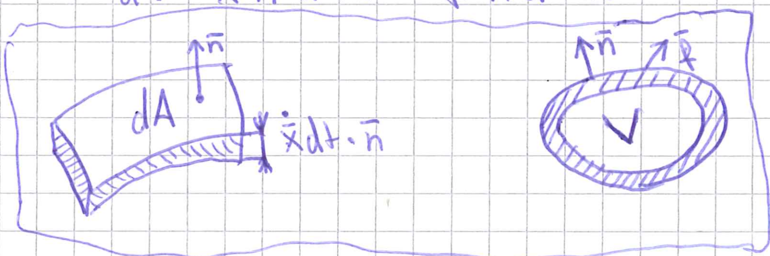
For what parameter values is the Lorenz equation volume contracting?

$$\begin{cases} \frac{dx}{dt} = \sigma(y-x) \\ \frac{dy}{dt} = rx - y - xz \\ \frac{dz}{dt} = xy - bz \end{cases}$$

Let's consider general case $\dot{\bar{x}} = \bar{f}(\bar{x})$

Volume element is given by

$dV = \dot{\bar{x}} \cdot \bar{n}\, dA\, dt = \bar{f} \cdot \bar{n}\, dA\, dt$



thus $\frac{dV}{dt} = \dot{V} = \oint_S \bar{f} \cdot \bar{n}\, dS = \int_V \bar{\nabla} \bar{f}\, dV$ here we have used Gauß theorem. Let's now find divergence of $\bar{f}(\bar{x})$ for our system

$$\bar{\nabla}\bar{f} = \frac{\partial}{\partial x}(\sigma(y-x)) + \frac{\partial}{\partial y}(rx - y - xz) + \frac{\partial}{\partial z}(xy - bz) =$$

$= -\sigma - 1 - b$, if we want system to be volume contracting ($\dot{V} < 0$) we demand $\bar{\nabla}\bar{f} < 0$ and get the following condition $\boxed{1 + \sigma + b > 0}$

N6

Compute the Hausdorf dimension of 3 dimensional Cantor's dust set.

④ This structure is self-similar thus we can just find self-similarity dimension and it will be Hausdorff dimension

we get $m = 8$ copies of cubes scaled down by 3

thus Hausdorf dimension is

$$d = \frac{\log m}{\log r} = \log_3 8 \, ; \qquad \boxed{d = \log_3 8}$$