① # Theoretical part (Introduction class)

Ground for relativistic mechanics is the pain of einstein postulates:

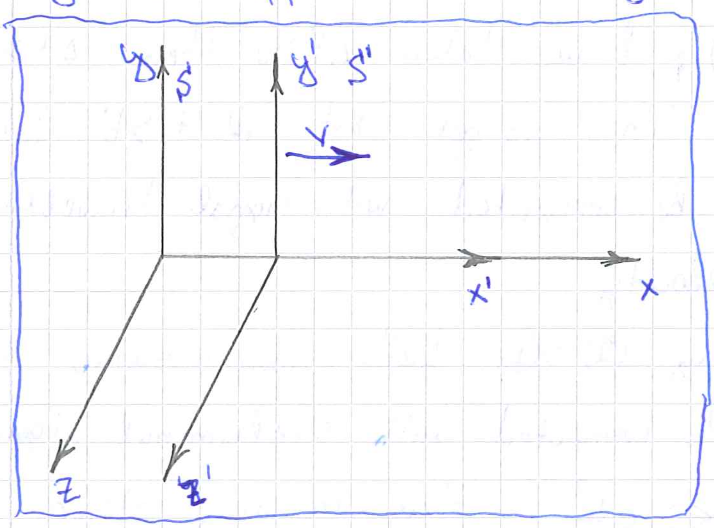
(I) The laws of physics are identical in any inertial frame

(II) The speed of light in a vacuum, c, is the same in any inertial frame.

Def An inertial frame is a reference frame with following properties:

* There is a universal time coordinate that can be synchronized.
* Euclidian spatial components
* A body with no forces acting on it will travel at constant velocity according to the clocks and measuring sticks in the inertial frame.

Starting from this 2 postulates we can derive transformation of time and coordinate in two inertial systems. Suppose $S'$ is moving with velocity $\bar{v} = v\hat{x}$ with respect to $S$. Then coordinates of some event can be related in this 2 systems in the following way




$$\boxed{\begin{aligned} &t' = \gamma\left(t - \frac{v}{c}x\right); \\ &x' = \gamma(x - vt);\ y' = y;\ z' = z; \end{aligned}}$$

This are the only basic formulas we need for this class

(2)

## Problem 3

Prove that the temporal order of 2 events is the same in all inertial frames if and only if they can be joined in one inertial frame by a signal travelling at or below the speed of light. Illustrate this result on a spacetime diagram.

So, we have 2 events:

$P_1 = (t_1; x_1; y_1; z_1)$ and $P_2 = (t_2; x_2; y_2; z_2)$

and we have 2 statements about this events:

I $t_2' > t_1'$ for any inertial system $S'$;

II $P_1$ and $P_2$ can be causally connected by a signal in $S$ travelling with luminal or subluminal speed.

We should prove that I is satisfied if and only if II is satisfied, i.e. we should prove that I $\Rightarrow$ II and II $\Rightarrow$ I;

* proof in one way I $\Rightarrow$ II Let's assume $\Delta t' > 0$. If we use Lorentz transformation formula:

$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$; if we take $\Delta t' > 0$ then $\Delta t > \frac{v}{c^2}\Delta x$ and $\frac{\Delta x}{\Delta t} < \frac{c^2}{v} < c$. Thus we get that if $t_2' > t_1'$ for any $S'$ then they can be connected with signal travelling with subluminal velocity.

* proof in another way II $\Rightarrow$ I Let's now assume that two events can be connected with subluminal signal, so that: $\frac{\Delta x}{\Delta t} < c < \frac{c^2}{v}$ thus in any other inertial system $S'$ we get

$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) = \gamma \Delta t \left(1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t}\right) > 0$, thus we have finally proved statements both way, q.e.d.

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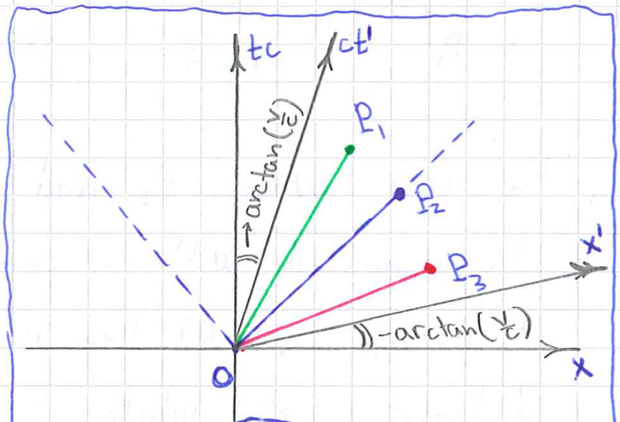
Now let's illustrate our result on the diagram. Here it is reasonable to make step aside and explain what is space-time diagram. This is just 2d plot of time vs. one of the spatial coordinates. Trajectory of point picted on the plot is called world-line




pic 2

Here on the picture we have drawn 3 trajectories $OP_1$, $OP_2$ and $OP_3$ wich can be classified withe the value of interval $\Delta S^2 = c^2\Delta t^2 - \Delta x^2$:

I if $\Delta S^2 = 0$ ($OP_2$) we call trajectory light-like

II if $\Delta S^2 > 0$ ($OP_1$) we call trajectory time-like. This type of trajectories corresponds to moving with subluminal velocities

III if $\Delta S^2 < 0$ ($OP_3$) we call trajectory space-like

Now it is reasonable to ask question:

How can we include axes of another inertial system $S'$ on the same diagram?

Times and coordinates of some event are related with Lorentz transformations:

$t' = \gamma(t - \frac{v}{c^2}x)$; $x' = \gamma(x - vt)$. Thus $x'$ axis is given by $t'=0$ or $t = \frac{v}{c^2}x$ and $t'$ axis is given by $x'=0$ thus $x = vt$ on the coordinate frame $(ct, x)$ we get 2 straight lines $ct = x^0 = \frac{v}{c}x^1$; and $x^0 = \frac{c}{v}x^1$ ($t'$-axis). This axes are drawn on pic. 2.

Now let's come back to our problem:

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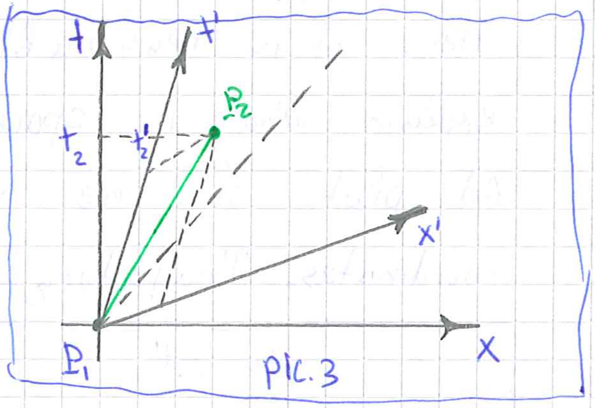
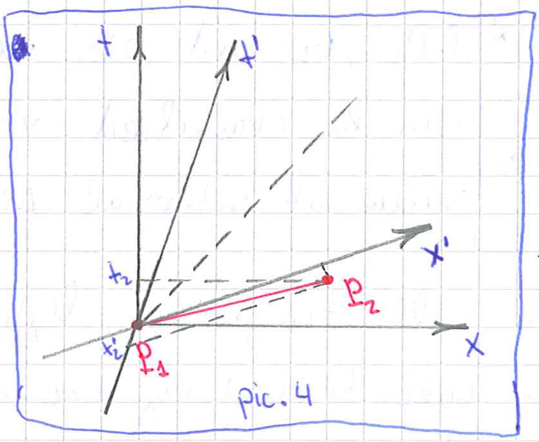
* First let's look what will happen if $P_1$ and $P_2$ are connected with the time-like trajectory (pic. 3)

We see from this picture that in both inertial systems $t_2' > t_1'$




pic. 3

* Now let's consider the case when $P_1$ and $P_2$ are connected with space-like trajectory (pic. 4) now in $S$ frame $t_2 > t_1$ while in $S'$ frame $t_2 < t_1$ (and we can always find such system $S'$ for any events that can be connected with superluminal signal)




pic. 4

## Problem 5

If 2 events occur at the same time in some inertial frame $S$, prove that there are no limit on the time separation in other frames but that their space separation varies from infinity to a minimum wich is measured in $S$

$S$: $\Delta t = 0$ ; $\Delta x \neq 0$ Now using Lorentz transformation we can write down in some other system $S'$:

$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) = -\frac{v\gamma}{c^2}\Delta x; \quad \Delta x' = \gamma(\Delta x - v\Delta t) = \gamma \Delta x$$

$\gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}} \geq 1$ minimum $\gamma = 1$ is obtained when $v = 0$ (wich corresponds to $S$ frame) thus

$$\Delta t' = -\frac{v\gamma}{c^2}\Delta x \in [0;\infty) : \text{as } v \to 0 \;\; \Delta t' \to 0 \quad \text{as } v \to c \; ; \Delta t' \to \infty$$

$$\Delta x' = \gamma \Delta x \in [\Delta x, \infty) : \text{as } v \to 0, \; \Delta x' \to \Delta x \;, \text{ as } v \to c \;\; \Delta x' \to \infty$$

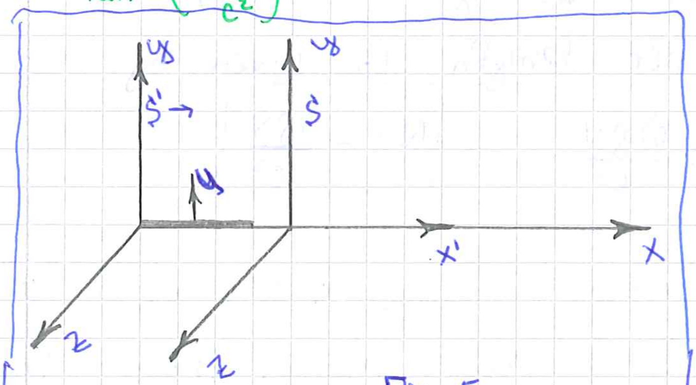
5 and that's exactely what we were asked to show.

Problem 6 In the inertial frame $S'$ the standart lattice clocks all emit "flash" at noon. Prove that in $S$ this flash occurs on a plane orthogonal to the x-axis and travelling in the positive x-direction. Let's $(\Delta t', \Delta x', \Delta y', \Delta z')$ be interval between 2 flashes in $S'$.
Let's assume $(\Delta t'; \Delta x'; \Delta y'; \Delta z')$ is interval between 2 flashes in $S'$. We take $\Delta t' = 0$ (flashes are simultaneous)
In $S$ we get

$$\Delta x = \gamma(\Delta x' + v\Delta t') = \gamma \Delta x'; \quad \Delta t = \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) = \gamma \frac{v}{c^2}\Delta x';$$

thus $\frac{\Delta x}{\Delta t} = \frac{c^2}{v}$ at the same time as boost is made along x-direction $\Delta y' = \Delta y$ and $\Delta z' = \Delta z$
Thus indeed observer in $S$-frame sees flashing plane travelling with the speed $u = \frac{c^2}{v}$ along x-direction.
q.e.d.

Problem 8 In $S'$ straight rod parallel to the $x'$-axis moves in the $y'$-direction with constant velocity $u$. Show that in $S$ the rod is inclined to the x-axis at an angle $-\tan^{-1}\left(\frac{\gamma u v}{c^2}\right)$



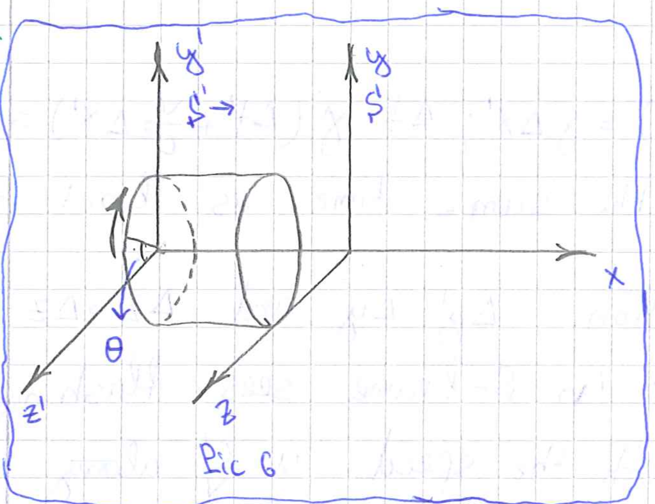
Pic. 5

Here we should first imagine what is rigid body for us. The answer is that this is set of points observed simultaneously. And as you remember "simultaneously" can in principle mean different in different reference frames. In $S'$ rod is moving with the speed $u = \frac{\Delta y'}{\Delta t'}$; and $\Delta y' = \Delta y$, as boost is made along x-axis. at the same time $\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$, $\Delta t = 0$ when we observe two points of rod simultaneously in $S$ thus in $S'$ this events differ by $\Delta t' = -\gamma \frac{v}{c^2}\Delta x$ time interval. angle in $S$ is given by $\tan\theta = \frac{\Delta y}{\Delta x} = \frac{\Delta y'}{\Delta x} \Rightarrow$

$\Rightarrow \tan\theta = -\frac{uv\gamma\,\Delta t'}{c^2\,\Delta t'} = -\frac{uv\gamma}{c^2}$ and thus:

$\boxed{\theta = -\arctan\left(\frac{uv\gamma}{c^2}\right)}$, q.e.d.

## Problem 9




Pic 6

Assume in $S'$ cylinder is rotationg about x-axis with angular speed $\omega$. Prove that in $S$ cylinder is twisted with the twist per unit length $\frac{\gamma\omega v}{c^2}$

Simultaneous observation in $S$ means $\Delta t = 0$ at the same time in $S'$ we get $\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) = -\frac{\gamma v}{c^2}\Delta x$. in $S'$ we have due to rotation $\Delta\theta' = \omega\Delta t'$ for all x. At the same time as $S'$ is moving along x-axis $y'z'$ is not transformed under this boost and we get $\Delta\theta' = \Delta\theta$;

Number of twists per length is given by:

$N = \frac{\Delta\theta}{\Delta x} = \frac{\omega\Delta t'}{\Delta x} = -\frac{\omega\gamma v}{c^2}$; $\boxed{N = -\frac{\omega\gamma v}{c^2}}$

## Problem 10

Two photons travel along x-axis of $S$ with constant distance $L$ between them. Prove that in $S'$ the distance between this photons is $L\sqrt{\frac{c+v}{c-v}}$

Solving this problem, we should be accurate with concept of simultanety again.

(7) * Let's first consider what we get in $S$ system: positions of photons are given by $x_1 = ct_1$ ; $x_2 = ct_2 + L$ and $t_2 = t_1$ - that what we mean by distance between photons - i.e. difference in their positions in the same moment of time.

* Now let's go to $S'$ frame. Here we get:

$$t_1' = \gamma\left(t_1 - \frac{v}{c^2}x_1\right) = \gamma t_1\left(1 - \frac{v}{c}\right); \quad t_2' = \gamma\left(t_2 - \frac{v}{c^2}x_2\right) =$$

$$= \gamma t_2\left(1 - \frac{v}{c}\right) - \frac{\gamma v}{c^2}L$$, now let's inverse this relations to express $t_1$ and $t_2$ through $t_1'$ and $t_2'$:

$$t_1 = \frac{ct_1'}{\gamma(c-v)}; \quad t_2 = \frac{ct_2'}{\gamma(c-v)} + \frac{vL}{c(c-v)};$$

Positions of photons in $S'$ are given by Lorenz transform.:

$$x_1' = \gamma(x_1 - vt_1) = \gamma t_1(c-v) = ct_1'; \quad \underline{x_1' = ct_1'}$$

$$x_2' = \gamma(x_2 - vt_2) = \gamma(c-v)t_2 + L\gamma = ct_2' + \frac{\gamma v}{c}L + \gamma L =$$

$$= ct_2' + L\sqrt{\frac{c+v}{c-v}}$$, thus $\underline{x_2' = ct_2' + L\sqrt{\frac{c+v}{c-v}}}$;

And distance between photons in $S'$ is given by difference of simultaneous $(t_1' = t_2')$ positions of this photons in $S'$:

$$L' = x_2'(t_2' = t') - x_1'(t_1' = t') = L\sqrt{\frac{c+v}{c-v}}; \quad \boxed{L' = L\sqrt{\frac{c+v}{c-v}}}$$ q.e.d.

**Problem 14** Let's define alternative coordinates $\xi = ct + x$; $\eta = ct - x$, whose axes are $\pm 45^\circ$ lines on diagram (pic 2). Prove that under Lorentz transformations the directions of these axes do not change, how do their calibration change?

After Lorentz transformation we get:

$$\xi' = ct' + x' = c\gamma\left(t - \frac{v}{c^2}x\right) + \gamma(x - vt) = \gamma t(c-v) + \gamma x\left(1 - \frac{v}{c}\right) =$$

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$$= \gamma\left(1-\frac{v}{c}\right)(ct+x) = \sqrt{\frac{c-v}{c+v}}\,\xi;$$

$$\eta' = ct' - x' = c\gamma\left(t - \frac{v}{c^2}x\right) - \gamma(x - vt) = \gamma ct\left(1+\frac{v}{c}\right) -$$

$$-\gamma x\left(1+\frac{v}{c}\right) = \gamma\left(1+\frac{v}{c}\right)(ct-x) = \sqrt{\frac{c+v}{c-v}}\,\eta;$$

Thus this coordinates, often called light-cone coordinates transform under Lorentz transformation in the following way:

$$\boxed{\xi' = \sqrt{\frac{c-v}{c+v}}\,\xi; \quad \eta' = \sqrt{\frac{c+v}{c-v}}\,\eta;}$$

As we can see they are not mixed by Lorentz transformation as it happens with $(t,x)$-coordinates, and thus, there is no rotation of axes. The only thing happening with this coordinates under Lorentz transformation is their rescaling. q.e.d.

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# Problem session N2 (Relativistic kimematics)

On this lecture we will consider some more effects related to frames.

I **Length contraction** Suppose we have bar of length $L$, in $S'$-frame it is stationary. What is it's length in frame $S$? When we measure length of rode in $S$ we fix positions of it's endpoints simultaneously, thus $\Delta t = 0$. Now if we use Lorentz transformation

$$\Delta t = 0 = \gamma\left(\Delta t' + \frac{V}{c^2}\Delta x'\right) \Rightarrow \Delta t' = -\frac{V}{c^2}\Delta x';$$

$$\Delta x = \gamma\left(\Delta x' + V\Delta t'\right) = \gamma\left(1 - \frac{V^2}{c^2}\right)\Delta x' = \frac{L_0}{\gamma}$$ thus we get

Lorentz contraction of length $\boxed{L = \frac{L_0}{\gamma}}$

II **Time dilation** Suppose clock at fixed point in $S'$ measures time interval $\Delta t' = \Delta\tau$ (proper time of clock). What is time interval measured by observer in $S$. As $\Delta x' = 0$ Lorentz transformation immideately give $\boxed{\Delta t = \gamma\Delta t'}$ as $\gamma > 1$ observer in $S$ measures a longer elapsed time than the proper time of clock.

III **Velocity transformation** Let's try to understand how does velocity transform when we go from one reference frame to other.

in $S'$: $u'_x = \frac{\Delta x'}{\Delta t'}$; $u'_y = \frac{\Delta y'}{\Delta t'}$; $u'_z = \frac{\Delta z'}{\Delta t'}$;

in $S$: $u_x = \frac{\Delta x}{\Delta t}$; $u_y = \frac{\Delta y}{\Delta t}$; $u_z = \frac{\Delta z}{\Delta t}$;

now we use Lorentz transformation formulas

$$\Delta x = \gamma\left(\Delta x' + V\cdot\Delta t'\right); \quad \Delta t = \gamma\left(\Delta t' + \frac{V}{c^2}\Delta x'\right); \quad \Delta y' = \Delta y; \quad \Delta z' = \Delta z;$$

thus we get

$$u_x = \frac{\Delta x' + V\Delta t'}{\Delta t' + \frac{V}{c^2}\Delta x'} = \frac{u'_x + V}{1 + \frac{u'_x V}{c^2}}; \quad u_y = \frac{\Delta y'}{\gamma\left(\Delta t' + \frac{V}{c^2}\Delta x'\right)} = \frac{u'_y}{\gamma\left(1 + \frac{V u'_x}{c^2}\right)};$$

$$u_z = \frac{\Delta z'}{\gamma\left(\Delta t' + \frac{V}{c^2}\Delta x'\right)} = \frac{u'_z}{\gamma\left(1 + \frac{V u'_x}{c^2}\right)};$$

(2)

$$\boxed{u_x = \frac{u_x' + V}{1 + \frac{u_x' V}{c^2}} ; \quad u_y = \frac{u_y'}{\gamma\left(1 + \frac{u_x' V}{c^2}\right)} ; \quad u_z = \frac{u_z'}{\gamma\left(1 + \frac{u_x' V}{c^2}\right)} ;}$$

It can be also showed that if $|\bar{u}'| < c$ then $|\bar{u}| < c$ (if $v < c$), i.e. speed of light can not be exceeded.

(IV) Acceleration We define proper acceleration ~~~~ $\alpha$ as acceleration in instantoneous rest frame. $S'(t)$ Then by definition $du' = \alpha dt'$; as $S'(t)$ is instantoneous rest frame we can use, $dt = \gamma_u dt'$. At the same time we assume that velocity in $S'(t)$ is $du'$ and velocity in $S$ is $u + du$. Using velocity transformation formula we get

$$du = \frac{du' + u}{1 + \frac{u du'}{c^2}} - u \approx (du' + u)\left(1 - \frac{u du'}{c^2}\right) - u \approx du' \cdot \left(1 - \frac{u^2}{c^2}\right)$$

thus acceleration in $S$ is given by:

$$a = \frac{du}{dt} = \frac{du'}{dt'} \cdot \left(1 - \frac{u^2}{c^2}\right)^{3/2} = \frac{\alpha}{\gamma_u^3} ;$$

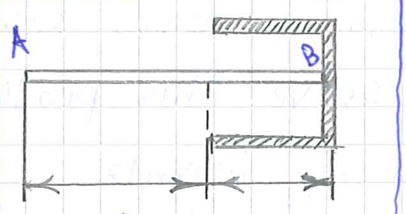
If we now say that proper acceleration is constant ($\alpha = \text{const}$) we can observe that $\frac{d}{dt}(\gamma_u u) = \alpha$, integrating this equation will give us trajectory of relativistic particle moving with constant proper acceleration $\alpha$:

$$x = \frac{c^2}{\alpha}\sqrt{1 + \frac{\alpha^2 t^2}{c^2}} + x_0 , \quad \text{or} \quad \boxed{x^2 - (ct)^2 = \frac{c^4}{\alpha^2} ;}$$

③

**Problem 1** In the pole-and-garage problem, what is the longest pole that can be run into a 12-foot garage at a speed v making $\gamma(v)=3$, assuming the elastic shock wave travels at the speed of light.

Let $S$ be the rest frame of garage and $S'$ - rest frame of pole.




pic. 1

Any phenomenon should have reasoning in any inertial frame, so it will be usefull to consider problem in both $S$ and $S'$

* in $S$-frame Let $x'$ be the proper length of the garage (12 ft.) and $x$ - the proper length of pole. Then length of pole in $S$ due to length contraction is $L=\frac{x}{\gamma}$. In $S$-frame rod hits the wall of the garage but the rear end of rod A is still moving as elastic shock wave travels with finite speed c and it takes $t=\frac{x'}{c}$ time for wave to propogate along rod so that it will meet A endpoint exactely where the rear wall of the garage is. This time should be equal to $\frac{L-x'}{v}$ (time A endpoint need to reach rear wall of the garage). Thus:

$\frac{x'}{c}=\frac{L}{v}-\frac{x'}{v}$ and $L=x'(1+\frac{v}{c}) \Rightarrow \boxed{x=\gamma x'(1+\frac{v}{c})=69,9 \text{ ft.}}$

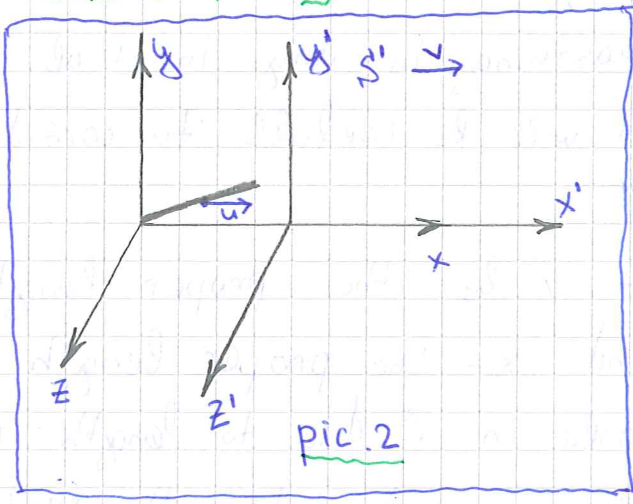
* in $S'$-frame in this frame length of the garage is $L'=\frac{x'}{\gamma}$. When wall of garage hits the rod deformation wave starts and should travel the whole longitude of rod $x$ which takes time $t=\frac{x}{c}$. During this time front end of the garage

should travel $(x-L')$ distance with velocity $v$

Thus $\frac{x}{c} = \frac{x-L'}{v}$; $x\left(\frac{1}{v}-\frac{1}{c}\right) = \frac{L'}{v} \Rightarrow x = x'\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$;

$\boxed{x = \gamma x'\left(1+\frac{v}{c}\right);}$ and we get the same result.

**Problem 3** A rod having slope $m$ relative to the x-axis of $S$, moves in the x-direction at speed $u$. What is the rod's slope in the usual second frame $S'$? I First let's solve this problem using Lorentz contraction formula




pic 2

Let's introduce third reference frame $S''$ - the proper frame of rod. Assume one end of rode is in the origin of all 3 systems. Another point is in $(\Delta x, \Delta y, 0)$ (or the same point with primed $\Delta x$ and $\Delta y$)

* in $S''$ we get $(\Delta x'', \Delta y'')$ point
* in $S$ due to Lorentz contraction we get $\Delta x = \frac{\Delta x''}{\gamma(u)}$; $\Delta y = \Delta y''$; and thus for the slope we get $\boxed{m = \frac{\Delta y}{\Delta x} = \frac{\Delta y''}{\Delta x''}\gamma(u);}$
* in $S'$ first of all rod's velocity in $S'$ is given by addition law:

$u' = \frac{u-v}{1-\frac{uv}{c^2}}$; thus in $S'$ we get:

$\Delta y' = \Delta y''$; $\Delta x' = \frac{1}{\gamma(u')}\Delta x''$. Let's calculate $\gamma(u')$

$$\gamma^{-2}(u') = 1 - \frac{(u-v)^2}{c^2\left(1-\frac{uv}{c^2}\right)^2} = \left\{c^2 - u^2 - v^2 + 2uv - 2uv + \frac{1}{c^2}u^2v^2\right\}\frac{1}{c^2\left(1-\frac{uv}{c^2}\right)^2}$$

(5) thus $\gamma^2(u') = \frac{1}{\left(1-\frac{uv}{c^2}\right)^2}\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)$; and thus

$\frac{1}{\gamma(u')} = \frac{1}{\gamma(u)} \cdot \frac{1}{\gamma(v)} \frac{1}{\left(1-\frac{uv}{c^2}\right)}$; finally $\boxed{\frac{\gamma(u')}{\gamma(u)} = \gamma(v)\left(1-\frac{uv}{c^2}\right)}$

for the slope of rod in $S'$-frame we get:

$$m' = \frac{\Delta y'}{\Delta x'} = \frac{\Delta y''}{\Delta x''}\gamma(u') = m \cdot \frac{\gamma(u')}{\gamma(u)} = m\gamma(v)\left(1-\frac{uv}{c^2}\right);$$

$$\boxed{m' = m\gamma(v)\left(1-\frac{uv}{c^2}\right);}$$

(II) Now let's consider the same problem in the style of previous chapter. When we observe rod we measure positions of it's points simultaneously. Thus:

* in $S$-frame we get $\Delta x$ and $\Delta y$ in $S'$ $\Delta t' = 0$ (simultaneous measurement). Using Lorentz transformation formulas $\Delta x' = \gamma(v)(\Delta x - v\Delta t)$; $\Delta t' = \gamma(v)(\Delta t - \frac{v}{c^2}\Delta x)$ thus $\Delta t = \frac{v}{c^2}\Delta x$. And thus while measurement is made rod is shifted with $u\Delta t$ distance:

$\Delta x = \Delta x_0 + u\Delta t$ thus we get:

$\Delta t = \frac{v}{c^2}\Delta x_0 + \frac{uv}{c^2}\Delta t \Rightarrow \boxed{\Delta t = \Delta x_0 \frac{v}{c^2}\frac{1}{1-\frac{uv}{c^2}}}$; then finally

$$\Delta x' = \gamma(v) \cdot \left\{\Delta x_0 + \frac{uv}{c^2}\frac{\Delta x_0}{\left(1-\frac{uv}{c^2}\right)} - \Delta x_0 \frac{v^2}{c^2}\frac{1}{\left(1-\frac{uv}{c^2}\right)}\right\} =$$

$$= \gamma(v) \cdot \Delta x_0 \cdot \left\{\frac{1-\frac{v^2}{c^2}}{1-\frac{uv}{c^2}}\right\}$$, thus finally $\Delta x' = \frac{\Delta x_0}{\gamma(v)\left(1-\frac{uv}{c^2}\right)}$

$m = \frac{\Delta y_0}{\Delta x_0}$ and $m' = \frac{\Delta y'}{\Delta x'} = \frac{\Delta y_0}{\Delta x_0}\gamma(v)\left(1-\frac{uv}{c^2}\right)$, thus finally we get the same answer:

$\boxed{m' = m\gamma(v)\left(1-\frac{uv}{c^2}\right);}$ as we see we get the same answer.

⑥

**Problem 5** (i) 2 particles move along the x-axis of $S'$ at velocities $0{,}8c$ and $0{,}9c$, respectively, the faster one momentarily 1 m behind the slower one. How many seconds elapse before collision?

This problem is solved just as usual Newtonian mechanics problem. If we choose time origin $t=0$ so that at $t=0$ distance between particles is $h=1m$. Then coordinates of particles are given by $x_1 = v_1 t + h$; $x_2 = v_2 t$; At the moment of impact $x_1 = x_2$ and thus $\boxed{t = \frac{h}{v_2 - v_1} \approx 3{,}3 \cdot 10^{-8}\, s.}$

(ii) A rod of proper length 10 cm moves longitudinally along the x-axis of $S$ at speed $\frac{1}{2}c$. How long (in $S$) does it take a particle, moving oppositely at the same speed, to pass the rod?

Length of rod is $L = \frac{L_0}{\gamma(\frac{c}{2})} = \frac{\sqrt{3}}{2} L_0$; Thus time it is take for particle to pass the rod is

$t = \frac{L}{c} = \frac{\sqrt{3}}{2}\frac{L_0}{c} = 2{,}89 \cdot 10^{-10}\, s$; $\boxed{t = \frac{\sqrt{3}}{2}\frac{L_0}{c} = 2{,}89 \cdot 10^{-10}\, s}$

**Problem 6** In a given inertial frame, 2 particles are shot out simultaneously from a given point, with equal speeds $v$, in orthogonal directions. What is the speed of each particle relative to the other?

in $S$-frame we have velocities of particles

$\bar{u}_1 = (v; 0; 0)$; $\bar{u}_2 = (0; v; 0)$

Let $S'$ be frame following first particle (so it moves with velocity $v$ along x-axis) then Lorentz transformation formulas give us

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$u'_{2x} = -v$ ; $u'_{2y} = \frac{v}{\gamma(v)}$ ; thus we get:

$\bar{u}'_1 = (0;0;0)$ ; $\bar{u}'_2 = (-v; \frac{v}{\gamma}; 0)$ and relative velocity is then given by

$$u^2_{rel} = |\bar{u}'_2 - \bar{u}'_1|^2 = v^2\left(1+\frac{1}{\gamma^2}\right) = v^2\cdot\left(1+1-\frac{v^2}{c^2}\right) =$$

$$= v^2\left(2-\frac{v^2}{c^2}\right); \quad \boxed{u_{rel} = v\sqrt{2-\frac{v^2}{c^2}};}$$

Problem 9 The rapidity $\varphi$, of a particle moving with velocity $u$, is defined by $\varphi = \operatorname{arctanh}\left(\frac{u}{c}\right)$. Prove that collinear rapidities are additive, i.e. if A has rapidity $\varphi$ relative to B, and B has rapidity $\psi$ relative to C, then A has rapidity $\varphi+\psi$ relative to C.

Let velocity of B with respect to A be $u$, and C has velocity "$v$" with respect to B. thus $\varphi = \operatorname{arctanh}\left(\frac{u}{c}\right)$ and $\psi = \operatorname{arctanh}\left(\frac{v}{c}\right)$. Velocity of C with respect to A is given by velocities addition formula $w = \frac{u+v}{1+\frac{uv}{c^2}}$, Corresponding rapidity is then $\chi = \operatorname{arctanh}\left(\frac{w}{c}\right)$; Another more convinient way to write "arctanh" is $\operatorname{arctanh}\left(\frac{x}{y}\right) = \frac{1}{2}\ln\left(\frac{y+x}{y-x}\right)$;

Thus $$\chi = \frac{1}{2}\ln\left\{\frac{c+\frac{u+v}{1+\frac{uv}{c^2}}}{c-\frac{u+v}{1+\frac{uv}{c^2}}}\right\} = \frac{1}{2}\ln\left\{\frac{1+\frac{u}{c}\cdot\frac{v}{c}+\frac{u}{c}+\frac{v}{c}}{1+\frac{u}{c}\cdot\frac{v}{c}-\frac{u}{c}-\frac{v}{c}}\right\} =$$

$$= \frac{1}{2}\ln\left\{\frac{\left(1+\frac{u}{c}\right)\left(1+\frac{v}{c}\right)}{\left(1-\frac{u}{c}\right)\left(1-\frac{v}{c}\right)}\right\} = \frac{1}{2}\ln\left(\frac{1+\frac{u}{c}}{1-\frac{u}{c}}\right) + \frac{1}{2}\ln\left(\frac{1+\frac{v}{c}}{1-\frac{v}{c}}\right) = \psi+\varphi,$$

thus, indeed, we get additivity of rapidities

$\boxed{\chi = \varphi+\psi \text{ ; q.e.d.}}$

(8)

**Problem 10** How many successive velocity increments of $\frac{1}{2}c$ from instantaneous rest frame are needed to produce resultant velocity of (i) $0,99c$; (ii) $0,999c$?

Hint: $\tanh 0,55 = 0,5$; $\tanh 2,65 = 0,99$; $\tanh 3,8 = 0,999$.

Here we will use results of previous problem and speak in terms of rapidity, which as we have show is additive. If one $\frac{1}{2}c$ increment corresponds to rapidity $\varphi = \operatorname{arctanh}(\frac{1}{2})$ then to get rapidity $\psi = N\varphi$ we need $N$ increments:

$$N = \frac{\psi}{\varphi}$$

(i) $\psi_1 = \operatorname{arctanh}(0,99) = 2,65$; $N_1 = \frac{2,65}{0,55} = 5$;

(ii) $\psi_2 = \operatorname{arctanh}(0,999) = 3,8$; $N_2 = \frac{3,8}{0,55} = 7$;

$$\boxed{N_1 = 5;\ N_2 = 7}$$

**Problem 12** In $S'$ a particle is momentarily at rest and has acceleration $\alpha$ in the $y'$-direction. What is the magnitude and direction of it's acceleration in $S$?

In a muon "storage ring" of radius 7m at the CERN laboratories in 1975, muons circled around at a speed $v = 0,9994c$. Find the magnitude of their proper acceleration.

First let's write down velocity transformation:

$$u_y = \frac{u_y'}{\gamma(v)} = \sqrt{1-\frac{v^2}{c^2}}\, u_y' \quad \text{and thus}$$

$$\frac{du_y}{dt'} = \frac{du_y'}{dt'}\sqrt{1-\frac{v^2}{c^2}}$$, and as we know from time dilation formulae $\frac{dt}{dt'} = \gamma(v)$ thus

$$\frac{du_y'}{dt'} = \frac{du}{dt}\gamma^2 \quad \text{thus} \quad \boxed{\frac{du}{dt} = \alpha \cdot \left(1-\frac{v^2}{c^2}\right)}$$

9

Now we can consider muons in the storage ring. If they are moving with velocity $v$ on the ring of radius $r$, they have acceleration $\alpha = \frac{v^2}{r}$ pointing to the center of ring i.e. transversal to velocity, and that's exacetely the example we have considered. Thus in lab frame we have $\alpha = \frac{v^2}{r}$ and in muon in instaneous fram $\alpha' = \frac{\alpha}{(1-v^2/c^2)} = \frac{v^2}{r(1-\frac{v^2}{c^2})} \approx 10^{19} \frac{m}{s^2}$;

$$\boxed{\alpha' = \frac{v^2}{r(1-\frac{v^2}{c^2})} \approx 10^{19} \frac{m}{s^2}}$$

**Problem 13** A certain piece of elastic breaks when it is stretched length. At time $t=0$, all points of it are accelerated longitudally with constant proper acceleration $\alpha$, from rest in unstretched state. Prove that the elastic breaks at $t = \frac{\sqrt{3}c}{\alpha}$;

If we use equation of motion of motion of point particle moving with constant proper acceleration we, as we have seen, get $x = \frac{c^2}{\alpha}\sqrt{\frac{\alpha^2 t^2}{c^2}+1} + x_0$; Thus in lab frame $S$ length of the rod is always the same as it was before rod started acceleration ($L_0$). Thus in $S'$ - frame following the rod we obtain, due to Lorentz contraction $L = \gamma L_0$, thus the rod is stretche, and when $\gamma$ becomes $\gamma = 2$ it breaks. From equations of motion we know that $\alpha t = \gamma(v) \cdot v$ thus $t = \frac{\gamma \cdot v}{\alpha}$. As critical $\gamma = 2$, critical velocity is $v = \frac{\sqrt{3}}{2}$ and time when rod breaks down is $\boxed{t = \frac{\sqrt{3}c}{\alpha}}$, q.e.d.

(10)

**Problem 16** Rocket moves from rest in an inertial frame $S$ with constant proper acceleration $g = 9{,}8\ m/s^2$

* Verify that in units of years and light years $g \simeq 1$;

$1\,yr = 3{,}2 \cdot 10^7\,s$ ; $1\,l.y. = 3{,}2 \cdot 10^7 \cdot s \cdot 3 \cdot 10^8 \frac{m}{s} = 9{,}6 \cdot 10^{15}\,m$;

then $g = 9{,}8 \frac{m}{s^2} = 9{,}8 \cdot \frac{(3{,}2 \cdot 10^7)^2}{3{,}2 \cdot 10^7 \cdot 3 \cdot 10^8} = 9{,}8 \frac{3{,}2 \cdot 10^7}{3 \cdot 10^8} = 1{,}05 \frac{ly}{yr^2}$

So indeed $\boxed{g = 1{,}05 \frac{ly}{yr^2}}$;

* Find it's Lorentz factor relative to $S$ when it's own clock indicates times $\tau = 1\,day, 1\,year, 10\,years$. Find also corresponding distances and times traveled in $S$.

We know equation of motion for accelerated particle : $\alpha t = \gamma(u) \cdot u$ , where $\alpha$ is proper acceleration. $t$ is time elapsed in $S$ frame then we get:

$\alpha t = \frac{u}{\sqrt{1 - \frac{u^2}{c^2}}}$ and $g^2 t^2 - \frac{u^2 g^2 t^2}{c^2} = u^2 \Rightarrow u = \frac{gt}{\sqrt{1 + \frac{g^2 t^2}{c^2}}}$; and

$\gamma(u) = \sqrt{1 + \frac{g^2 t^2}{c^2}}$ ; Now we should relate $S$-frame time $t$ with the rocket proper time $\tau$; as we have seen $d\tau = \frac{dt}{\gamma(u)} = \frac{dt}{\sqrt{1 + \frac{g^2 t^2}{c^2}}}$ and

$\tau = \int_0^t \frac{dt'}{\sqrt{1 + \frac{g^2 t'^2}{c^2}}} = \frac{c}{g} \operatorname{arcsinh} \frac{gt}{c}$ and then

$t = \frac{c}{g} \sinh \frac{g\tau}{c}$ ; $\gamma(u) = \sqrt{1 + \frac{g^2 t^2}{c^2}} = \cosh \frac{g\tau}{c}$; finally we want to know distance passed in $S$-frame.

$u = \frac{dx}{dt} = \frac{gt}{\sqrt{1 + \frac{g^2 t^2}{c^2}}}$ thus $x = \int_0^t \frac{dt' \cdot gt'}{\sqrt{1 + \frac{g^2 t'^2}{c^2}}} = \frac{c^2}{g}\left(\sqrt{1 + \frac{g^2 t^2}{c^2}} - 1\right) =$

$= \frac{c^2}{g}\left(\cosh \frac{g\tau}{c} - 1\right)$ So we have obtained

$$\boxed{t = \frac{c}{g} \sinh \frac{g\tau}{c}; \quad \gamma(u) = \cosh \frac{g\tau}{c}; \quad x = \frac{c^2}{g}\left(\cosh \frac{g\tau}{c} - 1\right)}$$

(11) If we substitute in numerics $\tau = \frac{1}{365}, 1, 10$ and $g = c = 1$ we get:

| | | | |
|---|---|---|---|
| $\tau$ | $\frac{1}{365}$ | $1$ | $10$ |
| $\gamma$ | $1,0000036$ | $\gamma = 1,5431$ | $11013$ |
| $x$ | $0,0000038$ | $0,5431$ | $11012$ |
| $t$ | $0,0027$ | $1,1752$ | $11013$ |

* If the rocket accelerates $\tau = 10$ yr of proper time and then decelerates $\tau = 10$ years., and then repeats the whole manoeuvre in the reverse direction, what is total time elapsed in $S$ during rocket absence?

The time spent on each accelertion and deceleraton as we have seen is 11013 years. Thus in total the journey will take $4 \cdot 11013$ yr. $= 4405$ years

$\boxed{t = 44052 \text{ years}}$

① **Problem session N3 (Relativistic Optics)**

Theory: There are two effects that we will consider on this session: Doppler effect and aberration.

* Doppler effect:

Light source P travels in S-frame with velocity $\bar{u}$ and radial velocity $u_r$ relative to O-origin of S. Let S' be the frame of moving light-source. Let the time between 2 pulses in S' be $dt_0$ and therefore in S time between pulses is $dt_0\gamma(u)$. In that time, the source has increased it's distance from O by $dt_0\gamma(u)u_r$. Thus interval between pulses arriving to O is $dt_0\gamma(u)\left(1+\frac{u_r}{c}\right)$. Interval between pulses is inversly proportional to corresponding frequencies. Thus

$$\frac{\nu_0}{\nu}=\frac{dt}{dt_0}=\frac{1+u_r/c}{\sqrt{1-\frac{u^2}{c^2}}}\ ; \quad \boxed{\frac{\nu_0}{\nu}=\frac{1+\frac{u_r}{c}}{\sqrt{1-\frac{u^2}{c^2}}}}$$

if movement, for example, is radial we get:

$$\frac{\nu_0}{\nu}=\sqrt{\frac{1+u/c}{1-u/c}}$$

; One more thing we can derive is relation between the frequencies $\nu$ and $\nu'$ ascribed by two observers O and O' to an incoming ray. $\alpha$ is the angle that negative direction of ray makes with x-axis. If we assume that ray originates from the source being in rest in S', then we should make following notations $\nu_0=\nu'$, $u=v$, $u_r=v\cdot\cos\alpha$; Thus

$$\boxed{\frac{\nu'}{\nu}=\frac{1+\frac{v}{c}\cos\alpha}{\sqrt{1-\frac{v^2}{c^2}}}}$$

* Aberration: Let's now consider incoming light signal whose negative direction makes angles $\alpha$ and $\alpha'$ with the x-axes of the usual frames S and S'. Let's say that $u_1=-c\cdot\cos\alpha$ and $u_1'=-c\cdot\cos\alpha'$;

② now we can use transformation of velocities formula:

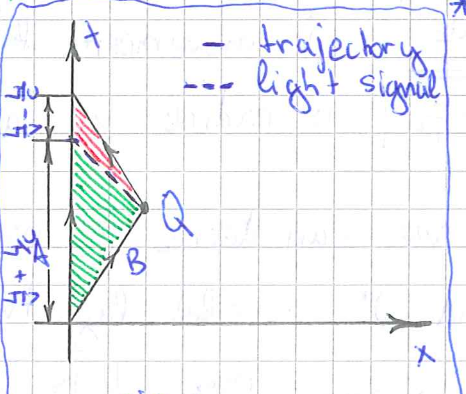
$$u_1' = \frac{u_1 - v}{1 - \frac{u_1 v}{c^2}}$$ and thus $$\boxed{\cos\alpha' = \frac{\cos\alpha + v/c}{1 + \frac{v}{c}\cos\alpha}; \quad \sin\alpha' = \frac{\sin\alpha}{\gamma\left[1 + \frac{v}{c}\cdot\cos\alpha\right]};}$$ and

in analogy we can obtain:

using trigonometric identity $\tan\frac{1}{2}\alpha' = \frac{\sin\alpha'}{1+\cos\alpha'}$ and obtain:

$$\boxed{\tan\tfrac{1}{2}\alpha' = \sqrt{\frac{c-v}{c+v}}\,\tan\tfrac{1}{2}\alpha;}$$ which will be usefull for us.

Problem 1 If the twins A and B, in the twin paradox "experiment" discussed on p.30 [Rindler] visually observe the regular ticking of each other's standard clocks, describe quantitatively what each sees as B travels to a distant point Q and back.




pic. 1

* First let's consider ticks that A recievs from B. Let L be the distance from the start point to Q. It takes $\frac{L}{v}$ time to travel to this point for B and light signal of tick emmited by B at Q will go back to A for $\frac{L}{c}$ time. So for $t_1 = \frac{L}{v} + \frac{L}{c}$ A recievs signals from receding B, and it recievs ticks with shift $\nu_1 = \nu_0 \sqrt{\frac{1 - v/c}{1 + v/c}}$. Total number of ticks recieved from B going towards Q is

$$N_{1A} = \nu_1 \cdot t_1 = \frac{L_0 \nu_0}{v}\sqrt{1 - \frac{v^2}{c^2}} = \frac{L_0 \nu}{v\,\gamma(v)}; \quad \boxed{N_{1A} = \frac{L_0 \nu}{v\,\gamma(v)};}$$

For remaining time $t_2 = \frac{L}{v} - \frac{L}{c}$ A recievs signals from impedent source B and thus total number recieved is $N_{2A} = t_2 \cdot \nu_2$ where $\nu_2 = \nu_0 \sqrt{\frac{1 + v/c}{1 - v/c}}$ and

(3) $N_{2A} = \frac{L\nu_0}{v}\sqrt{1-\frac{v^2}{c^2}} \Rightarrow N_{2A} = \frac{L\nu_0}{v\gamma(v)}$ thus total number of ticks recieved by A is $N_A = N_{1A} + N_{2A} = \frac{2L\nu_0}{v\gamma(v)}$

$\boxed{N_A = \frac{2L\nu_0}{v\cdot\gamma(v)}}$;

* Now let's consider ticks recieved by B from A. One-way journey in proper frame of B is, due to time dilation, takes $\tau_1 = \tau_2 = \tau = \frac{L}{v\gamma}$. For the first part of trip B is gowing away from A and thus it recievs ticks with the frequency $\nu_1 = \nu_0\sqrt{\frac{1-v/c}{1+v/c}}$. For the second part of jorney situation is opposite and frequency of recieved ticks is $\nu_2 = \nu_0\sqrt{\frac{1+v/c}{1-v/c}}$; thus total number of ticks is $N_B = (\nu_1+\nu_2)\tau = \frac{\nu_0 L}{v\gamma(v)}\left(\sqrt{\frac{1+v/c}{1-v/c}} + \sqrt{\frac{1-v/c}{1+v/c}}\right) = \frac{2\nu_0 L}{v\gamma(v)}\cdot\gamma(v) = \frac{2\nu_0 L}{v}$; $\boxed{N_B = \frac{2\nu_0 L}{v}}$

As $\gamma(v) \geq 1$ we conclude that $N_B > N_A$, so that A has recieved less ticks from B then B from A, and that means that B is indeed younger, as we know it from other justifications of twin paradox.

**Problem 5** A source of monochromatic light of frequency $\nu_0$ is fixed at the origin of a frame S. An observer travels through S with instantoneous velocity $u$ and radial velocity $u_r$ relative to the origin, when he observes the source. What Doppler shift $\nu/\nu_0$ does he observe?

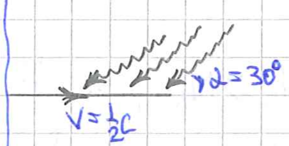
The situation in this problem is exactely opposite to the one considered in theory part of Doppler effect. I.e. now observer moves while source of light stays on the place. This corresponds to the change of variables $u \to -u$; $u_r \to u_r$. Note here

④ that radial velocity hasn't changed the sign because source and observer a still either coming closer or away from each other. Now if we apply formula we get:

$$\boxed{\frac{\nu_0}{\nu} = \frac{1+\frac{u_r}{c}}{\sqrt{1-\frac{u^2}{c^2}}}}$$

Note: So to make picture clear go to the proper frame of observer. What you get there is source moving with velocity $-u$ and radial velocity $u_r$.

**Problem 6** An observer moves into parallel radiation of frequency $\nu$ at the angle of $30°$ with the radiation and at speed $\frac{1}{2}c$. What frequency does he observe?



pic 2

Let's use already observed formula for relation of observed frequencies in standart frames $S$ and $S'$:

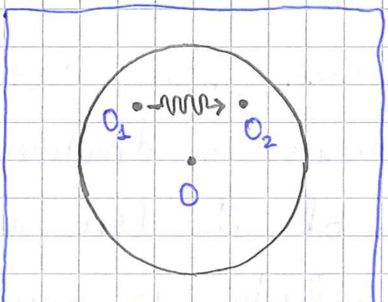
$$\frac{\nu'}{\nu} = \frac{1+\frac{v}{c}\cos\alpha}{\sqrt{1-\frac{v^2}{c^2}}}$$

here, just to reminod, $\nu'$ and $\nu$ are frequencies in $S'$ and $S$ respectively and $\alpha$ is the angle betwee velocity of $S'$ $\bar{v}$ and negative direction of incoming light frame ($30°$ in our case) $v = \frac{1}{2}c$ in our case. So if we substitute everything into this formula we get $\frac{\nu'}{\nu} = \frac{1+\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2}}$. Thus observer will observe radiation frequency equal to

$$\nu' = \nu \cdot \frac{4+\sqrt{3}}{2\sqrt{3}} = \left(\frac{2}{\sqrt{3}}+\frac{1}{2}\right)\nu \quad ; \text{ thus } \quad \boxed{\nu' = \left(\frac{2}{\sqrt{3}}+\frac{1}{2}\right)\nu}$$

(5)

**Problem 7** A large disc rotates at uniform angular speed $\omega$ in an inertial frame $S$. Two observers $O_1$ and $O_2$ ride on the disc at radial distances $r_1$ and $r_2$ from the centre (not necessarily on the same radius). They carry clocks $C_1$ and $C_2$ which they adjust so that they keep time with the clocks in $S$, i.e. they speed up their natural rates by the Lorentz factors $\gamma_1 = \left(1 - \frac{r_1^2\omega^2}{c^2}\right)^{-\frac{1}{2}}$, $\gamma_2 = \left(1 - \frac{r_2^2\omega^2}{c^2}\right)^{-\frac{1}{2}}$, respectively. Deduce that, when $O_2$ sends a light-signal to $O_1$, this signal suffers a Doppler shift $\frac{\nu_2}{\nu_1} = \frac{\gamma_2}{\gamma_1}$. There is no Doppler shift between any two observers equidistant from the centre.



Pic. 3

Assume in the rest frame of disc center (O) frequency of light signal is $\nu$. To go to $O_1$ rest frame we can use formula for relation between frequencies in different reference frames and you can check result yourself, but we will do simplier thing. All points are on the solid disk so that distance betwee all points remain constant, so that there is no relative radial velocity between the points.
Thus only effect of changing frequency is due to time dilation. I.e. if in $S$ time interval between 2 pulses is $\Delta t_0$ then in $S_1$ frame (proper frame $O_1$) we get $\Delta\tau_1 = \frac{\Delta t_0}{\gamma_1}$ and in $S_2$-frame $\Delta\tau_2 = \frac{\Delta t_0}{\gamma_2}$ where $\gamma_{1,2} = \left(1 - \frac{r_i^2\omega^2}{c^2}\right)^{-\frac{1}{2}}$; and then $\boxed{\frac{\nu_2}{\nu_1} = \frac{\Delta t_1}{\Delta t_2} = \frac{\gamma_2}{\gamma_1}}$; q.e.d.

(6) now if we take a look on this formula we get that if $r_1 = r_2$ then $\nu_1 = \nu_2$, q.e.d.

**Problem 9** From (17.3) and (18.2) [Rindler] derive the following relation between Doppler shift and abberation:

$$\frac{\nu'}{\nu} = \frac{\sin\alpha}{\sin\alpha'};$$

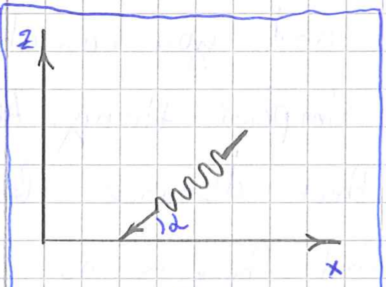
We have two equations:

$$\frac{\nu'}{\nu} = \frac{1 + \frac{v}{c}\cos\alpha}{\sqrt{1 - \frac{v^2}{c^2}}}; \qquad \sin\alpha' = \frac{\sin\alpha}{\gamma\left[1 + \frac{v}{c}\cos\alpha\right]}$$

as we see $\frac{\nu'}{\nu} = \gamma(v)\left(1 + \frac{v}{c}\cos\alpha\right) = \frac{\sin\alpha}{\sin\alpha'}$; $\boxed{\frac{\nu'}{\nu} = \frac{\sin\alpha}{\sin\alpha'}}$, q.e.d

**Problem 10** Let $\Delta t$ and $\Delta t'$ be the time separations in the usual two frames $S$ and $S'$ between two events occuring at a freely moving photon. If the photon has frequencies $\nu$ and $\nu'$ in these frames, prove that $\frac{\nu}{\nu'} = \frac{\Delta t}{\Delta t'}$

Let's assume that photon moves in direction making $\alpha$ angle with x-axis of $S$. (see pic 4)




pic. 4

So that it's velocity is given by $\bar{v} = (-c\cdot\cos\alpha; -c\cdot\sin\alpha; 0)$; and let the time separation between 2 events be $\Delta t$. In this case space separation in x-coordinate is $\Delta x = -c\cdot\cos\alpha\cdot\Delta t$. Using Lorentz transformation formulas we get in $S'$ frame the following time separation between this two events:

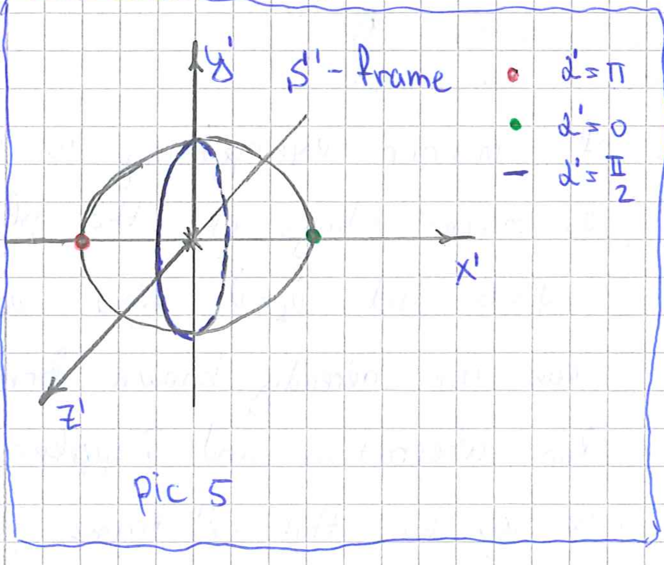
$$\Delta t' = \gamma(v)\left(\Delta t - \frac{v}{c^2}\Delta x\right) = \gamma(v)\,\Delta t\left(1 + \frac{v}{c}\cdot\cos\alpha\right)$$ and due to formula (17.3) from [Rindler] $\gamma(v)\cdot\left(1 + \frac{v}{c}\cdot\cos\alpha\right) = \frac{\nu'}{\nu}$.

Thus we observe desired result $\boxed{\frac{\nu'}{\nu} = \frac{\Delta t'}{\Delta t}}$; q.e.d.

7

**Problem 13** A source of light is fixed in $S'$ and in that frame it emits light uniformly in all directions. Show that for large $v$, the light in $S$ is mostly concentrated in a narrow forward cone; in particular, half the photons are emitted into a cone whose semi-angle is given by $\cos\theta = \frac{v}{c}$. This is called the headlight effect. Is the situation essentially different in the classical theory?

Let's consider ray going out of the source at angle $\alpha'$ (angle between ray direction and direction of $x'$-axis) in $S'$-frame. As light is emitted uniformly $\alpha' \in (0, \pi]$ (this corresponds to polar angle in spherical coordinates with center of sphere placed in the light source position and "North" of sphere placed in positive direction of $x'$-axis) $\alpha' = 0$ corresponds to emition of light in direction of motion and $\alpha' = \pi$ – in opposite one $\alpha' = \frac{\pi}{2}$ – emition of light in the direction of "equator" of the sphere (half of light is emmited in the angle $0 < \alpha' < \frac{\pi}{2}$) (see pic 5)



pic 5

Now we can see at what angle is photon emmited in $S$-frame. For this we will use formula, derived in the begining of lecture (or formula (18.3) from [Rindler]) Note formula is valid for incoming ray, ⑧ While in the problem we have out going ray. So we need to substitute "$-c$" instead of "$c$"

$\tan\frac{\alpha}{2} = \sqrt{\frac{c-v}{c+v}}\tan\frac{\alpha'}{2}$ we see that if $v \to c$ $\tan\frac{\alpha}{2} \to 0$

and thus $\boxed{\alpha \to 0}$ which corresponds to light emmited in positive x-axis direction (i.e. in direction of movement)

Now let's understand where half of light is emmited.

In $S'$ half of photons are emmited in North hemisphere $0<\alpha'<\frac{\pi}{2}$. Now if we substitute $\alpha' = \frac{\pi}{2}$ to the formula written above we get:

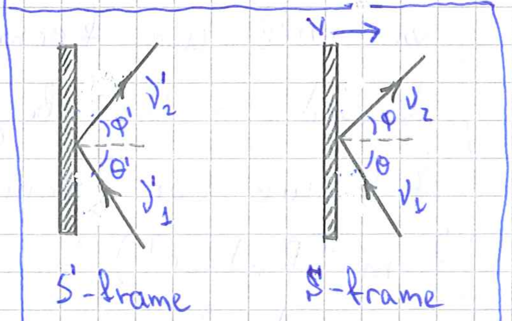
$\tan\frac{\alpha}{2} = \sqrt{\frac{c-v}{c+v}}\tan\frac{\pi}{4} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$. Now if we use following trigonometrical identity: $\tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$ we get: $\boxed{\cos\alpha = \frac{v}{c}}$ which is desired answer.

Problem 18 A plane mirror moves in the direction of its normal with uniform velocity $v$ in a frame $S$. A ray of light of frequency $\nu_1$ strikes the mirror at an angle of incidence $\theta$, and is reflected with frequency $\nu_2$ at an angle of reflection $\phi$. Prove that

$\frac{\tan\frac{1}{2}\theta}{\tan\frac{1}{2}\phi} = \frac{c+v}{c-v}$, and

$$\frac{\nu_2}{\nu_1} = \frac{\sin\theta}{\sin\phi} = \frac{c\cdot\cos\theta + v}{c\cdot\cos\phi - v} = \frac{c+v\cdot\cos\theta}{c-v\cdot\cos\phi};$$

In $S'$ - proper frame of mirror everything is simple as mirror stays on the place $\phi = \theta$ and $\nu_1 = \nu_2$. Now we can use already known formulas for aberration and Doppler shift to go to the $S$ frame, in which mirror is moving with velocity "$v$"



pic. 6

⑨ For incoming ray of light we have $\tan\frac{1}{2}\theta' = \sqrt{\frac{c-v}{c+v}}\tan\frac{1}{2}\theta$ and for out going ray:

$\tan\frac{\varphi}{2} = \sqrt{\frac{c-v}{c+v}}\tan\frac{1}{2}\varphi' = \sqrt{\frac{c-v}{c+v}}\tan\frac{1}{2}\theta' = \frac{c-v}{c+v}\tan\frac{1}{2}\theta$. So we have got desired relation

$$\boxed{\frac{\tan\frac{\theta}{2}}{\tan\frac{\varphi}{2}} = \frac{c+v}{c-v}}$$

Now let's go for frequencies. For incident light

$\frac{\nu_1'}{\nu_1} = \frac{1+\frac{v}{c}\cdot\cos\theta}{\sqrt{1-\frac{v^2}{c^2}}}$ ; $\frac{\nu_2'}{\nu_2} = \frac{1-\frac{v}{c}\cos\phi}{\sqrt{1-\frac{v^2}{c^2}}}$ as $\nu_1' = \nu_2'$ we immedeately get by deviding one equation with another:

$\frac{\nu_1}{\nu_2} = \frac{1-\frac{v}{c}\cos\varphi}{1+\frac{v}{c}\cos\theta}$ ⇒ and we now obtain

$$\boxed{\frac{\nu_2}{\nu_1} = \frac{c+v\cdot\cos\theta}{c-v\cdot\cos\varphi}};$$ q.e.d.

1

# Problem session N4,5 (Tensors)

## Theory

### Formal definition

Assume we are making some coordinate transformation from coordinates $\{x^i\} = \{x^1, x^2, x^3, \ldots, x^N\}$ to the coordinates $\{x^{i'}\} = \{x^{1'}, x^{2'}, x^{3'}, \ldots, x^{N'}\}$;

* An object having components $A^{ij\ldots n}$ in the $x^i$ system of coordinates and $A^{i'j'\ldots n'}$ in $x^{i'}$ system of coordinates is said to behave as a contravariant tensor under the transformation $\{x^i\} \to \{x^{i'}\}$ if:

$A^{i'j'\ldots n'} = A^{ij\ldots n} p^{i'}_{\ i} p^{j'}_{\ j} \ldots p^{n'}_{\ n}$, here $p^{i'}_{\ i}$ is defined in the following way $p^{i'}_{\ i} = \dfrac{\partial x^{i'}}{\partial x^i}$;

* Similarly, $A_{ij\ldots n}$ is said to behave as a covariant tensor under $\{x^i\} \to \{x^{i'}\}$ if

$A_{i'j'\ldots n'} = A_{ij\ldots n} p^i_{\ i'} p^j_{\ j'} \ldots p^n_{\ n'}$;

* Lastly, $A^{i\ldots k}_{\ell\ldots n}$ is said to behave as a mixed tensor (contravariant in $i\ldots k$ and covariant in $\ell\ldots n$) under $\{x^i\} \to \{x^{i'}\}$ if

$A^{i'\ldots k'}_{\ell'\ldots n'} = A^{i\ldots k}_{\ell\ldots n} p^{i'}_{\ i} \ldots p^{k'}_{\ k} p^{\ell}_{\ \ell'} \ldots p^{n}_{\ n'}$;

So tensor are just defined as objects transforming in some particular way under coordinate transformations. The particular coordinate transformations we are usually interested in this course are Lorentz transformations.

For Lorentz transfomation we usually write $\Lambda^{\mu'}_{\ \mu}$ instead of $p^{i'}_{\ i}$, where general form of $\Lambda^{\mu'}_{\ \mu}$ is

$$\Lambda^{\mu'}_{\ \mu} = \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_z\beta_x}{\beta^2} \\ -\gamma\beta_y & (\gamma-1)\frac{\beta_y\beta_x}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & (\gamma-1)\frac{\beta_z\beta_x}{\beta^2} & (\gamma-1)\frac{\beta_z\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{bmatrix}$$

②

here as usually $\gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ ; $\beta_i = \frac{v_i}{c}$; $\beta = \frac{v}{c}$;

All expressions in Joe's notes and in Rindler book are special cases of the formulae above

For example in case of boost along x-axis $\beta_x = \beta$, $\beta_y = \beta_z = 0$. and we get:

$$\Lambda^{\mu'}_{x\,\mu} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

Usefull property of Lorentz transformation is that Lorentz transformations form $SO(3,1)$ group

$$SO(\underbrace{3}_{\text{space rotations}},\underbrace{1}_{\text{boost}})$$

That means that for rotations $\Lambda^T = \Lambda^{-1}$ and for boosts $\Lambda^T = \Lambda$; or in other words $\boxed{\Lambda_{\nu\mu'}\Lambda^{\nu}{}_{\nu'} = \eta_{\mu'\nu'}}$

here $\eta_{\mu\nu}$ is metric. Minkowski flat metric is

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

As you know from lectures in notatio $\Lambda_{\nu\mu'}\Lambda^{\nu}{}_{\nu'}$ we summ over "$\nu$" index (repeating index). This summation indicies are called "dummy". If we have some kind of product of different rank tensors, then rank of resulting tensor equals number of not dummy indicies.

For example $A_\mu B^\mu$ is scalar product and is Lorentz scalar.

Examples of 4-vectors from relativistic mechanics

4-vectors are rank 1 tensor in 4-dimensional Minkowski space

covariant vector $\qquad$ contravariant tensor

$A_\mu = (A^0; -\bar{A})$ $\qquad$ $A^\mu = (A^0; \bar{A})$ here $\bar{A}$ is just 3-vector in euclidian space.

③

**Problem 2** An inertial observer bounces a radar signal off an arbitrary event $\mathcal{P}$. If the signal is emitted and received by him at times $\tau_1$ and $\tau_2$, respectively, as indicated by his standard clock, prove that the squared interval $\Delta s^2$ between his origin-event $\tau=0$ and $\mathcal{P}$ is $c^2\tau_1\tau_2$. This, in fact, constitutes a uniform method for assigning $\Delta s'^2$ to any pair of events.

Time for the signal to reach $\mathcal{P}$ is half of time difference between emition and receiving signal, i.e. $(\tau_2-\tau_1)\cdot\frac{1}{2}$. Thus distance to $\mathcal{P}$ is $\Delta x = \frac{\tau_2-\tau_1}{2}\cdot c$. Time interval between origin event $\tau=0$ and receiving signal by $\mathcal{P}$ is $\Delta\tau = \tau_1 + \frac{\tau_2-\tau_1}{2} = \frac{1}{2}(\tau_1+\tau_2)$; and thus, finally interval between $\mathcal{P}$ and observer origin-event is

$\Delta s^2 = c^2\Delta\tau^2 - \Delta x^2 = \frac{c^2}{4}(\tau_1+\tau_2)^2 - \frac{c^2}{4}(\tau_2-\tau_1)^2 = c^2\tau_1\tau_2$; and thus

$\boxed{\Delta s^2 = c^2\tau_1\tau_2;\ \text{q.e.d.}}$

**Problem 7** An antisymmetric tensor $F^{\mu\nu}$ has the following components in a frame $S$:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix}$$

(i) Find the values of all components $F^{\mu'\nu'}$ in the usual second frame $S'$

$S'$ is as usually boosted along x-axis.

Thus $F^{\mu\nu}$ is transformed under Lorentz transformations in the following way

$F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\mu}\Lambda^{\nu'}{}_{\nu}F^{\mu\nu}$ where $\Lambda^{\mu'}{}_{\mu} = \begin{bmatrix} \gamma(v) & -\frac{v}{c}\gamma(v) & 0 & 0 \\ -\frac{v}{c}\gamma(v) & \gamma(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

④ First of all to simplify our task let's note that electromagnetic tensor $F^{\mu\nu}$ is antisymmetric in all frames. Indeed:

$$F^{\mu'\nu'} = \Lambda^{\mu'}_{\ \mu}\Lambda^{\nu'}_{\ \nu}F^{\mu\nu} \quad ; \quad F^{\nu'\mu'} = \Lambda^{\mu'}_{\ \mu}\Lambda^{\nu'}_{\ \nu}F^{\nu\mu} = -\Lambda^{\mu'}_{\ \mu}\Lambda^{\nu'}_{\ \nu}F^{\mu\nu} = -F^{\mu'\nu'}$$

So we don't need to find all components of $F^{\mu\nu}$, but only it's upper triangle

$$F^{0'1'} = \Lambda^{0'}_{\ \mu}\Lambda^{1'}_{\ \nu}F^{\mu\nu} = \Lambda^{0'}_{\ 0}\Lambda^{1'}_{\ 1}F^{01} + \Lambda^{0'}_{\ 1}\Lambda^{1'}_{\ 0}F^{10} = -\gamma^2 E_1 + \frac{v^2}{c^2}\gamma^2 E_1 =$$
$$= -\gamma^2\left(1-\frac{v^2}{c^2}\right)E_1 = -E_1;$$

$$F^{0'3'} = \Lambda^{0'}_{\ \mu}\Lambda^{3'}_{\ \nu}F^{\mu\nu} = \Lambda^{0'}_{\ 0}\Lambda^{3'}_{\ 3}F^{03} + \Lambda^{0'}_{\ 1}\Lambda^{3'}_{\ 3}F^{13} = \gamma(-E_3) - \frac{v}{c}\gamma\cdot B_2 =$$
$$= -\gamma(v)\left(E_3 + \frac{v}{c}B_2\right);$$ similarly we find $F^{0'2'} \equiv -E_2' = -\gamma\left(E_2 - \frac{v}{c}B_3\right)$;

$$F^{1'2'} = \Lambda^{1'}_{\ \mu}\Lambda^{2'}_{\ \nu}F^{\mu\nu} = \Lambda^{1'}_{\ 0}\Lambda^{2'}_{\ 2}F^{02} + \Lambda^{1'}_{\ 1}\Lambda^{2'}_{\ 2}F^{12} = -\frac{v}{c}\gamma(-E_2) + \gamma(-B_3) =$$
$$= \gamma\left(\frac{v}{c}E_2 - B_3\right);$$

$$F^{1'3'} = \Lambda^{1'}_{\ 0}\Lambda^{3'}_{\ 3}F^{03} + \Lambda^{1'}_{\ 1}\Lambda^{3'}_{\ 3}F^{13} = -\frac{v}{c}\gamma(-E_3) + \gamma B_2 = \gamma\left(B_2 + \frac{v}{c}E_3\right)$$

$F^{2'3'} = \Lambda^{2'}_{\ 2}\Lambda^{3'}_{\ 3}F^{23} = -B_1$ thus we finally get:

$$F^{\mu'\nu'} = \begin{bmatrix} 0 & -E_1 & \gamma(\frac{v}{c}B_3 - E_2) & -\gamma(E_3 + \frac{v}{c}B_2) \\ E_1 & 0 & \gamma(\frac{v}{c}E_2 - B_3) & \gamma(B_2 + \frac{v}{c}E_3) \\ -\gamma(\frac{v}{c}B_3 - E_2) & -\gamma(\frac{v}{c}E_2 - B_3) & 0 & -B_1 \\ +\gamma(E_3 + \frac{v}{c}B_2) & -\gamma(B_2 + \frac{v}{c}E_3) & B_1 & 0 \end{bmatrix}$$

The thing here is that $E^i$ and $B^i$ don't really transform as 3-components of 4-vector (indeed $E_1$ and $B_1$ components don't even change). Really they are components of tensor $F^{\mu\nu}$ and transform in a way derived above

(ii) Verify directly that $\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \bar{B}^2 - \bar{E}^2$ is an invariant.

First of all

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{bmatrix} \quad ; \quad F_{\mu\nu} = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{bmatrix}$$

(5) From this 2 expressions we can easily check that indeed

$\frac{1}{2} F_{\mu\nu} F^{\mu\nu} = |\bar{B}|^2 - |\bar{E}|^2;$

Now let's show that this is Lorentz invariant

As we have already mentioned as there are no not dummy indicies expression should be scalar. Let's check this explicitely on this particular example:

$F^{\mu'\nu'} F_{\mu'\nu'} = \eta_{\mu'\rho'} \eta_{\nu'\lambda'} F^{\rho'\lambda'} F^{\mu'\nu'} = \eta_{\mu'\rho'} \eta_{\nu'\lambda} \Lambda^{\rho'}{}_{\rho} \Lambda^{\lambda'}{}_{\lambda} \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} F^{\rho\lambda} F^{\mu\nu} =$

$= \Lambda_{\mu'\rho} \Lambda_{\nu'\lambda} \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} F^{\rho\lambda} F^{\mu\nu}; \quad \Lambda_{\mu'\rho} \Lambda^{\mu'}{}_{\mu} = \eta_{\rho\mu};$

thus $F_{\mu'\nu'} F^{\mu'\nu'} = \eta_{\rho\mu} \eta_{\lambda\nu} F^{\rho\lambda} F^{\mu\nu} = F_{\mu\nu} F^{\mu\nu}$; so we have shown that $\boxed{F_{\mu'\nu'} F^{\mu'\nu'} = F_{\mu\nu} F^{\mu\nu}}$, and thus $F_{\mu\nu} F^{\mu\nu}$ is indeed Lorentz invariant, q.e.d.

(iii) Show that $\overset{*}{F}_{\mu\nu} \overset{*}{F}{}^{\mu\nu} = - F_{\mu\nu} F^{\mu\nu}$ where $\overset{*}{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$

$\overset{*}{F}_{\mu\nu} \overset{*}{F}{}^{\mu\nu} = \frac{1}{4} \varepsilon_{\mu\nu\alpha\beta} \varepsilon^{\mu\nu\gamma\delta} F^{\alpha\beta} F_{\gamma\delta}$, $\varepsilon_{\mu\nu\alpha\beta}$ is absolutely antisymmetric tensor:

$$\varepsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & \text{if } \mu\nu\alpha\beta \text{ is even permut. of } 0123 \\ -1 & \text{if } \mu\nu\alpha\beta \text{ is odd permut. of } 0123 \\ 0 & \text{if some of indicies are the same} \end{cases}$$

For it we can check that (see exercise 19 from appendix)

$\varepsilon_{\mu\nu\alpha\beta} \varepsilon^{\mu\nu\gamma\delta} = 2e (\delta^{\gamma}{}_{\alpha} \delta^{\delta}{}_{\beta} - \delta^{\gamma}{}_{\beta} \delta^{\delta}{}_{\alpha})$ where $e = \eta = \det \eta_{\alpha\beta}$ is determinant of the metric. In our case $e = -1$;

thus $\overset{*}{F}_{\mu\nu} \overset{*}{F}{}^{\mu\nu} = \frac{1}{2} e (\delta^{\gamma}{}_{\alpha} \delta^{\delta}{}_{\beta} - \delta^{\gamma}{}_{\beta} \delta^{\delta}{}_{\alpha}) F^{\alpha\beta} F_{\gamma\delta} =$

$= \frac{1}{2} e F^{\alpha\beta} F_{\alpha\beta} - \frac{1}{2} e F^{\alpha\beta} F_{\beta\alpha} = e \cdot F_{\alpha\beta} F^{\alpha\beta} = - F_{\mu\nu} F^{\mu\nu}$

$\boxed{\overset{*}{F}_{\mu\nu} \overset{*}{F}{}^{\mu\nu} = - F_{\mu\nu} F^{\mu\nu}}$ q.e.d.

**Problem 8** Consider rotation-translation transformations (i.e. rotations and translations of euclidian 3-space)

(i) Prove that if RT transformations are applied to a 4-vector $A^{\mu} = (A^0; \bar{a})$, $A^0$ transforms as scalar and

6 $\bar{a}$ as a three-vector

* Translations are acting on 3-vector only so that $A^\mu \to A^\mu + c^\mu$, where general form of $c^\mu$ looking like $c^\mu = (0, \bar{c})$ thus we see that translation transformations act in the following way on 4-vector components:

$A^0 \to A^0$ : scalar-like transformation

$\bar{a} \to \bar{a} + \bar{c}$ : vector-like transformation

* Rotation

general form of rotation transformation:

$$A^{\mu'} = R^{\mu'}{}_{\nu} A^{\nu} \quad \text{where} \quad R^{\mu'}{}_{\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & R^1{}_1 & R^1{}_2 & R^1{}_3 \\ 0 & R^2{}_1 & R^2{}_2 & R^2{}_3 \\ 0 & R^3{}_1 & R^3{}_2 & R^3{}_3 \end{bmatrix}$$

Particular example is rotation by angle $\theta$ in positive direction about z-axis.

$$R_z(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

thus if we consider transformation of $A^\mu$ components $A^{0'} = R^{0'}{}_0 A^0 = A^0$

$A^{0'} = A^0$ : scalar-like transformation

$a^{i'} = R^{i'}{}_i a^i$ - 3-vector transformation.

(ii) of an RT transformation is applied to a four-tensor $T^{\mu\nu}$, then $T^{00}$ transforms as scalar, $T^{0i}$ and $T^{i0}$ as three-vectors and $T^{ij}$ as three-tensor

Let's see how all this tensor components are transformed

$T^{0'0'} = R^{0'}{}_0 R^{0'}{}_0 T^{00} = T^{00}$ - transforms as scalar

$T^{0'i'} = R^{0'}{}_0 R^{i'}{}_i T^{0i} = R^{i'}{}_i T^{0i}$ - transforms as 3-vector

$T^{i'0'} = R^{0'}{}_0 R^{i'}{}_i T^{i0} = R^{i'}{}_i T^{i0}$ - transforms as 3-vector

$T^{i'j'} = R^{i'}{}_i R^{j'}{}_j T^{ij}$ - transforms as 3-tensor

the same we can say about translation transformations

(7) $T^{0'0'} = T^{00}$ – scalar transformation

$T^{0'i'} = T^{0i} + c_1^i$ – 3-vector transformation

$T^{i'0'} = T^{i0} + c_2^i$ – 3-vector transformation

$T^{i'j'} = T^{ij} + c_1^i \otimes c_2^j$ – tensor transformation

Problem 9 An inertial observer O has 4-velocity $U_0$ and a particle P has 4-acceleration A. If $U_0 \cdot A = 0$, what can you conclude about speed of P in O's rest frame.

---

Theory Here we meet examples of 4-vectors with physical meaning - 4-velocity and 4-acceleration

If we parametrise trajectory of particle with the proper time $\tau$ we can introduce 4-velocity $U^\mu = \frac{dx^\mu}{d\tau}$ and 4-acceleration $A^\mu = \frac{dU^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2}$;

To relate 4-velocity $U^\mu$ with 3-velocity $\bar{u}$ remember that $\frac{dt}{d\tau} = \gamma(u)$ due to dilation of time. Thus $U^\mu = \frac{dx^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dx^\mu}{dt} = \gamma(u)(c, \bar{u})$. Now 4-acceleration is given by $A^\mu = \frac{dU^\mu}{d\tau} = \gamma\frac{dU^\mu}{dt} = \gamma\frac{d}{dt}\gamma(u)\cdot(c, \bar{u})$;

Let's look what are values of Lorentz invariants $U^\mu U_\mu$; $A^\mu \cdot A_\mu$ and $U_\mu \cdot A^\mu$. As this all are Lorentz invariants we can obtain their values in some particular reference frame. Convinient one is instantoneous rest frame. There we get $\bar{u} = 0$ thus $\gamma(u) = 1$ and $U^\mu = (c, \bar{0})$, $A^\mu = (0, \bar{a})$, $\bar{a}$ being proper acceleration. So in instantoneous rest frame we easily get:

$$\boxed{U^\mu \cdot U_\mu = c^2;\ A^\mu \cdot A_\mu = -\bar{a}^2;\ U^\mu \cdot A_\mu = 0;}$$

and this equations are valid for all reference frames.

⑧ Now let's return to the problem we are solving. We have just obtained that $U_0 \cdot A$ is Lorentz invariant so that it's equal zero in all reference frames. In O rest frame $U_0' = (c, 0)$. Now if $\bar{u}'$ is speed of P in O rest frame we get

$A' = \gamma(u') \frac{d}{dt}\{\gamma(u')(c, \bar{u}')\}$ thus as $U_0' \cdot A' = 0$:

$c \cdot \gamma(u') \frac{d}{dt} \cdot c \cdot \gamma(u') = 0$; as $\frac{d\gamma}{dt} \sim \dot{u} \cdot u = 0 \Rightarrow u = 0$ i.e. P has momentarely zero velocity in O rest frame.

**Problem 15** In a given inertial frame $S_0$ moving with 4-velocity $U_0^\mu$ a tensor $T^{\mu\nu}$ has but a single non-vanishing component: $T^{00} = c^2$. Find the components of this tensor in the general frame $S$, relative to which $U_0^\mu = \gamma(u)(c, \bar{u})$

There are 2 ways of solving this problem. First one is straightforward: making Lorentz transformation

$T^{\mu'\nu'} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} T^{\mu\nu} = \Lambda^{\mu'}{}_{0} \Lambda^{\nu'}{}_{0} c^2$ (only nonzero component is $T^{00} = c^2$)

of general form $\Lambda^{\mu'}{}_{\mu} = \begin{bmatrix} \gamma & +\gamma\beta_x & +\gamma\beta_y & +\gamma\beta_z \\ +\gamma\beta_x & \cdots & \cdots & \cdots \\ +\gamma\beta_y & \cdots & \cdots & \cdots \\ +\gamma\beta_z & \cdots & \cdots & \cdots \end{bmatrix}$ written in theoretical introduction

we easily obtain:

$T^{0'0'} = \gamma^2 c^2$; $T^{0'i'} = \gamma^2 c \cdot u^{i'}$; $T^{i'j'} = \gamma^2 u^{i'} u^{j'}$

This components in more compact form can be written as $\boxed{T^{\mu\nu} = U_0^\mu U_0^\nu}$

Second way of solving this problem is just guessing the only tensor structure we can construct from information we have is $T^{\mu\nu} = a \cdot U_0^\mu U_0^\nu$ where "a" is some scalar. In $S_0$ reference frame we get:

$T^{00} = c^2$ so $a = 1$ and we get the same result

$$\boxed{T^{\mu\nu} = U_0^\mu U_0^\nu;}$$

(9)

**Problem 16** A particle perfoms a helical motion described by $x = R\cos\omega t$; $y = R\sin\omega t$; $z = ut$ ($R, \omega, u$ being constants). Find it's proper acceleration.

4-velocity of particle is $u^\mu = \gamma(v)(c, -\omega R\sin\omega t; \omega R\cos\omega t; u)$

$v$ here is absolute velocity $v = \sqrt{u^2+\omega^2R^2} = const$;

Then 4-acceleration is given by

$$A^\mu = \frac{du^\mu}{d\tau} = \gamma\frac{du^\mu}{dt} = -\gamma^2(0; \omega^2R\cos\omega t; \omega^2R\sin\omega t; 0)$$

the proper acceleration can be easely found by

$\alpha^2 = -A^\mu A_\mu = +\gamma^4\omega^4R^2$, thus finally $\boxed{\alpha = \gamma^2\omega^2R;}$

**Problem 17** A particle moves rectilinearly with constant proper acceleration $\alpha$. $U$ and $A$ it's 4-velocity and 4-acceleration, $\tau$ its proper time, and units are chosen to make $c=1$, prove $\frac{dA^\mu}{d\tau} = \alpha^2U^\mu$

In this part of problem we are asked to use known form of rectilinear motion. along x-axis

$$V = c\tanh\left(\frac{\tau\alpha}{c}\right); \quad \gamma(u) = \cosh\left(\frac{\alpha\tau}{c}\right); \quad \frac{\alpha t}{c} = \sinh\left(\frac{\alpha\tau}{c}\right); \quad \frac{\alpha x}{c^2} = \cosh\left(\frac{\alpha\tau}{c}\right);$$

as $A^1 = \frac{d^2x}{d\tau^2} = \alpha\cosh\left(\frac{\alpha\tau}{c}\right)$; $U^1 = \frac{dx}{d\tau} = c\cdot\sinh\left(\frac{\alpha\tau}{c}\right)$;

we see that indeed $\frac{dA}{d\tau} = \frac{\alpha^2}{c}\sinh\left(\frac{\alpha\tau}{c}\right) = \frac{\alpha^2}{c^2}U$

if we take $c=1$; $\frac{dA}{d\tau} = \alpha^2U$, indeed.

Prove, conversly, that this equation, without the information that $\alpha$ is the proper acceleration, or constant, implies both these facts.

Let's assume that equation $\frac{dA}{d\tau} = \alpha^2U$ is valid

Now let's interpret $\alpha$ using equation $A\cdot U = 0 \Rightarrow$

$\frac{dA}{d\tau}\cdot U + \frac{dU}{d\tau}\cdot A = 0 \Rightarrow \frac{dA}{d\tau}\cdot U = -A\cdot A = \alpha^2$ where $\alpha$ is proper acceleration. Now let's check that $\alpha$ is constant: $\alpha^2 = -A_\mu\cdot A^\mu \Rightarrow \alpha\frac{d\alpha}{d\tau} = -A_\mu\frac{dA^\mu}{d\tau} = -\alpha^2A_\mu U^\mu = 0$; so

(10) So we conclude that $\alpha = \text{const}$, q.e.d.

Finally show, by integration, that the equation implies rectilinear motion in a suitable inertial frame, and thus, in fact, hyperbolic motion. Consequently $\frac{dA}{d\tau} = \alpha^2 U$ is the tensor equation characteristic of hyperbolic motion.

Let's solve $\frac{d^2 U^\mu}{d\tau^2} = \alpha^2 U^\mu$, as $\alpha$ is constant

$U^\mu(\tau) = U_1^\mu e^{\alpha\tau} + U_2^\mu e^{-\alpha\tau}$. If $U^\mu_{1,2} = 0$ for $\mu = 2,3$ (motion along x-axis) we get

$\frac{dx}{d\tau} = a e^{\alpha\tau} + b e^{-\alpha\tau}$, where $a$ and $b$ are constants.

we know else that $(U^0)^2 - (U^1)^2 = 1$. So we can use the following parametrisation $U^0 = \cosh x$ $U^1 = \sinh x$ if we look on expression for $\frac{dx}{d\tau}$ we can conclude that $\frac{dx}{d\tau} = \sinh \alpha\tau \Rightarrow \boxed{x = \frac{1}{\alpha}\cosh \alpha\tau + C_1}$ where $C_1$ is integration constant. So we see that we have indeed obtained formulae for rectlinear motion.

# Appendix problems.

**Problem 1** (i) A vector $A^i$ has components $\dot{x}, \dot{y}$ in rectangular Cartesian coordinates; what are its components in polar coordinates.

$A^i = (\dot{x}, \dot{y})$ in Cartesian coordinates. Polar coordinate system basis vectors in cartesian coordinates are written in the following form: $\bar{e}_r = (\cos\theta, \sin\theta)$; $\bar{e}_\theta = (-\sin\theta; \cos\theta)$

$\dot{\bar{e}}_r = \dot{\theta}(-\sin\theta; \cos\theta)$; $\dot{\bar{e}}_\theta = -\dot{\theta}(\cos\theta; \sin\theta) = -\dot{\theta}\,\bar{e}_r$, now we can find $\bar{A} = \dot{\bar{r}} = (\dot{r\bar{e}_r}) = \dot{r}\bar{e}_r + r\dot{\bar{e}}_r = \dot{r}\bar{e}_r + r\dot{\theta}\bar{e}_\theta$;

Thus in polar coordinates $\boxed{\bar{A} = (\dot{r}; r\dot{\theta});}$

(ii) The same for vector $\bar{B} = \ddot{\bar{r}}$

$\ddot{\bar{r}} = \ddot{r}\bar{e}_r + \dot{r}\dot{\bar{e}}_r + \dot{r}\dot{\theta}\bar{e}_\theta + r\ddot{\theta}\bar{e}_\theta + r\dot{\theta}\dot{\bar{e}}_\theta = \ddot{r}\bar{e}_r + 2\dot{r}\dot{\theta}\bar{e}_\theta + r\ddot{\theta}\bar{e}_\theta -$
$- r\dot{\theta}^2\bar{e}_r$ ; thus $\bar{B} = (\ddot{r} - r\dot{\theta}^2)\bar{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\bar{e}_\theta$;

$\boxed{B = (\ddot{r} - r\dot{\theta}^2; 2\dot{r}\dot{\theta} + r\ddot{\theta});}$

**Problem 3** Use the quotient rule to show that $\delta^i_{\ j}$ is a tensor?

Quotient rule says that if a set of components when combined by a given type of multiplication with arbitrary tensor of a given valence yields a tensor, then the set constitutes a tensor. For example if $A^i$ and $B^i$ are vectors $A^i = C^i_{\ j} B^j$ and as we know for $\delta^i_{\ j}$: $A^i = \delta^i_{\ j} A^j$ so $\delta^i_{\ j}$ is indeed tensor

**Problem 5** (i) If $C_{ij}A^iA^j$ is a scalar for an arbitrary vector $A^i$, prove that $(C_{ij} + C_{ji})$ is a tensor

Let's look how $C_{ij}$ transforms. As $C_{ij}A^iA^j$ is scalar
$C_{i'j'}A^{i'}A^{j'} = C_{ij}A^iA^j$ and as $A^{i'} = A^i p^{i'}_{\ i}$ we get

$(C_{i'j'}p^{i'}_{\ i}p^{j'}_{\ j} - C_{ij})A^iA^j = 0$ and as $A^i$ is arbitrary vector and we can take first vector with one nonzero component thus eliminating summation, then vector with 2 nonzero components and so on we get:

$C_{ij} = C_{i'j'}p^{i'}_{\ i}p^{j'}_{\ j}$ which is rule of tensor transformation.

thus $C_{ij}$ is tensor. Now we can make standart procedure of symmetrisation: $C_{ij}A^iA^j = \frac{1}{2}(C_{ij} + C_{ji})A^iA^j$

Thus object $(C_{ij} + C_{ji})$ can be treated on equal footing with $C_{ij}$ and thus it is tensor too, q.e.d.

(ii) If $C_{ij}A^iB^j$ is a scalar for two arbitrary vectors $A^i, B^i$, prove that $C_{ij}$ is a tensor.

(12) The line of proof is just the same:

$C_{ij} A^i B^j = C_{i'j'} A^{i'} B^{j'} = C_{i'j'} p^{i'}_i p^{j'}_j A^i B^j$ and thus

$(C_{ij} - C_{i'j'} p^{i'}_i p^{j'}_j) A^i B^j = 0$ and as in the previous part of problem we conclude

$C_{ij} = C_{i'j'} p^{i'}_i p^{j'}_j$ and we conclude that $C_{ij}$ is tensor.

Problem 9 If $g_{ij} = 0$ for $i \neq j$, prove that $g^{ij} = 0$ for $i \neq j$, and $g^{ii} = \frac{1}{g_{ii}}$

By definition $g^{ij} g_{jk} = \delta^i_k$ as $g_{ij} = 0$ for $i \neq j$

we write $g^{ij} g_{jj} = \delta^i_j$ (no summation over $j$ here)

Thus $g^{ij} g_{jj} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$ so if $i \neq j$ $g^{ij} = 0$ and if $i = j$ $g^{ii} = \frac{1}{g^{ii}}$, q.e.d.

Problem 17 For any antisymmetric tensor $F_{ij}$ we define a dual tensor $\overset{*}{F}_{ij} = \frac{1}{2} \varepsilon_{ijkl} F^{kl}$. Prove that the dual of the dual is plus or minus the original tensor

$\frac{1}{2} \varepsilon_{ijkl} \overset{*}{F}^{kl} = e F_{ij}$

Let's go streightforward $\frac{1}{2} \varepsilon_{ijkl} \overset{*}{F}^{kl} = \frac{1}{4} \varepsilon_{ijkl} \varepsilon^{klmn} F_{mn} =$

$= \frac{1}{4} \varepsilon_{ijkl} \varepsilon^{mnkl} F_{mn} = \frac{1}{2} e (\delta^m_i \delta^n_j - \delta^m_j \delta^n_i) F_{mn} = \frac{1}{2} e (F_{ij} - F_{ji}) =$

$= \frac{1}{2} e F_{ij} \cdot 2 = e F_{ij}$. here we have used equation (ii) from exercise 16: $\varepsilon_{ijkl} \varepsilon^{ijmn} = 2e (\delta^m_k \delta^n_l - \delta^m_l \delta^n_k)$

and antisymmetric property of $F_{ij}$: $F_{ij} = -F_{ji}$;

so we indeed get $\boxed{\frac{1}{2} \varepsilon_{ijkl} \overset{*}{F}^{kl} = e F_{ij}}$, q.e.d.

① 

# Seminar 6 (relativistic particle mechanics)

**Theory** In previous lecture we have introduced some 4-vectors that will be usefull in seminars 6 and 7. This is mainly 4-momentum vector which is introduced in the way analogeous to the usual 3-momentum $\underset{\text{nonrelativistic}}{p^i = m_0 u^i}$ and $\underset{\text{relativistic.}}{p^\mu = m_0 u^\mu}$

$m_0$ is calle the rest mass. If we write it in the components form: $p^\mu = m_0 \gamma(u)(c, \bar{u})$. There are 2 conclusions can be drawn from this expression

* first of all if we look on spatial components we get following expression for 3-momentum:
$\bar{p} = m_0 \gamma(u) \cdot \bar{u} = m\bar{u}$ the last form looks very much like usual nonrelativistic expression, in which we have introduced velocity-dependent relativistic mass of particle $\boxed{m(u) = m_0 \gamma(u)}$

* second we should interpret somehow $p^0$. Right interpretation coming from nonrelativistic limit is that $p^0 = \frac{E}{c}$, i.e this is energy of particle. Difference with nonrelativistic mechanics is that particle posses energy even being at rest $(\boxed{E_0 = m_0 c^2})$ Kinetic energy is then given by $\boxed{T = E - E_0 = m_0 c^2(\gamma - 1)}$;

Important and usefull law we will widely use is 4-momentum conservation

$\sum\limits_{\text{before react}} p_\mu = \sum\limits_{\text{after react}} p_\mu$. Using just this single relation a lot of problems can be solved.

Another thing that will be usefull to us is relation for $p_\mu p^\mu$: $p_\mu p^\mu = m_0^2 c^2$ and invariance of scalar product.

2

**Problem 1** How fast must particle move before its kinetic energy equals it's rest energy?

Let $m_0$ be the rest mass of particle. Then as we have already written energy is given by $E = m_0 c^2 \gamma$, rest energy is $E_0 = m_0 c^2$ and kinetic is $T = m_0 c^2 (\gamma - 1)$. So if we want kinetic energy to be equal to rest energy we should obtain:

$$T = E_0 \Rightarrow \gamma - 1 = 1 \Rightarrow \boxed{\gamma = 2}$$

**Problem 2** How fast must a 1kg cannon ball move to have the same kinetic energy as a cosmic-ray proton moving with $\gamma$-factor $10^{11}$?

To solve this we should know proton mass:

$m_p = 938 \frac{MeV}{c^2}$ or in SI units we get: $m_p = 1{,}6 \cdot 10^{-27}$ kg.

Thus kinetic energy is $T_p = m_p c^2 (\gamma - 1) \approx m_p c^2 \gamma$ as $\gamma \gg 1$

canon ball mass is 27 orders bigger then the proton's one, so cannon ball will be nonrelativistic for sure. and we can approximate it's kinetic energy by standart non-relativistic formulae. $T_b = \frac{1}{2} m_b v^2$ where $m_b = 1$kg is canon ball rest mass. thus we get

$$\frac{1}{2} m_b v^2 = \gamma m_p c^2 \Rightarrow v = \sqrt{\frac{2\gamma m_p c^2}{m_b}} = \sqrt{\frac{2 \cdot 10^{11} \cdot 1{,}6 \cdot 10^{-27}}{1}} \, c =$$

$$= 1{,}79 \cdot 10^{-8} \cdot 3 \cdot 10^8 \frac{m}{s} = 5{,}36 \frac{m}{s}; \text{ thus } \boxed{v = \sqrt{\frac{2\gamma m_p c^2}{m_b}} = 5{,}36 \frac{m}{s}}$$

**Problem 3** What is the $\gamma$-factor of a proton accelerated to an energy of 20 TeV?

The rest mass of the proton is $m_p = 938 \frac{MeV}{c^2}$

Energy of the proton is $\mathcal{E}_p = m_p \cdot \gamma \cdot c^2$ thus $\gamma = \frac{\mathcal{E}_p}{m_p c^2}$

$$\gamma = \frac{20 \cdot 10^{12}}{938 \cdot 10^6} \approx 21300 \qquad \boxed{\gamma = \frac{\mathcal{E}_p}{m_p c^2} \approx 21300}$$

(3)

Problem 4 The mass of hydrogen atom is 1.00814 a.m.u., that of a neutron is 1.00898 a.m.u., and that of a helium atom (2 hydrogen atoms and 2 neutrons) is 4.00388 a.m.u. Find the binding energy as a fraction of the total energy of a helium atom.

$m_H = 1{,}00814$ a.m.u. ; $m_n = 1{,}00898$ a.m.u. ; $m_{He} = 4{,}00388$ a.m.u. ;

mass excess equals $\Delta m = 2m_p + 2m_n - m_{He} = 2\cdot 1{,}00814$ a.m.u. + $+ 2\cdot 1{,}00898$ a.m.u. $- 4{,}00388$ a.m.u. $= 0{,}03036$ a.m.u.

mass excess is related to the bonding energy directly: $E_{Bond} = \Delta m \cdot c^2$; thus

$$\frac{E_{Bond}}{E_{He}} = \frac{\Delta m \cdot c^2}{m_{He} c^2} = \frac{\Delta m}{m_{He}} = \frac{0{,}03036}{4{,}00388} = 0{,}00758 \; (0{,}76\%)$$

$$\boxed{\frac{E_{Bond}}{E_{He}} = 2\,\frac{m_p + m_n}{m_{He}} - 1 = 0{,}76\%}$$

Problem 6 A rocket propels itself rectilinearly by giving portions of its mass a constant (backward) velocity $U$ relative to its instantaneous rest frame. It continious to do so until it attains a velocity $V$ relative to its initial rest frame. Prove that the ratio of the initial to the final rest mass of the rocket is

$\frac{M_i}{M_f} = \left(\frac{c+V}{c-V}\right)^{\frac{c}{2U}}$ Note that the least expenditure of mass needed to attain a given velocity occurs when $U=c$, i.e. when the rocket propels itself with a jet of photons

Let $S$ be the initial rest frame and $S'$ - instantaneous rest frame at some time $t$ measured in $S$ and corresponding to the proper time $\tau$ (proper time of ship)

Let's consider system rocket + ejected gas in instentoneous rest frame between moments of time

④

$\tau$ and $\tau+d\tau$. During this time ship ejects mass $(-dM)$ fuel which gives it momentum $-dM\cdot U = M\,du'$ where $du'$ is velocity in instanteneous frame.

In initial rest frame in corresponding moment of time we get due to velocity transformation formulae:

$$V+dV = \frac{V+du'}{1+\frac{V du'}{c^2}} \Rightarrow V+dV+\frac{V}{c^2}dV du' + \frac{V^2}{c^2}du' = V+du' \quad \text{thus}$$

we finally get: $dV = du'\left(1-\frac{V^2}{c^2}\right) = -\frac{dM}{M}u\left(1-\frac{V^2}{c^2}\right)$

on the second step we have used relation obtained from momentum conservation law. Thus we get:

$$-\frac{dM}{M}\cdot u = \frac{dv}{1-\frac{V^2}{c^2}} = \frac{1}{2}\frac{dv}{1+\frac{V}{c}} + \frac{1}{2}\frac{dv}{1-\frac{V}{c}}$$

This equation can be easily integrated:

$-u\cdot\ln\frac{M_f}{M_i} = \frac{1}{2}c\log\frac{c+V}{c-V}$ and thus we finally get desired answer: $\boxed{\frac{M_i}{M_f} = \left(\frac{c+V}{c-V}\right)^{\frac{c}{2u}}, \text{ q.e.d.}}$

(ii) By reference to exercise II (15), prove that, if the rocket moves with constant proper acceleration $\alpha$ for a proper time interval $\tau$, then $\frac{M_i}{M_f} = \exp\left(\frac{\alpha\tau}{u}\right)$. If $u=c$, $\alpha=g$ and $\tau = n$ yr, prove $\frac{M_i}{M_f}\approx e^n$;

By definition proper acceleration is $\alpha = \frac{du'}{d\tau} = -\frac{u}{M}\frac{dM}{d\tau}$;

Integration of this equation gives us

$\tau\alpha = u\log\frac{M_i}{M_f}$; and thus $\frac{M_i}{M_f} = \exp\frac{\alpha\tau}{u}$, substituting here $u=c\approx 1$, $\tau = n$ years; $\alpha = g\approx 1\,\frac{\text{l.yr.}}{\text{yr.}}$ we get:

$$\boxed{\frac{M_i}{M_f} = e^n, \text{ q.e.d.}}$$

⑤

**Problem 7** Two particles with rest masses $m_1$ and $m_2$ move collinearly in some inertial frame, with uniform velocities $u_1$ and $u_2$ respectively. They collide and form a single particle with rest mass $m$ moving at velocity $u$. Prove that

$m^2 = m_1^2 + m_2^2 + 2 m_1 m_2 \gamma(u_1)\gamma(u_2)\left(1 - \frac{u_1 u_2}{c^2}\right)$ and also find $u$.

Here we will see how convinient 4-momentum conservation is. Here it takes form

$P^\mu = P_1^\mu + P_2^\mu$; Let's square both sides. We get:

$P^2 = (P_1 + P_2)^2$ (as usually by $p^2$ we mean $p_\mu p^\mu$)

Here in particular $P^\mu = \gamma(u) m (c, \bar{u})$; $P_1^\mu = \gamma(u_1) m_1 (c; \bar{u}_1)$ and $P_2^\mu = \gamma(u_2) m_2 (c; \bar{u}_2)$ as initial movement of particles 1 and 2 is collinear we conclude that $\bar{u}_1 \parallel \bar{u}_2 \parallel \bar{u}$ (collinearity of $\bar{u}$ with inital velocities follows from conservation of 3-momentum. Transverse momentum can't appear after collision due to conservation). So we get

$P^2 = P_1^2 + P_2^2 + 2 P_1 \cdot P_2$ $\quad P_i^2 = m_i^2 c^2$ and $P_1 \cdot P_2 = \gamma(u_1)\gamma(u_2) m_1 m_2 \times c^2\left(1 - \frac{u_1 u_2}{c^2}\right)$

thus

$$\boxed{m^2 = m_1^2 + m_2^2 + 2\gamma(u_1)\gamma(u_2) m_1 m_2 \left(1 - \frac{u_1 u_2}{c^2}\right);} \quad (*)$$

to find $u$ let's consider conservation of energy

$P^0 = P_1^0 + P_2^0$; $\Rightarrow \gamma(u) = \frac{m_1}{m}\gamma(u_1) + \frac{m_2}{m}\gamma(u_2)$; thus

$$\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{m_1 \gamma(u_1) + m_2 \gamma(u_2)}{m} \Rightarrow \frac{u^2}{c^2} = 1 - \frac{m^2}{(m_1\gamma_1 + m_2\gamma_2)^2};$$

now substituting $m^2$ from $(*)$ into this equation as $(1 - \gamma_i^2) = -\frac{u_i^2}{c^2}\gamma_i^2$

$$\frac{u^2}{c^2} = \frac{m_1^2(\gamma_1^2 - 1) + m_2^2(\gamma_2^2 - 1) + 2\gamma_1\gamma_2 m_1 m_2 \frac{u_1 u_2}{c^2}}{(m_1\gamma_1 + m_2\gamma_2)^2}$$

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$$\boxed{u = \frac{|m_1\gamma_1 u_1 + m_2\gamma_2 u_2|}{m_1\gamma_1 + m_2\gamma_2}}$$

**Problem 8** Consider a head-on elastic collision of a bullet of rest mass $M$ with a stationary target of rest mass $m$. Prove that the post-collision $\gamma$-factor of the bullet cannot exceed $\frac{(m^2+M^2)}{2mM}$. This means that for large bullet energies (with $\gamma$ factors much larger than this critical value), the relative transfer of energy from bullet to target is almost total. The situation is radically different in Newtonian mechanics, where the pre- and post-collision velocities of the bullet are related by $\frac{u}{u'} = \frac{M+m}{M-m}$. Prove this

* relativistic case

Let $P$ and $Q$ be 4 momentum of bullet and target correspondingly, before the collision. $P'$ and $Q'$ - corresponding 4-momenta after collision.

We will use time-like nature of 4-momentum. In fact we can consider $(P'-Q)^2$ and go to the center of momentum system where only nonzero component of $(P'-Q)$ is $(P'^0 - Q^0)$ so that we can say that in this system $(P'-Q)^2 = (P'^0 - Q^0)^2 > 0$ and thus as $(P'-Q)^2$ is Lorentz invariant it will be positive in all frames

in the rest frame of target we have

$P'^\mu = \gamma(u')(c;\bar{u}')m$; $Q^\mu = (cM;0)$;

$(P'-Q)^2 = M^2c^2 + m^2c^2 - 2P'_\mu Q^\mu$; substituting explicit form of 4-mometa into this expression we get:

$M^2c^2 + m^2c^2 - 2c^2 Mm\gamma(u') \geqslant 0$ thus $\boxed{\gamma(u') \leqslant \frac{M^2+m^2}{2Mm}, \text{ q.e.d}}$

(7)

* nonrelativistinc case

Here we just write momentum and energy conservation

$Mu = Mu' + mv'$

$\frac{1}{2}Mu^2 = \frac{1}{2}Mu'^2 + \frac{1}{2}mv'^2$

from the first equation we get:

$v' = \frac{M}{m}(u-u')$; substituting this into second equation we get: $\frac{1}{2}Mu^2 = \frac{1}{2}Mu'^2 + \frac{1}{2}\frac{M^2}{m}(u-u')^2$ thus

$mu^2 = mu'^2 + Mu^2 + Mu'^2 - 2Mu'u$ If we now denote $x = \frac{u}{u'}$; $(m-M)x^2 + 2Mx - (M+m) = 0$;

$D = 4M^2 + 4m^2 - 4M^2 = 4m^2$; $x = \frac{-M \pm m}{m-M}$ "+" sign give us trivial result. when collision doesn't effect the motion of bullet $x = \frac{m+M}{M-m}$ is solution we are interested in. so $\boxed{\frac{u}{u'} = \frac{M+m}{M-m}}$; difference with the relativistic case is that there we have obtained the limit for velocity of bullet after collision. Here in classical case there is no such limit and velocity after collision linearly depends on velocity before the collision.

Problem 11 Show that a photon cannot spontaneously disintegrate into an electron-positron pair. But in the presence of a stationary nucleus (acting as a kind of catalyst) it can. If the rest mass of nucleus is $N$, and that of the electron is $m$, what is the threshold frequency of photon? Verify that for large $N$ the efficiency is ~ 100 percent so that the nucleus then comes close to being a pure catalyst.

4-momentum conservation give us $P^\mu = P_1^\mu + P_2^\mu$; thus $(P_1 + P_2)^2 = 0$;

(8) $2m_e^2c^2 + 2P_1 \cdot P_2 = 0$; in electron-positron COM

$P_1^\mu + P_2^\mu = (P_1^0 + P_2^0; \bar{0})$ thus in this system for $(P_1+P_2)^2 = 0$ we should obtain $P_1^0 = -P_2^0$ which can be satisfied only if $P_1^0 = P_2^0 = 0$ which in turn can't be satisfied as $P_{1,2}^0 \geq m_e c$; Anothe way to show the same fact is write down $2m_e^2c^2 + 2P_1 \cdot P_2 = 2m_e^2c^2 + 2m_e^2c^2\gamma(u_1)\gamma(u_2)\left(1 - \frac{u_1u_2}{c^2}\right) = 0$ and as velocities of electron and positron are smaller the the speed of light we get $1 - \frac{u_1u_2}{c^2} \geq 0$ and thus $2m_e^2c^2 + 2m_e^2c^2\gamma(u_1)\gamma(u_2)\left(1 - \frac{u_1u_2}{c^2}\right) > 0$ always. We can say that decay is forbidden because we can't go to the photons rest frame.

Now assume we have one more body in the story, namely, nuclei

$$P^\mu + N^\mu = Q_1^\mu + Q_2^\mu + N'^\mu;$$

threshold energy occurs when in the COM frame of resulting particles all they have zero momentum, taking square of both sides we get:

$$(P_\mu + N_\mu)^2 = c^2N^2 + 2Nc\,h\nu = (Q_1^\mu + Q_2^\mu + N'^\mu)^2 = (2m_e^2 + N^2)c^2 + $$
$$+ 4m_eNc^2 + 2m_e^2 \cdot c^2$$ wher we have used following 4-momentas

$$\left.\begin{aligned} P^\mu &= \tfrac{1}{c}h\nu(1, \bar{n}); \\ N^\mu &= N \cdot (c, \bar{0}); \end{aligned}\right\} \text{lab. frame}$$

$$\left.\begin{aligned} N'^\mu &= N \cdot (c, \bar{0}); \\ Q_{1,2}^\mu &= m_e \cdot (c, \bar{0}); \end{aligned}\right\} \text{threshold in COM};$$

$\frac{1}{c}h\nu \cdot 2Nc = 4m_e^2c^2 + 4m_eNc^2$; $\boxed{\nu = \frac{2m_e^2c^2 + 2m_eNc^2}{hN}}$

(9) **Problem 13** A fast electron of rest mass $m$ decelerates in a collision with a heavy nucleus and emits a Bremsstrahlung photon. Prove that the energy of the photon can range all the way up to $(\gamma-1)mc^2$, the kinetic energy of electron

What we have is process:

$N+e^- \to N+e^-+\gamma$, 4-momentum conservation gives us

$N^\mu + p^\mu = N'^\mu + p'^\mu + q^\mu$

energy of photon is maximal when energy of system electron + nuclei is minimal and this happens when in COM frame of electron and nuclei both particles are in rest

---

**Note:** To understand this fact conside $(p_1+p_2)^\mu = \left(\frac{\mathcal{E}_1+\mathcal{E}_2}{c}; \bar{p}_1+\bar{p}_2\right)$

$(p_1+p_2)^2 = \frac{1}{c^2}(\mathcal{E}_1+\mathcal{E}_2)^2 + (\bar{p}_1+\bar{p}_2)^2 = m_1^2c^2 + m_2^2c^2 + 2p_1\cdot p_2$;

r.h.s. of this equation is minimal when $p_1 p_2$ is minimal. as $p_1 p_2$ is Lorentz invariant we can go into COM frame where we have $\bar{p}_1 = -\bar{p}_2 = \bar{p}$ and thus $p_1 p_2 = \frac{1}{c^2}\mathcal{E}_1\mathcal{E}_2 + |\bar{p}|^2$ which is minimal, in turn, when $\bar{p}=0$, i.e. both particles are in rest.

---

Now we write 4-momentum consevation in the following form: $N^\mu + p^\mu - q^\mu = N'^\mu + p'^\mu$ and square it

$(N+p-q)^2 = (N'+p')^2$. It is reasonable to consider l.h.s in lab. frame and r.h.s. in COM-frame of nuclei and electron. Then

$$\left.\begin{aligned} N &= (m_N c, \bar{0}) \\ p &= m_e\gamma(u)(c; \bar{u}) \\ q &= \frac{h\nu}{c}(1, \bar{n}) \end{aligned}\right\} \text{lab. fram. } \bar{n} \text{ is direction of photon velocity}$$

$$\left.\begin{aligned} N' &= (m_N c; \bar{0}) \\ p' &= (m_e c; \bar{0}) \end{aligned}\right\} \text{N-e COM-frame.}$$

Here $m_N$ and $m_e$ are masses of nuclei and electron

(10) respectively and $\nu$ is frequency of photon. Thus we get:

$$m_N^2 \cdot c^2 + m_e^2 c^2 + 2 m_N m_e c^2 \gamma(u) - 2 \frac{h\nu}{c} \cdot m_N c - 2 \frac{h\nu}{c} \gamma(u) m_e (c - u \cos\theta) =$$

$$= c^2 (m_e + m_N)^2 \text{ ; thus we get}$$

$$h\nu = \frac{m_e m_N c^2 (\gamma(u) - 1)}{m_N + m_e \gamma(u) (1 - \frac{u}{c} \cdot \cos\theta)} \leqslant \frac{m_e m_N c^2 (\gamma(u) - 1)}{m_N + m_e \gamma(u) (1 - \frac{u}{c})} \text{, maximum}$$

is obtained when $\cos\theta = 1$. Then maximal energy of photon is given by $\mathcal{E}_{max} = \dfrac{m_e m_N c^2 (\gamma(u) - 1)}{m_N + m_e \sqrt{\frac{1 - \frac{u}{c}}{1 + \frac{u}{c}}}}$ ; as

$\sqrt{\dfrac{1 - \frac{u}{c}}{1 + \frac{u}{c}}} \leqslant 1$ and $m_e \ll m_N$ we get $\boxed{\mathcal{E}_{max} = m_e c^2 (\gamma - 1)}$ , q.e.d.

# Seminar 7 (relativistic particle mechanics II)

## Theory

Some facts from previous class

The main kind of problems we will be solving on this class are collision problems and main tool is 4-momentum conservation

$$\sum_{\substack{\text{before}\\ \text{collision}}} p^\mu_i = \sum_{\substack{\text{after}\\ \text{collision}}} p^\mu_j$$ (latin indicies correspond to different particles)

Main trick here is to consider different 4-momentum conservation squared. As $p_\mu p^\mu$ is Lorentz invariant quantity we can consider it in convinient frame. For example, useful frame is centre of momentum (CM) frame, thus the frame in which $P^\mu = \sum_i p^\mu$ has no spatial components, thus $\sum_i \bar{p} = 0$. This are main things we need. For better understanding we should consider some concrete examples.

**Problem 14** A particle of rest mass $m$ decays from rest into a particle of rest mass $m'$ and photon. Find the separate energies of this end products.

Let's write down 4-momentum conservation:

$p^\mu = p'^\mu + q^\mu$ here we have introduced following notations:

$p^\mu = (mc; \bar{0})$ - 4-momentum of decaying particle

$p'^\mu = m'\gamma(u')(c, \bar{u}')$ - 4-momentum of massive product of decay

$q^\mu = h\nu(1, \bar{n})$ - 4-momentum of photon, $\bar{n}$ is photon's direction of movement, $\bar{u}'$ - massive particle velocity.

* first let's consider massive particle

Let's exclude photon by writing:

$q^\mu = p^\mu - p'^\mu$ Squaring this we obtain $(p-p')^2 = 0$ as

(2) $q^2=0$ for photon (<u>Note</u>: remember that for massive particle $p^2=m^2c^2$ and for massless particles $q^2=0$) thus

$(m^2+m'^2)c^2-2m\mathcal{E}'=0$ and finally we get

$$\boxed{\mathcal{E}'=\frac{m^2+m'^2}{2m}\cdot c^2;}$$

* now let's find energy of <u>photon</u>

$p^\mu-q^\mu=p'^\mu$ → in the same way as we excluded photon dynamics in previous part of the problem, in this part we exclude massive product of decay. Now we get: $(p-q)^2=m^2c^2-2\mathcal{E}_\gamma m$ ; $p'^2=m'^2c^2$; thus

$m^2c^2-2\mathcal{E}_\gamma m=m'^2c^2$ ; and finally we get:

$$\boxed{\mathcal{E}_\gamma=\frac{(m^2-m'^2)c^2}{2m};}$$

<u>Problem 15</u> An excited atom, of total mass $m$, is at rest in a given frame. It emits a photon and thereby loses internal (i.e. rest) energy $\Delta E$. Calculate the exact frequency of the photon, making due allowance for the recoil of the atom.

In fact in this problem we have just the same process as in the previous problem, i.e. 4-momentum conservation. $p^\mu=p'^\mu+q^\mu$ where $p^\mu$ is 4-momentum of atom of mass $m$, $p'^\mu$ – 4-momentum of atom of mass $m'$ and $q^\mu$ – 4-momentum of photon. In our case

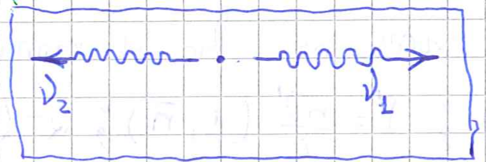
$m'=m-\frac{\Delta E}{c^2}$; So we can directly apply formulae obtained in previous problem:

$$\mathcal{E}_\gamma=h\nu=\frac{(m^2-m'^2)c^2}{2m}=\frac{c^2}{2m}\left(m^2-\left(m-\frac{\Delta E}{c^2}\right)^2\right)=\frac{c^2}{2m}\left(\frac{2\Delta E}{c^2}\cancel{m}-\frac{(\Delta E)^2}{c^4}\right)=$$

$=\Delta E\left(1-\frac{\Delta E}{2mc^2}\right)$, and thus $\boxed{\nu=\frac{\Delta E}{h}\left(1-\frac{\Delta E}{2mc^2}\right)}$ q.e.d.

③

**Problem 18** In an inertial frame $S$, 2 photons of frequencies $\nu_1$ and $\nu_2$ travel in the positive and negative x-directions respectively. Find the velocity of the CM frame of these photons.




; $p_1^\mu = \frac{h\nu_1}{c}(1;\hat{x})$

$p_2^\mu = \frac{h\nu_2}{c}(1;-\hat{x})$

here we have written down 4-momentum of photons, $\hat{x}$ is positive direction of x-axis. As we have written befor CM-frame is one were spatial components of 4-momentum equals zero. Total 4-momentum

$$P^\mu = p_1^\mu + p_2^\mu = \frac{h}{c}(\nu_1+\nu_2;\ \nu_1-\nu_2;0;0)$$

Now let's transform this 4-momentum with Lorentz transformation. $P^{\mu'} = \Lambda^{\mu'}{}_\nu P^\nu$, where $\Lambda^{\mu'}{}_\nu$ is standart lorentz transformation matrix

$$\Lambda^{\mu'}{}_\nu = \begin{bmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

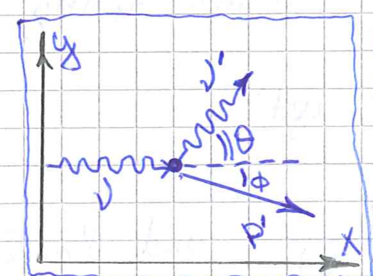
In particular:

$$P^{1'} = \Lambda^{1'}{}_\mu P^\mu = -\frac{v}{c}\gamma P^0 + \gamma P^1 = \frac{h}{c}\gamma(\nu_1 - \nu_2 - -\frac{v}{c}\nu_1 - \frac{v}{c}\nu_2)$$

$P^{2'} = \Lambda^{2'}{}_\mu P^\mu = P^2 = 0$; $P^{3'} = \Lambda^{3'}{}_\mu P^\mu = P^3 = 0$; So as we can see if we want momentum to be zero we need

$P^{1'} = \frac{h}{c}\gamma(\nu_1 - \nu_2 - \frac{v}{c}\nu_1 - \frac{v}{c}\nu_2) = 0$ so that $\boxed{\frac{v}{c} = \frac{\nu_1-\nu_2}{\nu_1+\nu_2}}$, q.e.d.

**Problem 19** For the Compton collision discussed in [Rindler] section 33 prove the relation $\tan\phi = \left(1+\frac{h\nu}{mc^2}\right)^{-1}\cot\frac{1}{2}\theta$;




Let $P, P'$ are pre- and post-collision four-momenta of the photon and $Q, Q'$ those of electrons (this are notations of Rindler). Let first recall the main results of Compton scattering derived in book.

Then 4-momentum conservation is:

(4) $P^\mu + Q^\mu = P'^\mu + Q'^\mu$; To find frequency $\nu'$ after collission we should read off electon after collision

$P^\mu + Q^\mu - P'^\mu = Q'^\mu$ squaring this we get:

$(P+Q-P')^2 = Q'^2$ thus $P^2 + Q^2 + P'^2 + 2P\cdot Q - 2P\cdot P' - 2P'\cdot Q = Q'^2$

Now we use the following expressions for 4-momentum: in lab. frame: $P = \frac{h\nu}{c}(1;\hat{x})$; $P' = \frac{h\nu'}{c}(1;\bar{n})$; $Q = (mc;\bar{0})$; $Q' = m\gamma(u)(c;\bar{u})$; where $\bar{n}$ is vector pointing in direction of outgoing photon momentum; $P^2 = P'^2 = 0$; $Q^2 = Q'^2 = m^2c^2$;

Then we get:

$h(\nu - \nu')m - \frac{1}{c^2}h^2\nu\nu'(1-\cos\theta) = 0$ where $\theta$ is the angle between $\bar{n}$ and $\hat{x}$, using trigonometric identity $1-\cos\theta = 2\sin^2\frac{\theta}{2}$; $\boxed{\sin^2\frac{\theta}{2} = \frac{mc^2}{2h}\left(\frac{1}{\nu'} - \frac{1}{\nu}\right);}$ q.e.d. (*)

Now to find angle between electron's direction of motion after collision let's consider separately conservations of all components of momentum, assuming that reaction plane coinsides with x-y plane

x: $P^1 + Q^1 = P'^1 + Q'^1 \Rightarrow \frac{h\nu}{c} = p'\cos\varphi + \frac{h\nu'}{c}\cos\theta$;

y: $P^2 + Q^2 = P'^2 + Q'^2 \Rightarrow p'\sin\varphi = \frac{h\nu'}{c}\cdot\sin\theta$;

so we get $p'\sin\varphi = \frac{h\nu'}{c}\sin\theta$; $p'\cos\varphi = \frac{h\nu}{c} - \frac{h\nu'}{c}\cos\theta$;

Deviding one equation with another we get:

$\tan\varphi = \frac{\nu'\sin\theta}{\nu - \nu'\cos\theta} = \frac{\sin\theta}{\frac{\nu}{\nu'} - \cos\theta}$; now from (*) we know that $\frac{\nu}{\nu'} = 1 + \frac{2h\nu}{mc^2}\sin^2\frac{\theta}{2}$; substituting this expression into equation above we get:

$\tan\varphi = \frac{\sin\theta}{1-\cos\theta + \frac{2h\nu}{mc^2}\sin^2\frac{\theta}{2}} = \frac{\sin\theta}{2\sin^2\frac{\theta}{2}}\frac{1}{1+\frac{h\nu}{mc^2}}$ and finally

$\boxed{\tan\varphi = \cot\frac{\theta}{2}\left(1 + \frac{h\nu}{mc^2}\right)^{-1}, \text{ q.e.d.}}$

⑤

**Problem 21** Taking $h = 6{,}63 \cdot 10^{-27}$ ergs and $c = 3 \cdot 10^5 \frac{km}{s}$; calculate how many photons of wavelength $5 \cdot 10^{-5}$ cm must fall per second on a blackened plate to produce a force of one dyne.

Force is just momentum absorbed per unit time: $F = \frac{\Delta p}{\Delta t}$. Let N denote total number of photons absorbed. Then total absorbed momentum is $\frac{Nh\nu}{c} = \frac{Nh}{\lambda}$ thus we get that

$$\frac{N}{\Delta t} = \frac{F\lambda}{h} = \frac{5 \cdot 10^{-5}\,cm \cdot 1\,dyne}{6{,}63 \cdot 10^{-27}\,erg} = 7{,}5 \cdot 10^{21}\,s^{-1};$$

$$\boxed{\frac{N}{\Delta t} = \frac{F\lambda}{h} = 7{,}5 \cdot 10^{21}\,s^{-1};}$$

**Problem 23** A point source of light moves at constant velocity directly towards (or away from) an observer O. In it's rest frame it radiates isotropically with total luminosity (power) L. Prove that, if it was at distance r when it emitted the light whereby it is now seen, the energy flux due to it at O is $\frac{L}{4\pi r^2}\mathcal{D}^4$, where $\mathcal{D} = \frac{(\text{frequency of reception})}{(\text{frequency of emition})}$ for a typical spectral line.

It is instructive to work the problem in two ways: once in the rest frame of the source, and once in the rest frame of the observer.

We will consider only source moving away from an observer O. First of all let's find out how luminosity changes when we go from one frame to another. Luminosity is proportional to the energy emmited per second by the light source. Let's consider this process from corpuscular point of

⑥ view. Let's say that source emits 1 photon of frequency $\nu'$ in $\Delta t'$ time in $O'$-frame moving with source. In $O$ we get due to Dopler effect $\nu = \nu'\sqrt{\frac{c-v}{c+v}}$ and due to time dilation and movement of source (remember here the way of Doppler effect derivation)

$\Delta t = \gamma(\Delta t' + \Delta t' \cdot \frac{v}{c}) = \Delta t' \cdot \gamma(1+\frac{v}{c}) = \Delta t' \cdot \sqrt{\frac{c+v}{c-v}}$. So when we go from one frame to another luminosity changes in the following way:

$\frac{L}{L'} = \frac{\nu}{\nu'}\frac{\Delta t'}{\Delta t} = \frac{c-v}{c+v} = \left(\frac{\nu}{\nu'}\right)^2$; So $\boxed{L = \frac{c-v}{c+v}L' ; \nu = \sqrt{\frac{c-v}{c+v}}\nu' ;}$

Now we go further and try to find change of flux while going from one frame to another. This can be done in two frames:

Ⅰ O'-frame In this frame it is quite easy to do. As in $O'$ radiation is isotropic we get in $O'$ flux equal to

$F = \frac{L}{4\pi x^2}$ where $L = \frac{c-v}{c+v}L'$ - luminosity that observer $O$ observes, and $x$ is distance to observer in $O'$-frame

$x = \gamma(r + v\Delta t)$ wher $\Delta t$ is time in $O$-frame essential for photon to reach observer $O$: $\Delta t = \frac{r}{c}$. So we get $x = r \cdot \gamma \cdot (1+\frac{v}{c}) = r \cdot \sqrt{\frac{c+v}{c-v}}$; Then we get

$F = \frac{L'}{4\pi r^2}\left(\frac{c-v}{c+v}\right)^2$; as $\nu = \nu' \cdot \sqrt{\frac{c-v}{c+v}}$ thus $D = \sqrt{\frac{c-v}{c+v}}$ and indeed $\boxed{F = \frac{L'}{4\pi r^2} D^4 \text{, q.e.d.}}$

Note: First of all notations are a bit messy. What we call $L'$ here is $L$ in notations of Rindler Book. Second - all reasonings remain the same for source of light moving towards observer (we should just make change $v \to -v$) and conclusions remain the same.

⑦ II Now let's find flux $F'$ considering problem in O frame
In this frame source doesn't emmit light isotropic due to "headlight" effect (see problem 13 from chapter III). So let's find the ratio $\frac{d\Omega}{d\Omega'}$ where $d\Omega$ is element of solid angle in O and $d\Omega'$ - solid angle in O'. In problem 13 (chapter III) we have derived relation between polar angles $\tan\frac{\theta}{2} = \sqrt{\frac{c-v}{c+v}}\tan\frac{\theta'}{2}$; If source is moving away from observer, this corresponds to angles close to $\pi$: $\theta' = \pi - \alpha'$ where $\alpha'$ is small. Then we get by taking derivative:

$$\frac{d\theta}{\cos^2\frac{\theta}{2}} = \sqrt{\frac{c-v}{c+v}}\frac{d\theta'}{\cos^2\frac{\theta'}{2}}$$

, now using $\cos^2\theta = \frac{1}{1+\tan^2\theta}$ we get:

$$d\theta = d\theta' \cdot \sqrt{\frac{c-v}{c+v}}\frac{1+\tan^2\frac{\theta'}{2}}{1+\tan^2\frac{\theta}{2}} = d\theta'\sqrt{\frac{c-v}{c+v}}\cdot\frac{1+\tan^2\frac{\theta'}{2}}{1+\frac{c-v}{c+v}\cdot\tan^2\frac{\theta'}{2}},$$

now as $\theta \to \pi$ we get: $\boxed{d\theta = d\theta'\cdot\sqrt{\frac{c+v}{c-v}};}$

Solid angle is given by $d\Omega = \sin\theta\, d\theta\, d\varphi$
$d\varphi$ doesn't change after Lorentz transformations, because it corresponds to rotations transversal to boost direction (boost is made along x-axis) so we need else to find relation between "$\sin\theta$" and "$\sin\theta'$"

$\tan\frac{\theta'}{2} = \tan\left(\frac{\pi}{2} - \frac{\alpha'}{2}\right) \approx \frac{2}{\alpha'}$; then we get

$\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{c-v}{c+v}\left(\frac{2}{\alpha'}\right)^2 = A \to \infty$ so that $A - A\cos\theta = 1+\cos\theta$

and $\cos\theta = \frac{1-A}{1+A}$ and $\sin^2\theta = \frac{4A}{(1+A)^2} \approx \frac{4}{A} = \frac{c+v}{c-v}\alpha'^2 = \frac{c+v}{c-v}\cdot\sin^2\alpha' =$

$= \frac{c+v}{c-v}\sin^2\theta'$ and finally $\sin\theta = \sqrt{\frac{c+v}{c-v}}\sin\theta'$. Thus

$\frac{d\Omega'}{d\Omega} = \frac{\sin\theta' d\theta'}{\sin\theta\, d\theta} = \frac{c-v}{c+v}$, and for flux we get

$\frac{F}{F'} = \frac{d\Omega'}{d\Omega} = \frac{\sin\theta' d\theta'}{\sin\theta\, d\theta} = \frac{c-v}{c+v} = D^2$ here $F'$ is flux without taking into account "headlight" effect

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i.e. $F' = \frac{L}{4\pi r^2}$ so we get $\boxed{F = \frac{L'}{4\pi r^2} D^4 \quad , \text{q.e.d.}}$

Problem 27 A particle moves rectilinearly under a rest-mass preserving force in some inertial frame. Show that the product of its rest mass and its instanteneous proper acceleration equals the magnitude of the relativistic 3-force acting on the particle in that frame. Show also that this is not necessarily true when the motion is not rectilinear.

First let's recall some theory from lectures

By definition 4-force is $F^\mu \equiv \frac{dp^\mu}{d\tau}$; $F^\mu = \gamma(v)\left(\frac{1}{c}\frac{dE}{dt}; \bar{f}\right)$, and $\bar{f} = \frac{d\bar{p}}{dt}$. Mass-preserving or pure force is one that doesn't change the rest energy. Let's show some properties of pure force $U_\mu \cdot F^\mu = \gamma^2\left(\frac{dE}{dt} - \bar{f}\cdot\bar{u}\right)$, from the other point of view $U_\mu F^\mu$ is Lorentz invariant and thus can be considered for example in particle rest frame where we get $U_\mu F^\mu = \gamma c^2 \frac{dm_0}{dt}$ thus for pure force $U_\mu F^\mu = 0$ and $\frac{dE}{dt} = \bar{f}\cdot\bar{u}$ and finally $F^\mu = m_0 A^\mu = \gamma(u)\left(\frac{1}{c}\bar{f}\cdot\bar{u}; \bar{f}\right)$

thus $A^\mu = \frac{\gamma(u)}{m_0}\left(\frac{1}{c}\bar{f}\cdot\bar{u}; \bar{f}\right)$, finally

$\alpha^2 = -A_\mu A^\mu = -\frac{\gamma^2(u)}{m_0^2}\left(\frac{1}{c^2}(\bar{f}\cdot\bar{u})^2 - |\bar{f}|^2\right)$ then we use

$\bar{f}\cdot\bar{u} = |\bar{f}|\,|\bar{u}|\cdot\cos\theta$ where $\theta$ is angle between $\bar{f}$ and $\bar{u}$ 3-vectors. then we finally get:

$\alpha^2 m_0^2 = \gamma^2(u)\cdot|\bar{f}|^2\left(1 - \frac{u^2}{c^2}\cdot\cos^2\theta\right)$. If motion is rectilinear we get $\bar{f} \parallel \bar{u}$, thus $\theta = 0, \cos\theta = 1$ and we get $\alpha^2 m_0^2 =$
$= \gamma^2(u)\left(1 - \frac{u^2}{c^2}\right)|\bar{f}|^2 = |\bar{f}|^2$ where we have used $\gamma^2(u) = \left(1 - \frac{u^2}{c^2}\right)^{-1}$

if motion is not rectilinear $\cos^2\theta < 1$ $\left(1 - \frac{u^2}{c^2}\cos^2\theta\right) > \left(1 - \frac{u^2}{c^2}\right)$ and thus $\alpha^2 m_0^2 > |\bar{f}|^2$; so we get

$\boxed{\text{rectilinear } \alpha^2 m_0^2 = |\bar{f}|^2 \;;\; \text{not rectilinear } \alpha^2 m_0^2 > |\bar{f}|^2}$ q.e.d.

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# Seminar 8 (electrodynamics I)

General note: We here use more convinient notations not the one in Rindler's book.

First of all we introduce antisymmetric tensor of electromagnetic field $F^{\mu\nu}$. It's defined by equation $\partial_\mu F^{\mu\nu} = \varkappa j^\nu$ where $j^\nu$ is current density $j^\nu = \rho_0 \gamma (c, \bar{v})$ where $\rho_0$ is charge density in the rest frame of charge flow, and $v$ is velocity of flow; Actually $\partial_\mu F^{\mu\nu} = \varkappa j^\nu$ can be shown to be just well known to us if $\varkappa = \frac{1}{c\varepsilon_0}$ and

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -cB_3 & cB_2 \\ E_2 & cB_3 & 0 & -cB_1 \\ E_3 & -cB_2 & cB_1 & 0 \end{bmatrix}$$

So we see that $E^i$ and $B^i$ are actualy components of tensor rather then vector. We else need Lorentz equations that can be written in 2 ways

* "relativistic" form $\boxed{\partial_\mu F^{\mu\nu} = \frac{1}{c\varepsilon_0} j^\nu ; \ \partial_\mu \tilde{F}^{\mu\nu} = 0;}$ where we have introduced dual tensor $\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$; where $\varepsilon^{\mu\nu\rho\sigma}$ is antisymmetric tensor.

* "usual" form

$$\boxed{\bar{\nabla} \bar{E} = \frac{1}{\varepsilon_0} \rho \ ; \quad -\frac{1}{c^2} \frac{\partial \bar{E}}{\partial t} + \bar{\nabla} \times \bar{B} = \mu_0 \bar{j}} \ ; \quad \varepsilon_0 \mu_0 = \frac{1}{c^2}$$

– this 2 equations are called "first pair of Maxwell equations" and correspond to equation $\partial_\mu F^{\mu\nu} = \frac{1}{c\varepsilon_0} j^\nu$

$$\boxed{\bar{\nabla} \cdot \bar{B} = 0 \ ; \quad \frac{\partial \bar{B}}{\partial t} + \bar{\nabla} \times \bar{E} = 0 \ ;}$$

second pair of Maxwell equations corresponding to equation $\partial_\mu \tilde{F}^{\mu\nu} = 0$;

Now from tensor properties of $F^{\mu\nu}$ we can derive how do electric and magnetic fields $\bar{E}$ and $\bar{B}$ transform under Lorentz transformation:

$$E_1' = E_1 \ ; \quad E_2' = \gamma (E_2 - v B_3); \quad E_3' = \gamma (E_3 + v B_2);$$

② $B_1' = B_1$ ; $B_2' = \gamma(B_2 + \frac{v}{c^2} E_3)$; $B_3' = \gamma(B_3 - \frac{v}{c^2} E_2)$;

- There are 2 Lorentz invariants:
  * $F_{\mu\nu} F^{\mu\nu}$ (electromagnetic field action);
  * $F_{\mu\nu} \tilde{F}^{\mu\nu}$ ;

The very last thing we will need is Lorentz force, i.e. force acting on the charged particle placed in external electromagnetic field.

$$\boxed{\bar{f} = q(\bar{E} + \bar{v} \times \bar{B});}$$

Problem 1 (i) A particle of rest mass m and charge q is injected at velocity $\bar{u}$ into a constant pure magnetic field $\bar{B}$ at right angles to the field lines. Use the Lorentz force law to establish that the particle will trace out a circle of radius $\frac{mu\gamma(u)}{qB}$ with period $\frac{2\pi m\gamma(u)}{qB}$.

In the case of pure magnetic field Lorentz force acting on the charged particle is given by: $\bar{f} = q\,\bar{u}\times\bar{B}$ where $\bar{u}$ is the velocity of the particle. We can now write "relativistic Newton law"

$\bar{f} = \gamma m \bar{a} + \frac{\bar{u}}{c^2}(\bar{f}\cdot\bar{u})$ but as $\bar{f}\cdot\bar{u} = \bar{u}\cdot(\bar{u}\times\bar{B})\cdot q = 0$

we get simply: $\gamma(u) m \frac{d\bar{u}}{dt} = \bar{f} = q(\bar{u}\times\bar{B})$

As we are free to make any convinient choice of axes we can direct $\bar{B}$ in positive $\hat{z}$ direction:

$\bar{B} = B\cdot\hat{z}$ , so that: $\bar{u}\times\bar{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_1 & u_2 & u_3 \\ 0 & 0 & B \end{vmatrix} = \hat{x}\, u_2 B - \hat{y}\, u_1 B =$

$= \begin{bmatrix} Bu_2 \\ -Bu_1 \\ 0 \end{bmatrix}$ , so we can now write down equation of motion in components:

③
$$\begin{cases} m\gamma(u)\dfrac{du_1}{dt} = qBu_2 ; \\ m\gamma(u)\cdot\dfrac{du_2}{dt} = -qBu_1 ; \\ m\gamma(u)\dfrac{du_3}{dt} = 0 ; \end{cases}$$

We know that magnetic field doesn't do any work so energy $\mathcal{E} = mc^2\gamma(u)$ is preserved. We can show it directly by considering time derivative of energy:

$\dfrac{d\mathcal{E}}{dt} = mc^2 \dfrac{\partial\gamma}{\partial u}\dfrac{du}{dt}$, at the same time

$\dfrac{d(u^2)}{dt} = 2\bar{u}\dfrac{d\bar{u}}{dt} = 2\bar{u}\cdot\dfrac{q}{m\gamma(u)}(\bar{u}\times\bar{B}) = \dfrac{2q}{m\gamma(u)}\bar{B}(\bar{u}\times\bar{u}) = 0$. So we get $\dfrac{d\mathcal{E}}{dt} = 0$ indeed and $u = const$. Taking one more derivative with respect to "$t$" we get:

$\ddot{u}_x = \dfrac{qB}{m\gamma}\cdot\dot{u}_y = -\left(\dfrac{qB}{m\gamma}\right)^2 u_x = -\omega^2 u_x$ where we have introduced

$\boxed{\omega = \dfrac{qB}{m\gamma}}$; In the same way $\ddot{u}_y = -\dfrac{qB}{m\gamma}\dot{u}_x = -\left(\dfrac{qB}{m\gamma}\right)^2 u_y = -\omega^2 u_y$

So we get
$$\ddot{u}_x = -\omega^2 u_x \Rightarrow u_x = A_1 e^{i\omega t} + B_1 e^{-i\omega t} = A_1' \cos\omega t + B_1' \sin\omega t;$$
$$\ddot{u}_y = -\omega^2 u_y \Rightarrow u_y = A_2 e^{i\omega t} + B_2 e^{-i\omega t} = A_2' \cos\omega t + B_2' \sin\omega t;$$
$$\dot{u}_z = 0 \Rightarrow u_z = C = const$$

Here $A_1, A_2, B_1, B_2$ and $C$ are integration constants to be determined by initial conditions.

In part (i) particle is injected at right angle to the field lines of magnetic field $\bar{B}$, i.e. $u_x(0) = u$; $u_y(0) = 0$; $u_z(0) = 0$

as $u_z(0) = 0$ we immideately get $u_z(t) = 0$

as $u_y(0) = 0 \Rightarrow u_y(t) = B_2' \sin\omega t$ (i.e. $A_2' = 0$)

as $u_x(0) = u$ and $u_x^2 + u_y^2 = u^2 = const$ we get $A_1' = u$

$$u^2\cos^2\omega t + B_1' u \cdot \sin 2\omega t + B_1'^2 \sin^2\omega t + B_2' \sin^2\omega t = u^2$$

$u^2 + B_1' u \cdot \sin 2\omega t + (B_1'^2 + B_2'^2 - u^2)\sin^2\omega t = u^2$ the easiest way to satisfy this equation is to take $B_1' = 0$ and $B_2' = u$ so that $\boxed{u_x(t) = u\cos\omega t;\ u_y(t) = u\sin\omega t;\ u_z(t) = 0;}$ integrating once more

$$\boxed{x(t) = \frac{u}{\omega}\sin\omega t + x_0;\ y(t) = -\frac{u}{\omega}\cos\omega t + y_0;\ z(t) = z_0;}$$

(4) This is circular motion with period $\boxed{T = \frac{2\pi}{\omega} = \frac{2\pi m \gamma(u)}{qB};}$

and the radius of the circle is $\boxed{R = \frac{u}{\omega} = \frac{u m \gamma(u)}{qB};}$ q.e.d.

(ii) If the particle is injected into the field with the same velocity but at an angle $\theta \neq \frac{\pi}{2}$ to the field lines, prove that the path is a helix, of smaller radius, but that the period for one complete cycle is the same as before.

In this case initial conditions take form $u_x(0) = u_r$; $u_y(0) = 0$; $u_z(0) = u_z$ and $u_r^2 + u_z^2 = u^2$. The difference in solution is that now velocity in the plane transverse to magnetic field lines is $u_r < u$ and velocity along z-axis is constant

$$\boxed{\begin{gathered} u_x(t) = u_r \cos\omega t;\ u_y(t) = u_r \sin\omega t;\ u_z(t) = u_z; \\ x(t) = \frac{u_r}{\omega}\sin\omega t + x_0;\ y(t) = -\frac{u_r}{\omega}\cos\omega t + y_0;\ z(t) = u_z t; \end{gathered}}$$

and integration gives this is indeed helix motion with period $T = \frac{2\pi}{\omega}$ but now radius of helix is now $\boxed{r_H = \frac{u_r}{\omega} < r}$ as $u_r < u$;

Problem 2 (i) A particle of rest mass $m$ and charge $q$ is released from rest in a frame $S'$ in which there are constant and orthogonal $\bar{E}$ and $\bar{B}$ fields say $\bar{E} = (0; E_0; 0)$ and $\bar{B} = (0; 0; B_0)$ such that $0 < E_0 < cB_0$. How much time elapses before the particle is momentarely at rest again

First of all remember that we have 2 Lorentz invariants for electromagnetic field:

$\frac{1}{2} F_{\mu\nu} F^{\mu\nu} = c^2 B^2 - E^2 > 0$ and $\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = c(\bar{B} \cdot \bar{E})$. In particular as $E_0 < cB_0$ we can alway go to the frame where we obtain only magnetic field. Using Lorentz transformation formulas we can find the velocity of this frame:

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$E_1' = E_1 = 0$; $E_2' = \gamma(E_2 - B_3 V)$; $E_3' = \gamma(E_3 + VB_2) = 0$

if we want $\bar{E}' = 0$: $V = \frac{E_2}{B_3} = \frac{E_0}{B_0}$, i.e. $\boxed{V = \frac{E_0}{B_0}}$;

Magnetic field in this frame is

$B_1' = B_1 = 0$; $B_2' = \gamma(B_2 + \frac{V}{c^2}E_3) = 0$; $B_3' = \gamma(B_3 - \frac{V}{c^2}E_2) = \gamma(B_0 - \frac{E_0^2}{c^2 B_0})$

$\gamma = \left(1 - \frac{E_0^2}{c^2 B_0^2}\right)^{-\frac{1}{2}} \Rightarrow B_3' = B_0\left(1 - \frac{E_0^2}{c^2 B_0^2}\right)^{\frac{1}{2}} = \frac{B_0}{\gamma}$

So in $S'$ frame we have particle that in initial moment has velocity $u' = (-V; 0; 0)$. As we have seen this corresponds to circular movement in $S'$-frame, which is periodic and period is given by $T' = \frac{2\pi m \gamma(V)}{qB'} = \frac{2\pi m \gamma^2}{qB}$, due to time dilation $T = \gamma T'$ so we finally get $T = \frac{2\pi m \gamma^3}{qB_0} \Rightarrow \boxed{T = \frac{2\pi m c^3 B_0^2}{q(c^2 B_0^2 - E_0^2)^{3/2}}}$; q.e.d.

(ii) Show also that if $0 < cB_0 < E_0$, the particle ultimately moves with constant proper acceleration $\frac{q}{m}(E_0^2 - c^2 B_0^2)^{\frac{1}{2}}$ in a direction making an angle $\cos^{-1}\left(\frac{cB_0}{E_0}\right)$ with the x-axis.

Now as $0 < cB_0 < E_0$ we can go to the frame such that $\bar{B}' = 0$. Using Lorentz transformation formulas $B_1' = B_1 = 0$; $B_2' = \gamma(B_2 + \frac{V}{c^2}E_3)$; $B_3' = \gamma(B_3 - \frac{V}{c^2}E_2)$; this is boos along x-axis. To make all components zero we should assume $B_3 = \frac{V}{c^2}E_2 \Rightarrow \boxed{V = c^2\frac{B_0}{E_0}}$;

Electric field in this frame is given by

$E_1' = E_1 = 0$; $E_2' = \gamma(E_2 - VB_3) = \gamma E_0\left(1 - \frac{c^2 B_0^2}{E_0^2}\right) = \frac{E_0}{\gamma}$; $E_3' = \gamma(E_3 + VB_2) = 0$;

So that $\bar{E}' = (0, \frac{E_0}{\gamma}, 0)^T$ And 3-force acting on charged particle is $\bar{f}' = qE'$; and equation of motion is $\dot{p}_x' = 0$ and $\dot{p}_y' = qE'$. Initial conditions in this frame

(6) (let's call it $S'$-frame) correspond to the particle moving in negative direction of x-axis. So solution to this equations is

$\boxed{p'_x = p'_0 \, ; \, p'_y = qE't \, ;}$ the energy of particle is given by $\mathcal{E}' = \sqrt{(p'c)^2 + (m'c^2)^2}$ where $m' = m\gamma(v)$ is the "relativistic mass" of particle in $S'$-frame

Substituting obtained solutions into this expression we get $\mathcal{E}' = \sqrt{\mathcal{E}_0'^2 + (qE'c\,t')^2}$ Where $\mathcal{E}_0'^2 = (p'_0 c)^2 + (m'c^2)^2$

If $t'$ is big enough (we need to describe ultimate movement) we get: $\mathcal{E}' = qE't\left(1 + \frac{\mathcal{E}_0'^2}{2(qE't)^2}\right)$; Another thing we need velocity of particle. As $\bar{p} = \gamma(v)m\bar{v}$; $\mathcal{E} = \gamma(v)mc^2$; we can conclude that $\bar{v} = \frac{c^2\bar{p}}{\mathcal{E}}$ In our case we get

$$u'_x = \frac{p'_0 c^2}{\mathcal{E}'} \, ; \, u'_y = \frac{qE't'c^2}{\mathcal{E}'} \, ; \, \mathcal{E}' = \sqrt{\mathcal{E}_0'^2 + (qE'ct)^2} \, ; \, \mathcal{E}_0'^2 = (p'_0 c)^2 + (m'c^2)^2$$

then $$\frac{1}{\gamma^2(u)} = 1 - \frac{u_x'^2 + u_y'^2}{c^2} = 1 - \frac{(p'_0 c)^2 + (qE't'c)^2}{\mathcal{E}'^2} = \frac{(m'c^2)^2}{\mathcal{E}'^2}$$

$\boxed{\gamma(u) = \frac{\mathcal{E}'}{m'c^2}}$ here $m' = \gamma(v)\,m$ - is mass of particle in $S'$-frame

4-force is given by:

$$F^{\mu'} = \gamma(u)\left(\frac{1}{c}\frac{d\mathcal{E}'}{dt'} \, ; \, \bar{f}'\right) \quad \text{and} \quad \frac{d\mathcal{E}'}{dt'} = \frac{(qE'c)^2 t'}{\mathcal{E}'}$$

now we can find proper acceleration using formula

$$\alpha^2 = -A_\mu A^\mu = -\frac{1}{m^2}F_{\mu'}F^{\mu'} = -\frac{\gamma^2(u)}{m^2}\left(\frac{1}{c^2}\left(\frac{d\mathcal{E}'}{dt'}\right)^2 - (\bar{f}')^2\right) =$$

$$= -\frac{\gamma^2(u)}{m^2}(qE')^2\left(\frac{(qE'c\,t')^2}{\mathcal{E}'^2} - 1\right) = \left(\frac{\mathcal{E}'}{m'c^2}\right)^2 \frac{(qE')^2}{m^2}\frac{\mathcal{E}_0'^2}{\mathcal{E}'^2}$$ now substituting

$E' = \frac{E_0}{\gamma(v)}$; $m' = \gamma(v)m$; $\mathcal{E}'_0 = m\gamma(v)c^2$, thus

$$\alpha^2 = \frac{(qE_0)^2}{m^2\gamma^2(v)} = \left(\frac{q}{m}\right)^2(E_0^2 - c^2B_0^2)$$ (we have used here

$\gamma(v) = \left(1 - \left(\frac{cB_0}{E_0}\right)^2\right)^{\frac{1}{2}}$, thus, finally: $\boxed{\alpha = \frac{q}{m}\sqrt{E_0^2 - c^2B_0^2}\text{; q.e.d.}}$

To determine direction of particles movement we should find velocity direction in $S$-frame

(7) We know that

$$u'_x = \frac{p'_0 c^2}{\mathcal{E}'} \; ; \; u'_y = \frac{qE_0 t' c^2}{\gamma \mathcal{E}'}$$

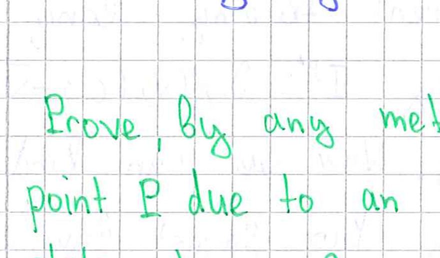
Using formulas of Lorentz transformation of velocities we get:

$$v_x = \frac{v'_x + V}{1 + \frac{u'_x V}{c^2}} \; ; \; v_y = \frac{v'_y}{\gamma(V)\left(1 + \frac{u'_x V}{c^2}\right)} \; ;$$

thus direction of motion makes $\varphi$ angle with x-axis such that: $\tan\varphi = \frac{v_y}{v_x} = \frac{v'_y}{\gamma(V)(v'_x + V)}$ When time is large enough we can take limit $v'_x \to 0$; $v'_y \to c$ and thus $\tan\varphi = \frac{c}{V\gamma(V)}$ and $\cos^2\varphi = \frac{1}{1+\tan^2\varphi} \Rightarrow$

$\Rightarrow 1 + \tan^2\varphi = 1 + \frac{c^2}{V^2}\left(1 - \frac{V^2}{c^2}\right) = \frac{c^2}{V^2}$ and $\cos\varphi = \frac{V}{c} = \frac{cB_0}{E_0}$ so we finally get $\boxed{\varphi = \arccos\frac{cB_0}{E_0}}$, q.e.d.

Problem 4 Prove, by any method, than the electric field $\bar{E}$ at a point P due to an infinite straight-line distribution of static charge, $\lambda$ per unit length, is given by $\bar{E} = \frac{\lambda \bar{r}}{2\pi\varepsilon_0 r^2}$; where $\bar{r}$ is the perpendicular vector-distance of P from the line. Deduce, by transforming to a frame in which this lines moves, that the magnetic field $\bar{B}$ at P due to an infinitely long straight current $\bar{I}$ is given by $\bar{B} = \frac{\bar{I} \times \bar{r}}{2\pi\varepsilon_0 c^2 r^2}$;




To find electric field we should use For this purpose we consider flux through cylinder of radius r, surrounding wire. Due to the symmetry of problem electric field sholdn't have any components along wire. As $\bar{\nabla}\cdot\bar{E} = \rho$ integration over the volume of cylinder $\int dV \bar{\nabla}\cdot\bar{E} = \oint_{\partial V} \bar{E}\cdot d\bar{S} = \int dV \rho$ so that:

(7) $2\pi r l E_r = \lambda l$ where $l$ is length of cylinder and $E_r$ is electric field strength pointing out of cylinder as shown on picture. So we get $E_r = \dfrac{\lambda}{2\pi r \varepsilon_0}$ or

$$\boxed{\bar{E} = \frac{\lambda \bar{r}}{2\pi r^2 \varepsilon_0}, \text{ q.e.d.}}$$

Now let's go to the $S'$ frame moving with velocity $v$ along x-axis. In this frame we have magnetic field that can be found using Lorentz transformation.

$$B_1' = B_1 = 0; \quad B_2' = \gamma(v)\left(B_2 + \frac{v}{c^2}E_3\right) = \frac{v\gamma(v)}{c^2}\frac{\lambda x_3}{2\pi\varepsilon_0 r^2};$$

$$B_3' = \gamma(v)\left(B_3 - \frac{v}{c^2}E_2\right) = -\frac{v\gamma(v)}{c^2}\frac{\lambda x_2}{2\pi\varepsilon_0 r^2};$$

Now in $S'$-frame we get instead of static charge disturbed along the wire, current flowing along this wire. This current is given by $I^{\mu'} = \lambda\gamma(v)(c; -\bar{v})$ (see formula (6.5) in Joe's lectures). Now we can find magnetic field in $S'$-frame, using Bio-Savart law

$$\bar{B} = \frac{\bar{I} \times \bar{r}}{2\pi\varepsilon_0 c^2 r^2} \quad \text{as} \quad \bar{I}' = \lambda\gamma(v)(-v; 0; 0). \text{ Thus:}$$

$$\bar{I}' \times \bar{r}' = \begin{Vmatrix} \hat{x}' & \hat{y}' & \hat{z}' \\ -\lambda\gamma v & 0 & 0 \\ x' & y' & z' \end{Vmatrix} = \lambda\gamma(v)\cdot v \begin{bmatrix} 0 \\ z' \\ -y' \end{bmatrix};$$

Thus $\boxed{\bar{B}' = \dfrac{\lambda\cdot\gamma(v)\cdot v}{2\pi\varepsilon_0 c^2 r'^2}\begin{bmatrix} 0 \\ x_3' \\ -x_2' \end{bmatrix}}$ note that $x_3$ and $x_2$ are directions transverse to the boost direction. so that $x_3' = x_3$ and $x_2' = x_2$ and our answer coinsides with the one observed previously by Lorentz transformation of electric field from $S$-frame to $S'$-frame. Let's now check Bio-Savart law by direct substitution into Maxwell equations.

First of all in Maxwell equations we have current density instead of current and they are related in the

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following way : $\bar{j} = \frac{\bar{I}}{\pi r^2}$ so that

$\bar{B} = \frac{\bar{j} \times \bar{r}}{2\varepsilon_0 c^2}$ now $\bar{\nabla} \times \bar{B} = \frac{1}{2\varepsilon_0 c^2} \bar{\nabla} \times \bar{j} \times \bar{r} = \frac{1}{2\varepsilon_0 c^2} \varepsilon^{ijk} \varepsilon^{klm} \partial_j \partial_l r_m =$

$= (\delta^{il}\delta^{jm} - \delta^{im}\delta^{lj}) \frac{1}{2\varepsilon_0 c^2} \partial_j (j_l r_m) = \frac{1}{2\varepsilon_0 c^2} (\bar{j}(\bar{\nabla} \cdot \bar{r}) - \bar{j}) = 2\bar{j} \frac{1}{2\varepsilon_0 c^2} =$

$= \frac{\bar{j}}{\varepsilon_0 c^2}$ thus we have obtained $\bar{\nabla} \times \bar{B} = \frac{\bar{j}}{\varepsilon_0 c^2}$ which is just Maxwell equation, q.e.d.

**Problem 8** Obtain the Lienard - Wiechert potentials $\left(\varphi = \frac{Q}{r(1+\frac{u_r}{c})}; \ \bar{A} = \frac{Q}{c^2}\left[\frac{\bar{u}}{r(1+\frac{u_r}{c})}\right], \ Q = \frac{q}{4\pi\varepsilon_0};\right)$ of an arbitrary moving charge $q$ by the following alternative method: Assume, first, that the charge moves uniformely and that in it's rest frame the potential is given by $\varphi = \frac{Q}{r}$; $\bar{A} = \bar{0}$. Then transform this to the general frame, using the 4-vector property of $A^\mu$

Note here we use more common notations of 4-potential $A^0 = \varphi$ ; $A^i = cw^i$ → this is correspondence between our notations and notations of Rindler book.

In rest frame of charged particle (let's call this frame $S'$) we get

$A^{0'} = \frac{Q}{r'}$ ; $\bar{A} = \bar{0}$ ; $A^\mu$ transforms as 4-vector under Lorentz transformation: $A^\mu = \Lambda^\mu{}_{\nu'} A^{\nu'}$ where

$\Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & \frac{v}{c}\gamma & 0 & 0 \\ \frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ - we assume particle to be moving along x-direction, and here we represent Boost back to $S$-frame from $S'$-frame. In our case Lorentz trasformation is easy as only nonzero component of $A^\mu$ in $S'$-frame is $A^{0'}$ so that:

9 $A^0 = \Lambda^0{}_{0'} A^{0'} = \frac{\gamma Q}{r'}$ ; $A^1 = \Lambda^1{}_{0'} A^{0'} = \frac{\gamma Q}{r'}\frac{v}{c}$ ; $A^3 = \Lambda^3{}_{0'} A^{0'} = 0$ ;
$A^2 = \Lambda^2{}_{0'} A^{0'} = 0$. So we have got

$A^\mu = \gamma \frac{Q}{r'} \left(1; \frac{v}{c}; 0; 0\right)^T$. Now we should relate $r$ and $r'$
Distance from charge to observer is given by
$r = ct$ and $r' = ct'$. Using Lorentz transformation for time it takes light to travel

$t' = \gamma\left(t - \frac{\bar{u}\cdot\bar{r}}{c^2}\right) = \gamma t\left(1 + \frac{u_r}{c}\right)$ here we have used general form of Lorentz transformation. and
$\bar{u}\cdot\bar{r} = -u_r \cdot r$ where $u_r$ is radial velocity away from the observer. So what we finally get:

$$\boxed{\phi = A^0 = \frac{Q}{r\left(1 + \frac{u_r}{c}\right)} \quad \text{and} \quad \bar{w} = \frac{\bar{A}}{c} = \frac{Q\bar{u}}{c^2 r\left(1 + \frac{u_r}{c}\right)}, \text{ q.e.d.}}$$

Note that here we have written more general $\bar{A} \sim \bar{u}$ rather then just one component of 4-potential.

Problem 9 Obtain field $\bar{E} = \frac{Q\bar{r}_0}{\gamma^2 r_0^3\left[1 - \frac{u^2}{c^2}\sin^2\theta\right]^{3/2}}$ ; and $\bar{B} = \frac{1}{c^2}(\bar{u}\times\bar{E})$;
of a uniformly moving charge $q$ by the following alternative method: assume that the field in the rest frame $S'$ of the charge is given by:
$\bar{E}' = \frac{Q}{r'^3}(x', y', z')$ ; $\bar{B}' = 0$, $r'^2 = x'^2 + y'^2 + z'^2$ ;
then transform this field to the usual second frame $S$ at $t = 0$.

So in the rest frame of particle we have

$\bar{E}' = \frac{Q}{r'^3}(x', y', z')$ ; $\bar{B}' = 0$

Now without loss of generality let's assume that particle is moving along x-axis in observers rest frame $S$. So we can find electromagnetic field in $S$-frame simply doing boost in negative x direction

(10) $E_1 = E_1' = \frac{Q}{r'^3}x'$ ; $E_2 = (E_2' + vB_3')\gamma = \frac{Q}{r'^3} y' \gamma(v)$ ;

$E_3 = \gamma(E_3' - vB_2') = \frac{Q}{r'^3} z' \gamma(v)$ ;

$B_1 = B_1' = 0$ ; $B_2 = \gamma(v)\left(B_2' - \frac{v}{c^2}E_3'\right) = -\frac{v\gamma(v)}{c^2} E_3'$ ;

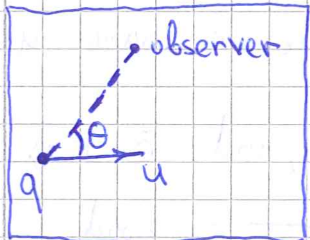
$B_3 = \gamma(v)\left(B_3' + \frac{v}{c^2}E_2'\right) = \frac{v\gamma(v)}{c^2} E_2'$

also we should transform coordinates in the following way:

$x' = \gamma x$ ; $y' = y$ ; $z' = z$ ; So that $r'^2 = x'^2 + y'^2 + z'^2 = \gamma^2 x^2 + y^2 + z^2 =$

$= \gamma^2(x^2+y^2+z^2) - (\gamma^2 - 1)(y^2+z^2) = \gamma^2 r^2\left(1 - \frac{u^2}{c^2}\frac{y^2+z^2}{r^2}\right) =$

$= \gamma^2 r^2\left[1 - \frac{u^2}{c^2}\sin^2\theta\right]$ , $\theta$ here is angle between $\bar{r}$ and $\bar{u}$, which in our case is directed along x-axis.




So we now conclude the following

$$E_1 = \frac{Q\gamma x}{\gamma^3 r^3\left[1 - \frac{u^2}{c^2}\sin^2\theta\right]^{3/2}} ; \quad E_2 = \frac{Qy}{\gamma^2 r^3\left[1 - \frac{u^2}{c^2}\sin^2\theta\right]^{3/2}} ;$$

$$E_3 = \frac{Qz}{\gamma^2 r^3\left[1 - \frac{u^2}{c^2}\sin^2\theta\right]^{3/2}} ;$$ or written in

vector form: $$\boxed{\bar{E} = \frac{Q\bar{r}}{\gamma^2 r^3\left[1 - \frac{u^2}{c^2}\sin^2\theta\right]^{3/2}} ;}$$ q.e.d.

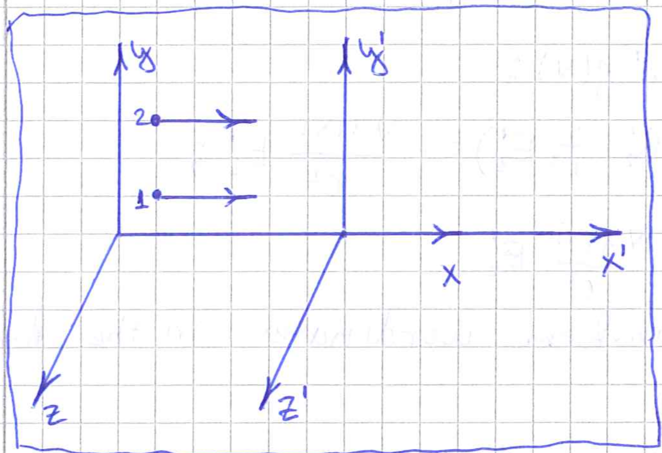
Now $\bar{B} = \frac{v\gamma(v)}{c^2}\begin{bmatrix} 0 \\ -E_3' \\ E_2' \end{bmatrix}$ ; $\bar{E} = \begin{bmatrix} E_1' \\ \gamma E_2' \\ \gamma E_3' \end{bmatrix}$ now we can

find that $\frac{1}{c^2}\bar{v}\times\bar{E} = \frac{1}{c^2}\begin{Vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v & 0 & 0 \\ E_1' & \gamma E_2' & \gamma E_3' \end{Vmatrix} = \frac{1}{c^2}\begin{bmatrix} 0 \\ -E_3' \\ E_2' \end{bmatrix} \cdot \frac{v\gamma(v)}{c^2} = \bar{B}$, so

we have directly shown that indeed $\boxed{\bar{B} = \frac{1}{c^2}\bar{v}\times E}$ , q.e.d.

Problem 10 In a fram S , two identical particles with electric charge q move abreast along lines parallel to the x-axis , a distance r apart and with velocity V . Determine the force , in S , that each

(11) exerts on the other.




There are two ways to determine force:

(i) Let's calculate electromagnetic fields fields created by charge 1 at the position of charge 2. For this we use formulas (41.5), (41.6), and (38.16) from [Rindler]. Force acting on charge is given by Lorenz force $\bar{F} = q(\bar{E} + \bar{u}\times\bar{B})$ and fields created by charge q are:

$$\bar{E} = \frac{Q\bar{r}}{\gamma^2 r^3\left[1-\frac{u^2}{c^2}\right]^{3/2}} \; ; \quad \bar{B} = \frac{1}{c^2}\bar{u}\times\bar{E}$$

here we have used $\sin\theta = 1$ as particles move abreast, and $\bar{r} = r\cdot\hat{y}$ and we get

$$\boxed{\begin{aligned}\bar{E} &= \frac{Q\gamma(u)}{r^2}\hat{y}; \\ \bar{B} &= \frac{Q\gamma(u)\cdot u}{r^2c^2}\hat{z};\end{aligned}} \qquad \bar{B} = \frac{1}{c^2}u\frac{Q\hat{x}\times\hat{y}}{\gamma^2r^2\left[1-\frac{u^2}{c^2}\right]^{3/2}}; \text{ and}$$

And now we are able to find force acting on second particle: $F = q(\bar{E} + \bar{v}\times\bar{B}) = q\left(\bar{E} - \frac{u^2}{c^2}\bar{E}\right) = q\bar{E}\frac{1}{\gamma^2} = \frac{qQ}{\gamma r^2}\hat{y}$

we have used here following formula for triple vector product $\bar{u}\times\bar{u}\times\bar{E} = \bar{u}\cdot\underbrace{(\bar{E}\cdot\bar{u})}_{0} - \bar{E}\cdot u^2$, so the force equals to

$$\boxed{\bar{F} = \frac{q^2}{4\pi\varepsilon_0 r^2\gamma(u)}\hat{y};}$$

(ii) Second way to derive same result is to consider problem in the rest frame of charges, i.e. frame S' moving with velocity v in positive $\hat{x}$-direction.

In S' particles are at rest and we need just Coulomb force $\bar{f}' = \frac{q^2\bar{r}'}{4\pi\varepsilon_0 r'^3} = \frac{q^2\hat{y}}{4\pi\varepsilon_0 r^2}$ we have taken here $r' = r$ as there are no changes in distancies in

(12) As for zero-component $F^{0'}$ it is vanishing $F^{0'}=0$ as Coulomb force is mass preserving.

So 4-force we get is $F^{\nu'} = (0; \bar{f}')$. It transforms as 4-vector under Lorentz transformations, so we get

$$f^1 = \frac{1}{\gamma} F^1 = \frac{1}{\gamma}\left(\Lambda^1{}_{0'} F^{0'} + \Lambda^1{}_{1'} F^{1'}\right) = \gamma^{-1}\gamma F^{1'} = f^{1'} = 0;$$

$$f^2 = \frac{1}{\gamma} F^2 = \frac{1}{\gamma} F^{2'} = \frac{q^2}{4\pi\varepsilon_0 \gamma r^2} \; ; \; f^3 = \frac{1}{\gamma} F^3 = 0; \text{ thus we get}$$

$$\boxed{\bar{f} = \frac{q^2}{4\pi\varepsilon_0 \gamma r^2}\hat{y}, \text{ q.e.d.}}$$

Problem 12 If $\bar{E}\cdot\bar{B} \neq 0$, prove there are infinitely many frames in which $\bar{E} \parallel \bar{B}$ precisely one of this moves in the direction $\bar{E}\times\bar{B}$, it's velocity being given by the smaller root of the quadratic $\beta^2 - R\beta + 1 = 0$, where $\beta = \frac{v}{c}$ and $R = \frac{E^2 + c^2B^2}{|\bar{E}\times c\bar{B}|}$. For the reality of $\beta$ $R>2$ inequality should be satisfied.

The frame in which $\bar{E}' \parallel \bar{B}'$ is one where $\bar{E}'\times c\bar{B}' = 0$.

Let's first assume that $\bar{E}'\times c\bar{B}' \neq 0$. We always can choose axes in such way that $\bar{E}'\times c\bar{B}' \parallel \hat{x}$, so that $(\bar{E}'\times c\bar{B}')^{2'} = (\bar{E}'\times c\bar{B}')^{3'} = 0$ and

$(\bar{E}'\times c\bar{B}')^1 = cE_2'B_3' - cE_3'B_2' = \{$using Lorentz transformation formulas$\} = \gamma^2\left((E_2 - vB_3)(cB_3 - \frac{v}{c}E_2) - (E_3 + vB_2)(cB_2 + \frac{v}{c}E_3)\right) =$

$= \gamma^2\left(cE_2B_3 - cE_3B_2 + \frac{v^2}{c^2}(cE_2B_3 - cB_2E_3) - \frac{v}{c}(|cB|^2 + E^2)\right) =$

$= \gamma^2\left(1 + \frac{v^2}{c^2}\right)(\bar{E}\times c\bar{B})^1 - \gamma^2\frac{v}{c}\left(|E|^2 + c^2|B|^2\right)$

Here we have made boost of velocity $v$ along x-axis direction. We want this expression to be zero. So that

$\left(1 + \frac{v^2}{c^2}\right)(\bar{E}\times c\bar{B})^1 - \frac{v}{c}\left(|E|^2 + c^2|B|^2\right) = 0$ we have chosen axes in such way that $(\bar{E}\times c\bar{B})^1 = |\bar{E}\times c\bar{B}|$ and we get:

$$\boxed{\beta^2 - R\beta + 1 = 0 \; ; \; R = \frac{|E|^2 + c^2|B|^2}{|\bar{E}\times c\bar{B}|}} ;$$ the solution of this

(13)

equation will give us the value of velocity $v$ of boost along $(\bar{E} \times c\bar{B})$ direction that will make $\bar{E} \parallel \bar{B}$. Starting from this frame we can do any boosts in $\bar{E}$ directon and $\bar{E}$ will stay parallel to $\bar{B}$ after all this boosts, q.e.d!!!

For the equation we have derived to have real solutions we want

$D = R^2 - 4 \geqslant 0$ so $R \geqslant 2 \Rightarrow |\bar{E}|^2 + c^2|\bar{B}|^2 \geqslant 2|\bar{E} \times c\bar{B}| = 2c|\bar{E}| \cdot |\bar{B}| |\sin\theta| <$

$< 2c|\bar{E}| \cdot |\bar{B}|$ So we get $|\bar{E}|^2 + c^2|\bar{B}|^2 \geqslant 2c|\bar{E}| \cdot |\bar{B}| \Rightarrow$

$\Rightarrow (|\bar{E}| - c|\bar{B}|)^2 \geqslant 0$ which is always satisfied q.e.d!!!

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# Seminar 9 (Combining analytical mechanics and special relativity)

Theory

First of all as usually in analytical mechanics we introduce action, which in case of relativistic massive particle is given by

$$S = -\int mc^2 d\tau = -\int mc\sqrt{dx_\mu \cdot dx^\mu} = -\int mc\sqrt{\frac{dx^\mu}{d\lambda}\cdot\frac{dx_\mu}{d\lambda}}\, d\lambda$$

The most used form of action is the last one

$\boxed{S = -\int mc\sqrt{\frac{dx^\mu}{d\lambda}\cdot\frac{dx_\mu}{d\lambda}}\, d\lambda;}$ This form of action possess reparamtrisation invariance, i.e.

we can go from one parametrisaton to another: $\lambda \to \lambda'(\lambda)$ and action remains the same. Lagrangian of relativistic particle is $L = -mc\sqrt{\frac{\partial x^\mu}{\partial\lambda}\frac{\partial x_\mu}{\partial\lambda}}$, equations of motion $\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0$ gives $-\frac{d}{d\lambda}\left(mc\frac{\dot{x}_\mu}{\sqrt{\dot{x}_\mu \cdot \dot{x}^\mu}}\right) = 0$ if we now take $\lambda = \tau$ we get $-\frac{d}{d\tau}(\dot{x}_\mu m) = 0$ or $\frac{dp^\mu}{d\tau} = 0$ which is just free relativistic particle equation of motion.

But all formulas above are aplicable only for massive particle. To include massless particles we should introduce more general action:

Problem 1

## Another formulation of relativistic action

(a) Show that the action

$$S = -\frac{1}{2}\int\left(e^{-1}\frac{\partial x^\mu}{\partial\lambda}\frac{\partial x_\mu}{\partial\lambda} + c^2 e m^2\right) d\lambda$$

is equivalent to the action above. Here, $e$ is a Lagrange multiplier.

To reduce this action to the previous one we should read off Lagrange multiplier $e$ using it's equations of motion: $\frac{\partial L}{\partial e} = 0$ which leads to

② $-\frac{1}{e^2}\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda} + m^2c^2 = 0 \Rightarrow \boxed{e = \frac{1}{mc}\sqrt{\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda}}\ ;}$

now substituting this value back to action we get:

$$S = -\frac{1}{2}\int\left(mc\sqrt{\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda}} + mc\sqrt{\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda}}\right)d\lambda \Rightarrow$$

$\Rightarrow S = -mc\int\sqrt{\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda}}\,d\lambda$; q.e.d. So, we have reduced initial action to the usual one. Why we need this new action at all is because it is usefull in the case of $m=0$, i.e. massless particle while initial action doesn't make any sence in $m=0$ limit.

ⓑ Find how $e$ must transform under the reparametrisation $\lambda \to \lambda'(\lambda)$ such that action is invariant under it.

If $\lambda \to \lambda'(\lambda)$; $\frac{\partial x^\mu}{\partial \lambda'} = \frac{\partial x^\mu}{\partial \lambda}\frac{d\lambda}{d\lambda'}$; $d\lambda' = \frac{d\lambda'}{d\lambda}d\lambda$; So under reparametrisation we get:

$$S = -\frac{1}{2}\int\left(\frac{1}{e}\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda} + e m^2c^2\right)d\lambda \to -\frac{1}{2}\int\left(\frac{1}{e'(\lambda')}\frac{\partial x^\mu}{\partial \lambda'}\frac{\partial x_\mu}{\partial \lambda'} + e'(\lambda')m^2c^2\right)d\lambda' =$$

$$= -\frac{1}{2}\int\left(\frac{1}{e'(\lambda')}\left(\frac{d\lambda}{d\lambda'}\right)^2\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda} + e'(\lambda')\,m^2c^2\right)\frac{d\lambda'}{d\lambda}\,d\lambda =$$

$$= -\frac{1}{2}\int\left(\frac{1}{e'(\lambda')}\frac{d\lambda}{d\lambda'}\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda} + e'(\lambda')\frac{d\lambda'}{d\lambda}m^2c^2\right)d\lambda =$$

$= -\frac{1}{2}\int\left(\frac{1}{e(\lambda)}\frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda} + e(\lambda)\,m^2c^2\right)d\lambda$ where in the last equation we supposed $e(\lambda) = e'(\lambda')\frac{d\lambda'}{d\lambda} \Rightarrow \boxed{e(\lambda)\,d\lambda = e(\lambda')\,d\lambda';}$

this is how Lagrange multiplier should be changed.

ⓒ Now let $m=0$ and "choose the gauge" $e=1$. Show that this gives the equations of motion for a massless particle.

If we take $m=0$ and $e=1$ we are left with the Lagrangian $L = \frac{\partial x^\mu}{\partial \lambda}\cdot\frac{\partial x_\mu}{\partial \lambda}$. Note here that we don't have reparametrisation invariance now as we fixed

③ parametrisation by choosing some particular "gauge" $e=1$. Equation of motion gives

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right)=0 \Rightarrow \boxed{\frac{d^2x^\mu}{d\lambda^2}=0;}$$ this is

indeed equation of motion for massless particle. (If we write down wave vector as $k^\mu = \frac{dx^\mu}{d\lambda}$ we get $\frac{d^2x^\mu}{d\lambda^2}=0$ as $dk^\mu=0$ - wave vector doesn't change along beam of light, or, in other words, particle trajectory)

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Theory

Now comes the question how can we couple charged particles to electromagnetic fields, so that we can describe their motion in external fields. There are usually two conditions that should be satisfied:

Ⓘ Lorentz invariance

Ⓘ gauge invariance.

Appropriate term is $S_{coup} = -\frac{1}{c}\int q A_\mu dx^\mu$; this term is obviously Lorentz invariant. As for gauge invariance, under gauge transformation we get:

$$\int q A_\mu dx^\mu \longmapsto \int q A_\mu dx^\mu + \int q\, \partial_\mu \Phi\, dx^\mu$$

this last term is total derivative and doesn't contribute to e.o.m. and physics.

So action for particle of mass m and charge q in external gauge field looks like:

$$\boxed{S = -\int mc^2 d\tau - \frac{1}{c}\int q A_\mu dx^\mu;}$$

④ The action and the magnetic flux.

Problem 3 Suppose we have a particle of charge $q$ in the presence of a magnetic field. Show that for motion around a loop, the contribution to the action from the magnetic field is $q\Phi$, where $\Phi$ is the magnetic flux through the loop.

Let's take only coupling term of action:

$S_c = -\frac{1}{c}\int q A_\mu dx^\mu$. As we have only magnetic field we can suppose $A_\mu = (0; -\bar{A})$, so that:

$S_c = \frac{1}{c}\int q \bar{A} d\bar{x}$. Trajectory is closed so that line integral goes to contour integral:

$$S_c = \frac{1}{c}\oint_{\partial S} q \bar{A} d\bar{x} = \frac{1}{c}\int_S q(\bar{\nabla}\times\bar{A}) d\bar{S} = q\int \bar{B} d\bar{S} = q\Phi$$

Here we have used Stokes formula when going from line to contour integral and definition of magnetic field:

$\bar{B} = \frac{1}{c}\bar{\nabla}\times\bar{A}$. So we have shown that:

$\boxed{S_c = q\Phi\text{, q.e.d.}}$

Problem 4 Consider the Pauli-Lubanski vector

$W_\mu \equiv \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma} p^\nu \Theta^{\lambda\sigma}$, where $\Theta^{\lambda\sigma} = -\Theta^{\sigma\lambda}$ - is generalized angular momentum. Assume $\Theta^{\lambda\sigma} = L^{\lambda\sigma} + S^{\lambda\sigma}$ where $L^{\lambda\sigma}$ is the orbital contribution:

$$L^{\lambda\sigma} = x^\lambda p^\sigma - x^\sigma p^\lambda;$$

and $S^{\lambda\sigma}$ is the contribution from an internal spin.

ⓐ Show that $p^\mu W_\mu = 0$

$p_\mu W^\mu = p^\mu W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma} p^\mu p^\nu \Theta^{\lambda\sigma}$, product of symmetric and antisymmetric tensors is always zero. Indeed

$\varepsilon_{\mu\nu\lambda\sigma} p^\mu p^\nu = \varepsilon_{\nu\mu\lambda\sigma} p^\nu p^\mu = -\varepsilon_{\mu\nu\lambda\sigma} p^\mu p^\nu = 0$ thus $\varepsilon_{\mu\nu\lambda\sigma} p^\mu p^\nu = 0$

here on the first step we have just renamed dummy indicies and on the second permuted them

⑤ So we have shown, that indeed $\boxed{p_\mu W^\mu = 0}$, q.e.d.

ⓑ Show that contribution of the orbital part to $W_\mu$ is zero.

Let's consider orbital momentum contribution

$$W_\mu^L = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma} p^\nu L^{\lambda\sigma} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma} p^\nu x^\lambda p^\sigma - \frac{1}{2}\varepsilon_{\mu\nu\lambda\sigma} p^\nu x^\sigma p^\lambda$$

Considering 2 terms separately we get:

* $\varepsilon_{\mu\nu\lambda\sigma} p^\nu p^\sigma = \varepsilon_{\mu\sigma\lambda\nu} p^\nu p^\sigma = \{3 \text{ permutations } \sigma \leftrightarrow \lambda;\ \sigma \leftrightarrow \nu;\ \lambda \leftrightarrow \nu\} = -\varepsilon_{\mu\nu\lambda\sigma} p^\nu p^\sigma = 0$

* $\varepsilon_{\mu\nu\lambda\sigma} p^\nu p^\lambda = \varepsilon_{\mu\lambda\nu\sigma} p^\nu p^\lambda = -\varepsilon_{\mu\nu\lambda\sigma} p^\nu p^\lambda = 0$, so that:

$$\boxed{W_\mu^L = 0 \text{ , q.e.d.}}$$

ⓒ In the particles rest frame, $S_{0i} = 0$; $S_{ij} = \varepsilon_{ijk} S_k$;
Assuming that $\bar{S}\cdot\bar{S} = \hbar^2 s(s+1)$, find $W_\mu W^\mu$ in terms of the particle's rest mass $m$ and spin $s$.

In the particle rest frame $p^\mu = (mc; \bar{0})$, so only nonzero component is $p^0$

$W_\mu = \frac{1}{2}\varepsilon_{\mu 0\lambda\sigma} mc \cdot S^{\lambda\sigma}$, if 2 indicies of $\varepsilon_{\mu\nu\alpha\beta}$ coincide it turns to zero so $\mu\lambda\sigma$ are all spatial. Thus

$W_i = \frac{1}{2}\varepsilon_{0ijk}\varepsilon^{jkp} S_p mc = -\frac{1}{2}\varepsilon_{ijk}\varepsilon^{jkp} \cdot S_p mc$, using $\varepsilon_{jki}\varepsilon^{jkp} = -2\delta_{pi}$, so that $W_i = S_i mc$ and $W_0 = 0$

then $W_\mu W^\mu = (W^0)^2 - |\bar{W}|^2 = -m^2c^2\,\bar{S}\cdot\bar{S} = -(mc\hbar)^2 s(s+1)$

So we get $\boxed{W_\mu W^\mu = -(mc\hbar)^2 s(s+1)}$

ⓓ Show that $hP_\mu = W_\mu$.

For massless particle $W_\mu W^\mu = 0$ so that $W^0 = |\bar{W}|$ and the same for $P^\mu$: $P^0 = |\bar{P}|$ then if $P_\mu W^\mu = 0$, as we have shown in ⓐ then $P^0 W^0 - P^i W^i = 0 \Rightarrow$
$\Rightarrow |\bar{P}|\cdot|\bar{W}| = (\bar{P}\cdot\bar{W}) \Rightarrow \bar{P} \parallel \bar{W}$ and so they are collinear and $W^\mu = P^\mu \cdot h$, where $h$ is constant called helicity

(1) Seminar 10 (exam 2011)

Problem 3 Consider a particle of mass m and charge e moving in the presence of electric and magnetic fields $\bar{E}=-\bar{\nabla}\Phi$ and $\bar{B}=\bar{\nabla}\times\bar{A}$. The Lagrangian of this particle is

$L=\frac{1}{2}m\dot{\bar{r}}^2-e\Phi+\frac{e}{c}\dot{\bar{r}}\bar{A}$ where $\bar{r}$ is the position of the particle.

(a) Derive the Euler-Lagrange equations of motion

Euler-Lagrange equations are $\frac{d}{dt}\frac{\partial L}{\partial\dot{r}_i}-\frac{\partial L}{\partial r_i}=0$

in index notation $L=\frac{1}{2}m\dot{r}_i^2-e\Phi(r_i)+\frac{e}{c}\dot{r}_iA_i$;

$\frac{\partial L}{\partial\dot{r}_i}=m\dot{r}_i+\frac{e}{c}A_i$; $\frac{\partial L}{\partial r_i}=-e\frac{\partial\Phi}{\partial r_i}+\frac{e}{c}\dot{r}_j\partial_jA_i$ (here in all expressions we assume summation over repeating index)

So that $\frac{d}{dt}\frac{\partial L}{\partial\dot{r}_i}=m\ddot{r}_i+\frac{e}{c}\dot{A}_i+\frac{e}{c}\dot{r}_j\partial_jA_i+e\frac{\partial\Phi}{\partial r_i}-\frac{e}{c}\dot{r}_j\partial_iA_j=0$

So that equations of motion we finally get:

$m\ddot{r}_i=-e\left(\frac{1}{c}\dot{A}_i+\frac{\partial\Phi}{\partial r_i}\right)+\frac{e}{c}\dot{r}_j(\partial_iA_j-\partial_jA_i)$

Now we use definitions of electric and magnetic field. $E_i=-\frac{1}{c}\dot{A}_i-\frac{\partial\Phi}{\partial r_i}$; $B_i=\varepsilon_{ijk}\partial_jA_k$ so that

$\partial_iA_j=\frac{1}{2}\varepsilon_{ijk}B_k \Rightarrow \dot{r}_j(\partial_iA_j-\partial_jA_i)=\frac{1}{2}(\varepsilon_{ijk}\dot{r}_jB_k-\varepsilon_{ijk}\dot{r}_kB_j)=$

$=\varepsilon_{ijk}\dot{r}_jB_k=\bar{v}\times\bar{B}$. So, finally, we get:

$\boxed{m\ddot{\bar{r}}=e(\bar{E}+\frac{1}{c}(\bar{v}\times\bar{B}))}$ So the force acting on charged particle is $\boxed{\bar{F}=e(\bar{E}+\frac{1}{c}(\bar{v}\times\bar{B}))}$ which is just Lorentz force and that's the answer for part (e) of the problem

(b) Compute the momentum $\bar{p}$ conjugate to $\bar{r}$

By definition $p_i=\frac{\partial L}{\partial\dot{r}_i}=m\dot{r}_i+\frac{e}{c}A_i$; $\boxed{\bar{p}=m\dot{\bar{r}}+\frac{e}{c}\bar{A};}$ so that $\dot{\bar{r}}=\frac{1}{m}\bar{p}-\frac{e}{mc}\bar{A}$;

(c) Compute the Hamiltonian, H as a function of $\bar{p}$ and $\bar{r}$.

②
$$H=\bar{p}\cdot\dot{\bar{r}}-L=\frac{\bar{P}^2}{2m}-\frac{e}{mc}\bar{p}\bar{A}+e\Phi-\frac{e}{mc}\bar{p}\bar{A}+\frac{e^2}{mc^2}\bar{A}^2;$$

so that $\boxed{H=\frac{1}{2m}\left(\bar{p}-\frac{e}{c}\bar{A}\right)^2+e\Phi;}$

ⓓ Check that the Hamiltonian equations of motion gives the same force law as in (a);

Canonical equations are

$$\dot{\bar{p}}=-\frac{\partial H}{\partial q}\Rightarrow\quad \dot{p}_i=\left(p_j-\frac{e}{c}A_j\right)\partial_i A_j\frac{e}{mc}-e\partial_i\Phi$$

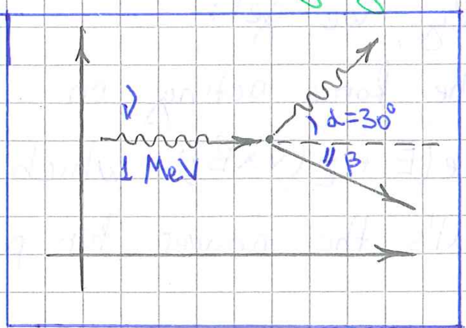
$$\dot{r}_i=\frac{\partial H}{\partial p^i}=\frac{1}{m}\left(p_i-\frac{e}{c}A_i\right)\quad\text{then}$$

$$m\ddot{r}_i=\dot{p}_i-\frac{e}{c}\dot{A}_i-\frac{e}{c}\dot{r}_j\partial_j A_i=\frac{e}{mc}\left(p_j-\frac{e}{c}A_j\right)\partial_i A_j-e\partial_i\Phi-\frac{e}{c}\dot{A}_i-$$
$$-\frac{e}{c}\dot{r}_j\partial_j A_i=\underbrace{\frac{e}{c}\dot{r}_j\left(\partial_i A_j-\partial_j A_i\right)}_{\frac{e}{c}(\bar{v}\times\bar{B})}-e\underbrace{\left(\frac{1}{c}\dot{A}_i+\partial_i\Phi\right)}_{-\bar{E}},$$

$\boxed{m\ddot{\bar{r}}=e\left\{\bar{E}+\frac{1}{c}(\bar{v}\times\bar{B})\right\};}$ which coinside with result obtained in part (a).

<u>Problem 4</u> A neutrino with energy 1 MeV scatters off an electron at rest ($m_e c^2=0{,}511$). The neutrino emerges from the interaction at a thirty degree angle away from its original trajectory. Assume that the neutrino has zero mass. You may also assume that the neutrino is not superluminal.

ⓐ Find the outgoing energy of the neutrino.




Let $P$ and $P'$ are momentum of neutrino before and after scattering $Q$ and $Q'$ momentum of electro before and after collision. As we take neutrino mass to be zero, this is just Compton scattering. Four-momentum conservation is:

$$P_\mu+Q_\mu=P'_\mu+Q'_\mu\Rightarrow Q'_\mu=P_\mu+Q_\mu-P'_\mu;\quad\text{taking square we get:}$$

$$Q'^2=P^2+Q^2+P'^2+2P\cdot Q-2P'Q-2P'P;\quad Q^2=Q'^2=m^2c^2;\quad P'^2=P^2=0;$$

So that we get $PQ-P'Q-P'P=0$;

③ Writing down all 4-momenta

$P = \frac{h\nu}{c}(1, \hat{x})$ ; $P' = \frac{h\nu'}{c}(1, \bar{n})$ ; $Q = (mc, 0)$ ; $Q' = (\frac{\mathcal{E}'}{c}, p')$ ;

where $\hat{x}$ is identity vector in positive x-direction, and $\bar{n}$ is vector along outgoing neutrino. So conservation equation becomes:

$$mh(\nu - \nu') - \frac{h^2}{c^2}\nu'\nu(1 - \bar{n}\hat{x}) = 0 \quad , \quad \bar{n}\hat{x} = \cos\alpha = \frac{\sqrt{3}}{2} \text{ so that}$$

$$\frac{mc^2}{h}\left(\frac{1}{\nu'} - \frac{1}{\nu}\right) = \frac{2-\sqrt{3}}{2} \Rightarrow \frac{1}{\mathcal{E}'} = \frac{1}{\mathcal{E}} + \frac{2-\sqrt{3}}{2mc^2}$$

, so finally neutrino energy is given by $\mathcal{E}' = \frac{2\mathcal{E}mc^2}{2mc^2 + (2-\sqrt{3})\mathcal{E}}$ if we substitute $\mathcal{E} = 1$ MeV and $mc^2 = 0{,}5$ MeV we get:

$\mathcal{E}' \approx 0{,}79$ MeV ;

$$\boxed{\mathcal{E}' = \frac{2\mathcal{E}mc^2}{2mc^2 + (2-\sqrt{3})\mathcal{E}} \approx 0{,}79 \text{ MeV}} \; ; \quad \mathcal{E}' = \frac{\mathcal{E}mc^2}{mc^2 + (1-\cos\alpha)\mathcal{E}} \; ;$$

ⓑ Find the outgoing angle of the electron.

To find angle of outgoing electron let's write down momentum conservation in X and Y axis

X-axis $\quad \frac{h\nu}{c} = \frac{h\nu'}{c}\cos\alpha + p'\cos\beta$

Y-axis $\quad \frac{h\nu'}{c}\sin\alpha = p'\sin\beta \; ; \Rightarrow p' = \frac{h\nu'}{c}\frac{\sin\alpha}{\sin\beta}$ ;

substitution to firs equation gives us:

$h\nu = h\nu'(\cos\alpha + \sin\alpha \cdot \mathrm{ctg}\,\beta)$ so we get

$$\mathrm{ctg}\,\beta = \frac{1}{\sin\alpha}\left(\frac{\mathcal{E}}{\mathcal{E}'} - \cos\alpha\right) ; \quad \frac{\mathcal{E}}{\mathcal{E}'} = 1 + \frac{\mathcal{E}}{mc^2}\left(1 - \frac{\sqrt{3}}{2}\right)$$

$$\mathrm{ctg}\,\beta = (2-\sqrt{3})\left(1 + \frac{\mathcal{E}}{mc^2}\right) = 6 - 3\sqrt{3}$$

$$\boxed{\beta = \mathrm{arccot}(6 - 3\sqrt{3})}$$

ⓒ finally energy of an electron:

We can find from energy conservation law.

$$\mathcal{E} - \mathcal{E}' = -mc^2 + \mathcal{E}_e \Rightarrow \mathcal{E}_e = mc^2 \quad \mathcal{E}\left(\frac{mc^2}{mc^2 + (1-\cos\alpha)\mathcal{E}} - 1\right)$$

$$\boxed{\mathcal{E}_e = \frac{(1-\cos\alpha)\mathcal{E}^2}{mc^2 + (1-\cos\alpha)\mathcal{E}} + mc^2 = 0{,}71 \text{ MeV}}$$

(4) Problems

Using Cartesian coordinates in the plane, the Lagrangian for the motion of the Foucault pendulum may be written

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} m\Omega^2 (x^2 + y^2) + \omega_z (x\dot{y} - y\dot{x}),$$

where $\Omega = \sqrt{\frac{g}{e}}$ is the usual pendulum frequency and $\omega_z$ represents the Coriolis force. In the $(x, y)$ coordinates, the motion is inseparable. Show that by going to polar coordinates in the plane, $(\rho, \varphi)$ the Hamiltonian becomes separable due to the cyclicity of $\varphi$. Solve the HJE, i.e. find Hamiltons principal function for this problem up to elementary integrals.

Let's go to polar coordinates $(\rho, \varphi)$

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases} \qquad \dot{x}^2 + \dot{y}^2 = \dot{\rho}^2 + \rho^2\dot{\varphi}^2 \ ; \ x^2 + y^2 = \rho^2 ;$$

$$x\dot{y} - y\dot{x} = \rho\dot{\rho}\cos\varphi \cdot \sin\varphi + \rho^2\dot{\varphi}\cos^2\varphi - \rho\dot{\rho}\sin\varphi \cdot \cos\varphi + \rho^2\dot{\varphi}\sin^2\varphi = r^2\dot{\varphi}$$

$$\begin{cases} \dot{x} = \dot{\rho}\cos\varphi - \rho\dot{\varphi}\sin\varphi \\ \dot{y} = \dot{\rho}\sin\varphi + \rho\dot{\varphi}\cos\varphi \end{cases}$$

So Lagrangian becomes

$$\boxed{L = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\varphi}^2 - \frac{1}{2}m\Omega^2\rho^2 + \omega_z\rho^2\dot{\varphi};}$$

*canonical momentum $p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}$ ; $p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi} + \omega_z\rho^2$

*inverting this expressions we get $\dot{\rho} = \frac{p_\rho}{m}$ ; $\dot{\varphi} = \frac{p_\varphi}{m\rho^2} - \frac{\omega_z}{m}$;

* making Legendre transformation we get:

$$H = p_\rho\dot{\rho} + p_\varphi\dot{\varphi} - L = \frac{p_\rho^2}{m} + \frac{p_\varphi^2}{m\rho^2} - \frac{p_\varphi\omega_z}{m} - \frac{p_\rho^2}{2m} - \frac{p_\varphi^2}{2m\rho^2} + \frac{p_\varphi\omega_z}{m} - \frac{\rho^2\omega_z^2}{2m} + \frac{1}{2}m\Omega^2\rho^2 - \frac{\omega_z p_\varphi}{m} + \frac{\omega_z^2\rho^2}{m};$$

$$\boxed{H = \frac{p_\rho^2}{2m} + \frac{1}{2m\rho^2}(p_\varphi^2 - 2\omega_z\rho^2 p_\varphi) + \frac{\rho^2\omega_z^2}{2m} + \frac{1}{2}m\Omega^2\rho^2}$$

Note that $\varphi$-coordinate is not presented in Hamiltonian so it is cyclic variable and we can put $p_\varphi = J = \text{const}$. and we get effective Lagrangian:

⑤ $$H=\frac{1}{2m}p_\rho^2+\frac{1}{2m\rho^2}J^2-\frac{\omega_z J}{m}+\frac{\rho^2\omega_z^2}{2m}+\frac{1}{2}m\Omega^2\rho^2;$$

Now we can introduce Hamilton's principal function $S$ and write down HJE

$$\frac{1}{2m}\left(\frac{\partial S}{\partial\rho}\right)^2+\frac{1}{2m\rho^2}J^2-\frac{\omega_z J}{m}+\frac{\rho^2\omega_z^2}{2m}+\frac{1}{2}m\Omega^2\rho^2+\frac{\partial S}{\partial t}=0$$

as Hamiltonian is time-independent we make ansatz: $S(\rho,t)=W(\rho)-\alpha_1 t$ and we get equation for characteristic equation

$$\frac{1}{2m}(W'_\rho)^2+\frac{1}{2m\rho^2}J^2-\frac{\omega_z J}{m}+\frac{\rho^2\omega_z^2}{2m}+\frac{1}{2}m\Omega^2\rho^2-\alpha_1=0$$

So that $$W'_\rho=\left[2m\alpha_1+2\omega_z J-m^2\Omega^2\rho^2-\omega_z^2\rho^2-\frac{J^2}{\rho^2}\right]^{\frac{1}{2}}$$

So we finally get Hamilton's principal function:

$$\boxed{S(\rho,t)=\int d\rho\left[2\alpha_1 m+2\omega_z J-m^2\Omega^2\rho^2-\omega_z^2\rho^2-\frac{J^2}{\rho^2}\right]^{\frac{1}{2}}-\alpha_1 t;}$$

Problem 6 The LHC is a collider between opposite beams of protons The protons in a beam are not actually equally spaced but come in "bunches". At present, the spacing between the bunches is about 50 ns ($5\cdot10^{-8}$ s) You may assume that each beam has 3,5 TeV protons and that the rest mass of the proton is 1 GeV.

ⓐ Find the distance between the bunches as measured by a physicist at the LHC

In lab frame distance is simply equal to the path passed by first bunch in 50 ns:

$$L_0=vt=ct\cdot\sqrt{1-\frac{1}{\gamma^2}}\approx ct=5\cdot10^{-8}\cdot3\cdot10^{8}=15\,m$$

Here we have used the fact that $\gamma$-factor equals to $\gamma=\frac{\mathcal{E}}{mc^2}=3{,}5\cdot10^3$ and velocity $v=c\sqrt{1-\frac{1}{\gamma^2}}\approx c$

So $\boxed{L_0=15\,m.}$

⑥ (b) Find the distance between the bunches as measured in the rest frame of the bunches.

* Let's consider coordinates of particles separated by $L_0$ distance. Now

$x_1 = ut_1$ $\qquad$ $x_2 = ut_2 + L_0$

let's go to some other frame moving with velocity $v$. In this frame velocity of bunches is given by $u' = \frac{u-v}{1-\frac{uv}{c^2}}$; Now times are conserved as

$$t_1' = \gamma\left(t_1 - \frac{v}{c^2}x_1\right) = \gamma t_1\left(1 - \frac{uv}{c^2}\right); \quad t_2' = \gamma t_2\left(1 - \frac{uv}{c^2}\right) - \gamma L_0 \frac{v}{c^2}$$

inverting this expressions we get

$$t_1 = \frac{t_1'}{\gamma\left(1-\frac{uv}{c^2}\right)}; \quad t_2 = \frac{t_2'}{\gamma\left(1-\frac{uv}{c^2}\right)} + \frac{v}{c^2}\frac{L_0}{1-\frac{uv}{c^2}};$$

now we can write down coordinates of particles

$$x_1' = \gamma(x_1 - vt_1) = \gamma(u-v)t_1 = \frac{u-v}{1-\frac{uv}{c^2}}t_1' = u't_1';$$

$$x_2' = \gamma(x_2 - vt_2) = \gamma(u-v)t_2 + \gamma L_0 = u't_2' + \gamma\frac{u'v}{c^2}L_0 + \gamma L_0$$

So we finally get

$$x_1' = u't_1'; \quad x_2' = u't_2' + \gamma L_0\left(1 + \frac{u'v}{c^2}\right) = u't_2' + L' \Rightarrow \boxed{L' = \gamma L_0\left(1 + \frac{u'v}{c^2}\right)}$$

Now if we substitute $u'=0$ (rest frame of bunches) we get $\boxed{L = \gamma L_0 \approx 52{,}5k}$. This is just formula given by Lorentz contraction, here proper length is $L$ and $L_0$ is length in lab frame.

(c) Find the distance between the bunches as measured in the rest frame of the bunches in the other beam.

Now we take frame moving with velocity "$-u$" using formula derived we get $L'' = \gamma L_0\left(1 + \frac{u'v}{c^2}\right)$ in this case $u' = \frac{2u}{1+\frac{u^2}{c^2}}$ so we get $L'' = \gamma L_0\left(1 - \frac{2u^2}{c^2}\frac{1}{1+\frac{u^2}{c^2}}\right) = \frac{L_0\left(1-\frac{u^2}{c^2}\right)}{\sqrt{1-\frac{u^2}{c^2}}\left(1+\frac{u^2}{c^2}\right)}$;

So

$$\boxed{L'' = \frac{L_0}{\gamma\left(1+\frac{u^2}{c^2}\right)} \approx \frac{L_0}{\gamma(u)} = 2\cdot 10^{-3}\,m = 2\,mm}$$

(7) The same result can be obtained from Lorentz contraction formula. Second bunch moves with velocity $u' = -\frac{2u}{1+\frac{u^2}{c^2}}$ So that

$$\frac{1}{\gamma(u')} = \sqrt{1 - \frac{4u^2}{c^2\left(1+\frac{u^2}{c^2}\right)^2}} \Rightarrow \gamma(u') = \frac{1+u^2/c^2}{1-u^2/c^2} = \gamma^2(u)\left(1+\frac{u^2}{c^2}\right)$$

Now we can use Lorentz contraction

$$L'' = \frac{L}{\gamma(u')} = \frac{\gamma(u)}{\gamma(u')} L_0 = \frac{L_0}{\gamma(u)\left(1+\frac{u^2}{c^2}\right)}$$ which coinsides with the formula obtained by previous methode.

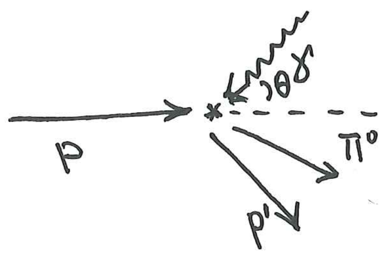
# Solution to the hand-in problem 2

We, in this problem, should find the threshold energy for the reaction:

$$\gamma + p \to p + \pi^0$$

with the following parameters given:

$\mathcal{E}_\gamma \approx 2{,}5 \cdot 10^{-10}\,\text{MeV}$; $m_p = 940\,\frac{\text{MeV}}{c^2}$; $m_\pi \approx 140\,\text{MeV}/c^2$;




- first of all threshold energy happens when the products of the reaction (proton + pion) are at rest in their common COM ref. frame:

$$\left.\begin{aligned} p^\mu &= \left(\frac{\mathcal{E}_p}{c}, \bar{p}\right); \\ q^\mu &= \frac{\mathcal{E}_\gamma}{c}(1, \bar{n}); \end{aligned}\right\} \text{lab. ref. frame} \qquad \left.\begin{aligned} p'^\mu &= (m_p c, 0); \\ \pi^\mu &= (m_\pi c, 0); \end{aligned}\right\} \text{p-}\pi \text{ COM ref. frame.}$$

- in the expressions above we can rewrite the 4-momentum of proton as $p^\mu = \frac{\mathcal{E}_p}{c}(1, \hat{x})$ because:
  1. We expect high energy of the proton so that $\mathcal{E}_p \gg m_p c^2$ and thus $\mathcal{E}_p \approx pc$;
  2. We direct the momentum of the proton along x-axis.

- We then write 4-mom. conservation:

$$p^\mu + q^\mu = p'^\nu + \pi^\mu$$

Squaring it we get

$$p^2 + q^2 + 2p^\mu \cdot q_\mu = p'^2 + \pi^2 + 2\pi^\mu \cdot p'_\mu$$

$$\Downarrow$$

$$\cancel{m_p^2 c^2} + 2\frac{1}{c^2} \cdot \mathcal{E}_p \cdot \mathcal{E}_\gamma (1 + \cos\theta) = \cancel{m_p^2 c^2} + m_\pi^2 \cdot c^2 + 2m_p m_\pi c^2$$

for the definition of "θ" see the picture.

then we obtain:

$$\mathcal{E}_p = \frac{m_\pi^2 c^4 + 2m_p m_\pi c^4}{2\mathcal{E}_\gamma(1 + \cos\theta)} \geqslant \frac{m_\pi^2 c^4 + 2m_p m_\pi c^4}{4\mathcal{E}_\gamma}$$

the minimal energy happens when $\cos\theta = 1 \Rightarrow \theta = 0$ (head on collision)

So we obtain:

$$\boxed{\mathcal{E}_p = \frac{m_\pi^2 c^4 + 2m_p m_\pi c^4}{4\mathcal{E}_\gamma} \approx 3 \cdot 10^{20}\,\text{eV};}$$