1 # Lecture 1: Electromagnetic field theory (short reminder)

- In terms of electric ($\bar{E}$) and magnetic ($\bar{B}$) fields electromagnetic theory is not Lorentz invariant.

For Lorentz invariant formulation one introduces:

ⓐ Four-potential $A^\mu = (\varphi, \bar{A})$ where φ is electric potential and $\bar{A}$ is vector-potential so that:

$$\bar{E} = -\bar{\nabla}\phi - \frac{\partial \bar{A}}{\partial t};$$

$$\bar{H} = \bar{\nabla} \times \bar{A};$$

Notice that here and further in this course we use $\hbar = c = 1$ units.

ⓑ Electromagnetic (or field strength) tensor $F_{\mu\nu}$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \text{ , then}$$

Notice: $F_{\mu\nu}$ is antisymmetric tensor $F_{\mu\nu} = -F_{\nu\mu}$;

$$\left.\begin{array}{l} F_{0i} = \partial_0 A_i - \partial_i A_0 = E_i; \\ F_{ij} = -\varepsilon_{ijk} H_k; \end{array}\right\} \Rightarrow \underset{\text{line} \nearrow \ \uparrow \text{column}}{F_{\mu\nu}} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -H_3 & H_2 \\ -E_2 & H_3 & 0 & -H_1 \\ -E_3 & -H_2 & H & 0 \end{pmatrix}$$

Notice: we use the metric with the signature $\eta_{\mu\nu} = \mathrm{diag}(+1 -1 -1 -1)$;

- The action of electromagnetic field is

$$S = -\frac{1}{4}\int d^4x \, F_{\mu\nu} F^{\mu\nu};$$

- Varying S one obtains:

$$\delta S = -\frac{1}{2}\int d^4x \cdot F_{\mu\nu}\,\delta F^{\mu\nu} = -\int d^4x \, F_{\mu\nu}\,\partial^\mu \delta A^\nu = \int d^4x \, \partial_\mu F^{\mu\nu} \delta A_\nu$$

As result we obtain:

$$\boxed{\partial_\mu F^{\mu\nu} = 0} \quad \text{eqivalent to} \begin{cases} \bar{\nabla}\cdot\bar{E} = 0 \\ \bar{\nabla}\times\bar{H} = \frac{\partial \bar{E}}{\partial t} \end{cases} \rightarrow \text{second pair of Maxwell equations.}$$

(2) • One more pair of Maxwell equations comes from the Bianchi identity

$$\partial_{[\mu} F_{\nu\alpha]} = \frac{1}{6}\left(\partial_\mu F_{\nu\alpha} - \partial_\mu F_{\alpha\nu} + \partial_\nu F_{\alpha\mu} - \partial_\nu F_{\mu\alpha} + \partial_\alpha F_{\mu\nu} - \partial_\alpha F_{\nu\mu}\right) =$$

$$= \frac{1}{3}\left(\partial_\mu F_{\nu\alpha} + \partial_\alpha F_{\mu\nu} + \partial_\nu F_{\alpha\mu}\right) = 0$$

Notice: here we use notation $[\mu_1 \mu_2 \dots \mu_n] = \frac{1}{n!} \sum_{\sigma} \overset{\text{sign in front of the corresponding term}}{(-1)^{P(\sigma)}} P(\mu_1, \mu_2, \dots \mu_n)$

This notation will be used in the future as well

σ ← pemutations

• Bianchi identity can be also rewritten in the following form

$$\boxed{\varepsilon^{\mu\nu\alpha\beta} \partial_\nu F_{\alpha\beta} = \partial_\nu \tilde{F}^{\mu\nu} = 0}, \text{ where } \varepsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & \text{if } \mu\nu\alpha\beta \text{ is even perm of } 0123 \\ -1 & \text{if } \mu\nu\alpha\beta \text{ is odd perm } - \text{"} - \end{cases}$$

and $\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$;

Bianchi identity is equivalent to $\begin{cases} \bar{\nabla} \cdot \bar{H} = 0; \\ \bar{\nabla} \times \bar{E} = -\frac{\partial \bar{H}}{\partial t}; \end{cases}$ first pair of Maxwell equations

• Gauge invariance

• Already when Maxwell formulated his equations it was noticed that the motion of the particles in external electromagnetic field is described by the interaction term in the action:

$$S_{int} = -e \int_{\gamma} A_\mu dx^\mu$$

γ → path of the particle

which leads to the equations of motion:

$$\frac{d\bar{p}}{dt} = \underbrace{e(\bar{E} + \bar{v} \times \bar{H})}_{\text{Lorentz force}}$$

• Lorentz force depends only on $\bar{E}$ and $\bar{H}$ but not on the four-potential A^μ, hence A^μ is not physical, while $\bar{E}$ and $\bar{H}$ are!

③ Now notice that $\bar{H} = \bar{\nabla} \times \bar{A}$ is invariant under $\bar{A} \to \bar{A}' = \bar{A} + \bar{\nabla} f$, where $f(\bar{x})$ is some scalar function.

Then $\bar{E} = -\bar{\nabla}\varphi - \frac{\partial \bar{A}}{\partial t} \to \bar{E}' = -\bar{\nabla}\varphi' - \frac{\partial \bar{A}}{\partial t} - \bar{\nabla}\frac{\partial f}{\partial t}$ → this is invarian if $\varphi' = \varphi - \frac{\partial f}{\partial t}$;

• Hence gauge symmetry transformations are

$$\boxed{\begin{matrix} \bar{A} \to \bar{A} + \bar{\nabla} f; \\ \varphi \to \varphi - \frac{\partial f}{\partial t}; \end{matrix} \Rightarrow A_\mu \to A_\mu - \partial_\mu f;}$$

• **Gauge symmetry means that physics stays invariant under gauge (i.e. coordinate-dependent) transformations!**

• Better way to test invariance of theory is to check the invariance of the action:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu - \cancel{\partial_\mu \partial_\nu f} - \partial_\nu A_\mu + \cancel{\partial_\nu \partial_\mu f} = F_{\mu\nu} \Rightarrow$$

$\Rightarrow$ $F_{\mu\nu}$ is gauge invariant!

Hence the action $S = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}$ is also gauge invariant!!!

• One interesting gauge invariant quantity is given by

$$\oint_C A_\mu dx^\mu$$ → integration here is along some contour C.

If the space-time is simply connected this quantity can be expressed through $\bar{E}$ and $\bar{H}$. In particular due to the Stokes' theorem for the space-like contour C:

$$\oint_C A_i dx^i = -\int_\Sigma (\bar{\nabla} \times \bar{A}) d\bar{s} = -\int_\Sigma \bar{H} \cdot d\bar{s}$$

④ • However if space-time is not simply connected, the even if $\bar{E}=\bar{H}=0$ in all space we can have $\oint_C \bar{A}\,d\bar{x} \neq 0$

This quantity can be obtained in quantum mechanical systems:

Aharonov-Bohm experiment.

• In external vector-potential particles w.f. acquires the phase:

$$e^{ie\int_{x_0}^{x} d\bar{y}\cdot\bar{A}(y)}$$

Indeed the Hamiltonian of the charged particle in external electromagnetic field is

$$\hat{H} = \frac{1}{2m}(\hat{\bar{p}} - e\hat{\bar{A}})^2 + \underset{\substack{\uparrow \\ \text{includes } \varphi(\bar{x})}}{V(\bar{x})}$$

In the absence of field we have $\hat{H}_0 = \frac{1}{2m}\hat{\bar{p}}^2 + V(\bar{x})$

Assume we found the solution for it in the form of $\Psi_0(x,t)$:

$i\frac{\partial}{\partial t}\Psi_0 = \hat{H}_0\Psi_0$, now if we turn on e.m. field the solution would become $\Psi = \exp(ie\int_{x_0}^{x} d\bar{y}\cdot\bar{A}(y))\,\Psi_0$

as now $$(\hat{\bar{p}} - e\hat{\bar{A}})\Psi = e\bar{A}(x)\Psi + \exp(ie\int^{x} d\bar{y}\,\bar{A}(\bar{y}))\,\bar{\nabla}\Psi_0(-i) - e\bar{A}\cdot\Psi = \exp(ie\int^{x} d\bar{y}\cdot\bar{A}(\bar{y}))(-i\bar{\nabla})\Psi_0$$

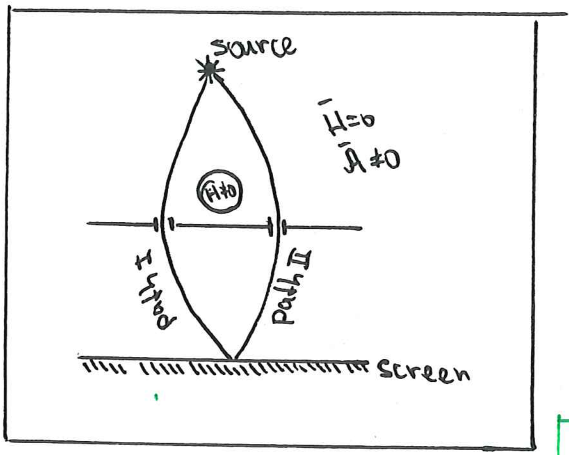
Hence Schrödinger equation is rewritten in the form:

$$\exp(ie\int^{x} d\bar{y}\cdot\bar{A})\left\{-\frac{1}{2m}\Delta + V(\bar{x})\right\}\Psi_0 = \exp(ie\int^{x} d\bar{y}\cdot\bar{A})\cdot i\frac{\partial}{\partial t}\Psi_0;$$

which is clearly satisfied.

Outline: Particles w.f. in magnetic field is
$$\Psi = \exp(i\int^{x} d\bar{y}\cdot\bar{A})\,\Psi_0(x,t);$$
where $\Psi_0(x,t)$ is the w.f. in the absence of field.

⑤ • This phase can be measured in the double-slit experiment



• There is solenoid between the slits. Magnetic field is nonzero only inside the solenoid.

• The difference of phases for the particles going along two pathes is given by

$$\boxed{\begin{aligned}\Delta\varphi &= \exp\Big(ie\Big(\int_{\text{path I}} - \int_{\text{path II}}\Big) d\bar{x}\cdot\bar{A}(\bar{x})\Big) = \\ &= \exp\big(ie\oint d\bar{x}\,\bar{A}(\bar{x})\big) = e^{ie\Phi}\end{aligned}}$$

where Φ is the flux of the magnetic field through the solenoid.

• Hence though electrons do not experience magnetic field directly (walls of the solenoid can be made unpenetrable for electrons) they acquire measurable phase!

• General solutions for the Maxwell equations.

To solve Maxwell equations let's perform Fourier transform:

$$A_\mu = \int_{k^0\geq 0} d^4k\,\big(a_\mu(k)e^{ik_\mu x^\mu} + \text{c.c.}\big),$$

here $a_\mu(k)$ is complex functon of 4-vector k^μ. c.c. is complex conjugate.

In this case

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \int_{k^0\geq 0} d^4k\cdot i\cdot\big(\{k_\mu a_\nu - k_\nu a_\mu\}e^{ik_\mu x^\mu} + \text{c.c.}\big)$$

$$\partial^\mu F_{\mu\nu} = \int d^4k\,(-1)\big(\{k^2 a_\nu - k^\mu a_\mu\cdot k_\nu\}e^{ik_\alpha x^\alpha} + \text{c.c.}\big)$$

and Maxwell equations reads:

$$\boxed{k^2 a_\nu - k^\mu k_\nu a_\mu = 0;}$$

⑥ • Here we need to consider separately two cases:

① $k^2 \neq 0$: in this case we see that a_μ is collinear with k^μ ⇨ $a_\mu = c(k) k_\mu$; Substituting this solution back to the Maxwell equations we see that $c(k)$ is not defined by any conditions. It is arbitrary function.

② $k^2 = 0$: in this case we obtain orthogonality condition:

$k^\mu a_\mu = 0$;

In four dimensions there are 3 4-vectors transverse to k^μ. One of them is k^μ itself (as $k^2 = k^\mu k_\mu = 0$). We choose two remaining vectors to be $e_\mu^{(\alpha)}$ ($\alpha = 1,2$) such that

- $e_0^{(\alpha)} = 0$ – pure space vectors (no time component)
- $e_i^{(\alpha)} \cdot k^i = 0$ – 3-vectors $\bar{e}^{(\alpha)}$ and $\bar{k}$ are orthogonal.
- $e_i^{(\alpha)} e^{i(\beta)} = \delta^{\alpha\beta}$ – 3-vectors $\bar{e}^{(\alpha)}$ $\bar{e}^{(\beta)}$ are orthogonal to each other.

So that the general solution for $k^2 = 0$ is

$a_\mu(k) = k_\mu \cdot c(\bar{k}) + e_\mu^{(\alpha)}(\bar{k})\, b_\alpha(\bar{k})$; where c, b_α are 3 arbitrary functions of $\bar{k}$;

• Uniting two cases we obtain:

Note: here and further we use notation $k \cdot x \doteq k_\mu x^\mu$;

$$A_\mu(x) = A_\mu^\perp(x) + A_\mu^{\parallel}(x);$$

$$A_\mu^\perp(x) = \int d^3k \left[e^{ik\cdot x} \cdot e_\mu^{(\alpha)}(\bar{k}) \cdot b_\alpha(\bar{k}) + c.c. \right]\Big|_{k^0 = |\bar{k}|};$$

$$A_\mu^{\parallel}(x) = \int d^4k \left[e^{ikx} \cdot k_\mu \cdot c(k) + c.c. \right];$$

Notice that $A_\mu^{\parallel}(x)$ is pure gauge as:

$$A_\mu^{\parallel}(x) = \partial_\mu \alpha(x), \text{ where } \alpha(x) = \int d^4x\, (-i) \left[c(k)\, e^{ik\cdot x} + c.c. \right];$$

• Non-trivial (i.e. not gauge) part of the solution is $A_\mu^\perp$ and it represents plane waves moving with the speed of light ($k^0 = |\bar{k}|$)

⑦ ## Gauge fixing.

Presence of gauge symmetry leads to the presence of disambiguity in the solution of Maxwell equations. This disambiguity is not physical and should be read off by implying some conditions on A_μ. This is cold gauge fixing.

Note: Important reason to fix the gauge is due to big problems it brings when one tries to quantize electromagnetic field.

Examples of gauge fixing.

① Coulomb gauge

$$\operatorname{div}\bar{A} \equiv \partial_i A^i = 0$$

Notice that this condition is not invariant under gauge transformations. However if A_μ satisfies $\partial_i A^i = 0$ then $A'_\mu = A_\mu + \partial_\mu \alpha$ satisfies the same condition only if:

$$\partial_i \partial^i \alpha = \Delta \alpha = 0;$$

This is the remaining gauge freedom.

This equation can be satisfied if α is constant or growing function. If the fields are decaying at infinity then gauge is fixed completely. Solution of Maxwell equations reads:

Coulomb gauge : $\bar{k}^2 c(k) = 0$, $A^{\parallel}_\mu(x) = 0$; $A_\mu = A^{\perp}_\mu(x)$; – solution of Maxwell equations.

② Lorentz gauge. $\partial_\mu A^\mu = 0$;

Remaining gauge symmetry $\partial_\mu \partial^\mu \alpha = 0$;

So the solution is up to the longitudal waves moving with the speed of light $\Rightarrow$ $c(k) = 0$ for $k^2 \neq 0$ but arbitrary for $k^2 = 0$;

⑧ Gauge $A_0=0$;

Remaining gauge: $\partial_0 \alpha=0$

General solution for Maxwell equation:

$$A_\mu(x)=A_\mu^{\perp}(x)+B_\mu(x)\text{ , where } \begin{array}{l} B_0=0 \\ B_i=\partial_i\alpha(x) \end{array}$$

hence $c(k)\neq 0$ only when $k^0=0$;

Problem I

Show that equations $\partial_\mu F^{\mu\nu}$ are equivalent to the pair of equations
$$\bar{\nabla}.\bar{E}=0$$
$$+\bar{\nabla}\times H=\frac{\partial\bar{E}}{\partial t}$$

while equations $\varepsilon^{\mu\nu\alpha\beta}\partial_\nu F_{\alpha\beta}=0$ corresponds to the pair of equations
$$\bar{\nabla}\cdot\bar{H}=0$$
$$\bar{\nabla}\times\bar{E}=-\frac{\partial\bar{H}}{\partial t};$$

Problem II

Find remaining gauge freedom and general solution of Maxwell equations for the axial gauge
$$\bar{n}\cdot\bar{A}=0$$
where $\bar{n}$ is some unity 3-vector.

① # Lecture 2: Scalar and vector fields. Noether's theorem.

- **Action of the scalar field**

Previous lecture - electromagnetic field, i.e. massless vector field. In nature we know following bosonic particles:
- photon γ: vector massless field.
- π^0, η mesons: real scalar fields.
- $\pi^\pm$ mesons: one complex scalar field.
- $W^\pm$ and Z bosons: massive vector fields.

Simpliest case is real scalar case.
We look for the action:
- Second order differential equations of motion $\Rightarrow$ action is quadractic in derivatives
- Lagrangian is Lorentz invariant
- Equations of motion are linear $\Rightarrow$ action is quadratic in fields.

$\Downarrow$

$$\boxed{S = \int d^4x \left(\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2\right);} \quad \text{where } (\partial_\mu\varphi)^2 \equiv \partial_\mu\varphi \cdot \partial^\mu\varphi;$$

- Equations of motion:

$$\delta S = \int d^4x \left[\cancel{2}\cdot\cancel{\tfrac{1}{2}}\partial^\mu\varphi\cdot\partial_\mu\delta\varphi - m^2\varphi\delta\varphi\right] = \int d^4x \left[-\partial_\mu\partial^\mu\varphi\cdot\delta\varphi - m^2\varphi\delta\varphi\right] +$$
$$+ \int_{\partial M} dS^\mu(\partial_\mu\varphi\cdot\delta\varphi);$$

Last term is over the boundary of space. We usually put these boundary terms to zero. Then the equations of motion are just Klein-Gordon equation (KG equation)

$$\boxed{\partial_\mu\partial^\mu\varphi + m^2\varphi = 0;}$$

- General solution of KG equation:

Fourier transforming: $\varphi(x) = \int_{k^0\geq 0} d^4k\,[\tilde\varphi(k)e^{ikx} + c.c.]$;

we get: $(k^2 - m^2)\tilde\varphi(k) = 0$;

Hence $\tilde\varphi(k)$ is arbitrary only if $k^2 = m^2$ ($k_0^2 = |\vec k|^2 + m^2$) and zero otherwise

② • Hence we have typical dispersion law of relativistic particle:
$k^0 = \sqrt{|\bar{k}|^2 + m^2}$ with m being the mass of field.

$$\varphi(x) = \int d^3k \left[\tilde{\varphi}(\bar{k}) e^{ik\cdot x} + c.c. \right] \Big|_{k^0 = \sqrt{k^2+m^2}} ; \; \varphi(\bar{k}) \text{ is arbitrary.}$$

• Energy of the scalar field:

Lagrangian is given by

$$L = \int d^3x \, \mathcal{L} = \int d^3x \left(\tfrac{1}{2}(\partial_\mu \varphi)^2 - \tfrac{m^2}{2}\varphi^2 \right) = \int d^3x \left(\tfrac{1}{2}\dot{\varphi}^2 - \tfrac{1}{2}(\bar{\nabla}\varphi)^2 - \tfrac{m^2}{2}\varphi^2 \right)$$

Notice that $\mathcal{L}$ is Lagrangian density rather then Lagrangian itself. However we sometimes refer it as the Lagrangian as well.

Then the energy of the scalar field can be derived using

$$E = \int \left(\frac{\delta L}{\delta \dot{\varphi}} \dot{\varphi} - L \right) d^3x \qquad \frac{\delta L}{\delta \dot{\varphi}(x)} = \dot{\varphi}(x); \text{ hence:}$$

$$\boxed{E = \int d^3x \left(\tfrac{1}{2}\dot{\varphi}^2 + \tfrac{1}{2}(\partial_i \varphi)^2 + \tfrac{1}{2} m^2 \varphi^2 \right);}$$

Comment on the choice of signs From the form of this expression for the energy of scalar field we can justify our choice of signs in the action.

• sign in front of the kinetic term $\partial_\mu \varphi \partial^\mu \varphi$ is chosen so that first two terms in the energy are positive definite and frequently oscillating fields have large positive energy.

• sign in front of the mass term is chosen so that the energy of large fields is positive and hence the energy is bounded from below.

③ Complex scalar field.

- $\varphi(x) = \mathrm{Re}\,\varphi(x) + i\,\mathrm{Im}\,\varphi(x)$ - another important example of the scalar field (it is very usefull for the description of charged scalar fields). Now on top of conditions we wanted to be satisfied before we also put reality condition. An appropriate action is then given by:

$$\boxed{S = \int d^4x \left(\partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi\right);}$$

- In order to find equations of motion corresponding to this action we should consider fields $\varphi(x)$ and $\varphi^*(x)$ as independent!

- varying w.r.t. $\varphi(x)$: $\delta S = \int d^4x \left(\partial_\mu \varphi^* \partial^\mu \delta\varphi - m^2 \varphi^* \delta\varphi\right) =$
$= -\int d^4x \left(\partial_\mu \partial^\mu \varphi^* + m^2 \varphi^*\right)\delta\varphi + \begin{bmatrix}\text{Bondary}\\ \text{terms}\end{bmatrix} \Rightarrow \boxed{\partial_\mu \partial^\mu \varphi^* + m^2 \varphi^* = 0;}$

- varying w.r.t. $\varphi^*(x)$: $\delta S = \int d^4x \left(\partial_\mu \delta\varphi^* \partial^\mu \varphi - m^2 \delta\varphi^* \cdot \varphi\right) =$
$= -\int d^4x \left(\partial_\mu \partial^\mu \varphi + m^2 \varphi\right)\delta\varphi^* + \begin{bmatrix}\text{Boundary}\\ \text{terms}\end{bmatrix} \Rightarrow \boxed{\partial_\mu \partial^\mu \varphi + m^2 \varphi = 0;}$

Hence we obtain two Klein-Gordon equations instead of one.

Comment: Instead of one complex scalar field we can introduce the pair of fields $\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$. Then the Leagrangian will become

$$\mathcal{L}_\varphi = \sum_a \left(\tfrac{1}{2}\partial_\mu \varphi^a \partial^\mu \varphi^a - \tfrac{1}{2} m^2 \varphi^a \cdot \varphi^a\right); \quad a = 1,2;$$

and equations of motion turn to be $\partial_\mu \partial^\mu \varphi^a + m^2 \varphi^a = 0;$
This description is completely equivalent to the one considered above.

- Theory of complex scalar field also has conserved current

$$j_\mu = -i\left(\varphi^* \partial_\mu \varphi - \varphi\, \partial_\mu \varphi^*\right).$$

Indeed it is easy to show that $\partial_\mu j^\mu = 0$ using e.o.m.:

$\partial_\mu j^\mu = -i\left(\partial_\mu \varphi^* \partial^\mu \varphi - \partial_\mu \varphi\, \partial^\mu \varphi^* + \varphi^* \partial_\mu \partial^\mu \varphi - \varphi\, \partial_\mu \partial^\mu \varphi^*\right) =$
$= -i\left(-m^2 \varphi^* \varphi + m^2 \varphi^* \varphi\right) = 0. \Rightarrow$ charged is conserved!

④ • Existence of conserved current ⇒ conserved charge

$Q \equiv \int d^3x\, j_0$, then $\partial_0 Q = \int d^3x\, \partial_0 j_0 = -\int d^3x\, \partial_i j^i = -\int d\Sigma_i\, j_i$

Boundary of the space

If we take bdry of the space at the spatial infinity and field decay there fast enough, then the last integral is zero and we obtain $\partial_0 Q = 0$ ⇒ conserved charge!

• Massive vector fields.

• Massive vector field should be described by the 4-vector $B_\mu(x)$ However just as in the case of e.m. field we can split vector field into really vector-like transverse component and longitudal gradient part:

$$B_\mu = B_\mu^\perp + \partial_\mu \chi; \text{ with } \partial_\mu B_\perp^\mu = 0;$$

• If the mass of field is nonzero then the expected dispersion relation is $k^0 = \sqrt{\vec{k}^2 + m^2}$; The easiest way to obtain this is to assume that every component of $B_\mu^\perp$ field satisfies KG equation:

$$(\partial_\mu \partial^\mu + m^2) B_\nu^\perp = 0 \rightarrow \text{goal!} + \partial_\mu B_\perp^\mu = 0; \quad (\mathrm{I})$$

An appropriate action fulfilling these conditions is:

$$\boxed{S = \int d^4x \left[-\tfrac{1}{4} B_{\mu\nu} B^{\mu\nu} + \tfrac{m^2}{2} B_\mu B^\mu\right];} \quad B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu;$$

Corresponding equations of motion have the following form

$$\partial_\mu B^{\mu\nu} + m^2 B^\nu = 0 \qquad (\mathrm{II})$$

this term is obtained similarly to Maxwell eq's

term from $\delta\left(\frac{m^2}{2} B_\nu B^\nu\right) = m^2 B_\nu \delta B^\nu$;

• Let's differentiate (II) w.r.t x^ν: $\partial_\mu \partial_\nu B^{\mu\nu} + m^2 \partial_\nu B^\nu = 0$

due to antisymmetry of $B^{\mu\nu}$ $\partial_\mu \partial_\nu B^{\mu\nu} = 0$ and hence $\partial_\nu B^\nu = 0$;

Now substituting this back into (II) we obtain:

⑤ $$\partial_\mu \partial^\mu B^\nu - \underbrace{\partial_\mu \partial^\nu B^\mu}_{\partial^\nu \partial_\mu B^\mu = 0} + m^2 B^\nu = 0 \Rightarrow \boxed{\begin{array}{c}\partial_\mu \partial^\mu B^\nu + m^2 B^\nu = 0; \\ + \\ \partial_\mu B^\mu = 0;\end{array}}$$ Just what we want

- Interaction with external sources.

In electrodynamics interaction of e.m. fields with charge currents is constructed using current 4-vector $j^\mu = (\underset{\substack{\uparrow \\ \text{charge density}}}{\rho}, \underset{\substack{\uparrow \\ \text{current density}}}{\vec{j}})$

Corresponding action is:

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu\right)$$

Variation of this gives

$$\delta S = \int d^4x \left(\partial_\mu F^{\mu\nu} - j^\nu\right) \delta A_\nu = 0 \Rightarrow \underline{\partial_\mu F^{\mu\nu} = j^\nu} \rightarrow$$ Maxwells equations with the source

$$\vec{\nabla} \cdot \vec{E} = \rho;$$
$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{E}}{\partial t} + \vec{j};$$

- Notice that Maxwell equation implies current conservation. Indeed:

$$\partial_\mu F^{\mu\nu} = j^\nu \Rightarrow \underbrace{\partial_\mu \partial_\nu F^{\mu\nu}}_{\substack{0 \text{ due to} \\ \text{antisymm. of } F^{\mu\nu}}} = \boxed{\partial_\nu j^\nu = 0}$$

- Conservation of the current leads in turn to the gauge invariance: $A'_\mu = A_\mu + \partial_\mu \alpha$

$$S[A'_\mu] = S[A_\mu] - \int d^4x\, j_\mu \partial^\mu \alpha = S[A_\mu] + \int d^4x\, \alpha \underbrace{\cancel{\partial_\mu j^\mu}}_{0} - \int d^4x\, \partial_\mu (\alpha j^\mu)$$

the last integral is integral over infinitely far surface (3d-surface)

$$\int d^4x\, \partial_\mu (j^\mu \alpha) = \int d\Sigma_\mu\, \alpha j^\mu = 0$$ if α decays at infinity fast enough.

Hence $S[A'_\mu] = S[A_\mu]$ and theory is gauge invariant.

- In analogy we can introduce source of the scalar field φ:

$$\boxed{S = \int d^4x \left[\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2\right] + \int d^4x\, \rho(x) \cdot \varphi(x);}$$

$\downarrow$

source is Lorentz scalar.

⑥ Interaction of fields. Scalar electrodynamics.

- Interaction terms in Leagrangians are terms with the powers of fields higher then two leading to the nonlinear terms in equations.
- In order for the action to be Lorentz invariant these interaction terms should also be Leorentz scalars.
- Simpliest example: interacting scalar field theory

$$S = \int d^4x \left[\tfrac{1}{2}(\partial_\mu\varphi)^2 - V(\varphi)\right]; \text{ where } V(\varphi) = \tfrac{1}{2}m^2\varphi^2 + V_{int}(\varphi);$$

with $V_{int}(\varphi)$ containing terms like $\varphi^3, \varphi^4, \ldots$ is polynomial of some finite degree.

Q: Do we still need $m^2>0$ condition or it should be modified somehow? How exactly?

Equations of motion obtained after varying $S[\varphi]$ are

$$\partial_\mu\partial^\mu\varphi + \frac{\partial V}{\partial\varphi} = 0;$$

- Now let's ask ourselvs how to construct the action for the scalar field interacting with e.m. field?

Hard way (straight forward).

- We described sources in electrodynamics, which leads to the term $\mathcal{L}_e = -j_\mu A^\mu$ in the Leagrangian.
- We have also seen that complex scalar field has conserved current $e j^{(0)}_\mu = -i(\varphi^*\partial_\mu\varphi - \partial_\mu\varphi\cdot\varphi^*)e$ ← charge of the scalar

Hence it is naturall to assume that in order to couple scalar and e.m. fields we just write down the action:

$$S = \int d^4x \left[\partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - e j^{(0)}_\mu A^\mu\right];$$

This leads to the following equations of motion

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$$\partial_\mu F^{\mu\nu} = e j^{(0)\nu} \quad (\text{III})$$

$$\partial_\mu \partial^\mu \varphi + m^2\varphi - 2ie\,\partial_\mu\varphi\, A^\mu - ie\,\varphi\,\partial_\mu A^\mu = 0 \quad (\text{IV})$$

$$\partial_\mu \partial^\mu \varphi^* + m^2\varphi^* + 2ie\,\partial_\mu\varphi^*\, A^\mu + ie\,\varphi^*\partial_\mu A^\mu = 0 \quad (\text{V})$$

Q: Derive these equations.

However using (IV) and (V) we find:

$$\partial_\mu j^{\mu(0)} = -i\,(\varphi^*\partial_\mu\partial^\mu\varphi - \varphi\,\partial_\mu\partial^\mu\varphi^*) = -i\cdot(2ie\,\varphi^*\,\partial_\mu\varphi\, A^\mu +$$

$$+ ie\,\varphi^*\varphi\,\partial_\mu A^\mu + 2ie\,\varphi\,\partial_\mu\varphi^*\, A^\mu + ie\,\varphi^*\varphi\,\partial_\mu A^\mu) =$$

$$= 2e\,\partial_\mu(\varphi^*\varphi A^\mu) \Rightarrow \boxed{\partial_\mu j^\mu_{(0)} = 2e\cdot\partial_\mu(\varphi^*\varphi A^\mu);} \rightarrow \text{the}$$

curren is not conserved any more! This contradicts (III)

- Now to fix this problem we modify the current as follows: $j_\mu = j^{(0)}_\mu + c\cdot\varphi^*\varphi A_\mu$ then (IV) and (V) become:

$$\partial_\mu\partial^\mu\varphi + m^2\varphi - 2ie\,\partial_\mu\varphi A^\mu - ie\,\varphi\,\partial_\mu A^\mu + ec\cdot\varphi A_\mu A^\mu = 0;$$

$$\partial_\mu\partial^\mu\varphi^* + m^2\varphi^* + 2ie\,\partial_\mu\varphi^* A^\mu + ie\,\varphi^*\partial_\mu A^\mu + e\cdot c\cdot\varphi^* A_\mu A^\mu = 0;$$

- Then still $\partial_\mu j^\mu_{(0)} = -i\,(\varphi^*\partial_\mu\partial^\mu\varphi - \varphi\,\partial_\mu\partial^\mu\varphi^*) = 2e\cdot\partial_\mu(\varphi^*\varphi A^\mu)$
- However now eq (III) turns into

$$\partial_\mu F^{\mu\nu} = e j^{(0)\nu} + 2e\cdot c\,\varphi^*\varphi A^\nu \Rightarrow 0 = \partial_\mu\partial_\nu F^{\mu\nu} = e\,\partial_\nu j^\nu_{(0)} +$$

$$+ 2e\cdot c\cdot\partial_\nu(\varphi^*\varphi A^\nu) = 0 \quad \text{if} \quad \underline{c = e}!$$

- Finally we can write the action:

$$\boxed{S = \int d^4x\,[\partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + ie(\varphi^*\partial_\mu\varphi - \varphi\,\partial_\mu\varphi^*)A^\mu - e^2\varphi^*\varphi A_\mu A^\mu];}$$

- We would also like our action to be gauge invariant

The action above is gauge inv. indeed. This can be checked by the direct calculation if we assume the following form of gauge transformations:

⑧

$$\boxed{\begin{aligned} &\varphi(x) \to \varphi'(x) = e^{i\alpha(x)}\varphi(x); \\ &\varphi^*(x) \to \varphi'^*(x) = e^{-i\alpha(x)}\varphi^*(x); \\ &A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x); \end{aligned}}$$

- Notice that transformations of φ, φ^* are symmetries of the free (no interaction) scalar field action only if $\alpha(x) = \alpha = \text{const}$ (global symmetry)
- This gives us recipie of introducing gauge invariant actions.

① Take the global symmetry for scalar fields (or fermions) and gauge it (make it x-dependent)

② Now notice that $\partial_\mu\varphi\,\partial^\mu\varphi^*$ term is not inv. under this local transformation anymore, because

$$\partial_\mu\varphi'(x) = e^{i\alpha(x)}[\partial_\mu\varphi(x) + i\partial_\mu\alpha(x)\cdot\varphi(x)];$$

To fix this introduce another derivative (covariant derivative) satisfying $(D_\mu\varphi)' = e^{i\alpha(x)}D_\mu\varphi;$

③ In order for this equation to work we need to compensate $i\partial_\mu\alpha\cdot\varphi$ term in derivative which can be done by $\boxed{D_\mu \equiv \partial_\mu - ieA_\mu}$

Then indeed $(D_\mu\varphi') = (\partial_\mu - ieA'_\mu)e^{i\alpha(x)}\varphi = ((\partial_\mu - ieA_\mu)\varphi)\cdot e^{i\alpha(x)} +$
$+ (i\partial_\mu\alpha\cdot\varphi - ie\cdot\frac{1}{e}\partial_\mu\alpha\cdot\varphi)e^{i\alpha(x)} = e^{i\alpha(x)}D_\mu\varphi \Rightarrow$ just as we want.

④ Then we directly observe the action with the gauge invariant interaction:

$$\boxed{S = \int d^4x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^* D_\mu\varphi - m^2\varphi^*\varphi\right];}$$

This is exactly the same action as the one derived before

9 Equations of Motion.

- Varying w.r.t A_μ:

$$\delta\left(\int d^4x \left(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)\right) = \int d^4x\, \partial_\mu F^{\mu\nu} \delta A_\nu \text{ as usually}$$

$$\delta\left(\int d^4x (D_\mu\varphi)^* D^\mu\varphi\right) = \delta \int d^4x\, (\partial_\mu + ieA_\mu)\varphi^* \cdot (\partial^\mu - ieA^\mu)\varphi =$$

$$= ie \int d^4x\, \delta A_\mu \left(\varphi^* D^\mu\varphi - \varphi D^\mu \varphi^*\right).$$

Hence we obtain the Maxwell equation

$$\boxed{\partial_\mu F^{\mu\nu} = j^\nu \quad \text{with} \quad j^\nu = -ie\left(\varphi^* D^\nu\varphi - \varphi D^\nu\varphi^*\right);}$$

- Varying w.r.t. φ^*:

$$\delta \int d^4x (\partial_\mu + ieA_\mu)\varphi^* (\partial^\mu - ieA^\mu)\varphi = -\int d^4x\, \delta\varphi^* (\partial^\mu - ieA^\mu)(\partial_\mu - ieA_\mu)\varphi =$$

$= -\int d^4x \cdot \delta\varphi^* D_\mu D^\mu \varphi$; leading to the equations of motion:

$$\boxed{\begin{array}{l} D_\mu D^\mu \varphi + m^2\varphi = 0; \\ (D_\mu D^\mu \varphi)^* + m^2\varphi^* = 0; \end{array}}$$ and in analogy:

- Notice that the current j^μ is conserved:

$$\partial_\mu j^\mu = -ie\left(\partial_\mu \varphi^* D^\mu\varphi - \partial_\mu\varphi \cdot D^\mu\varphi^* + \varphi^* \partial_\mu D^\mu \varphi - \varphi\, \partial_\mu D^\mu \varphi^*\right)$$

for simplicity we use $D_\mu\varphi^* = (\partial_\mu + ieA_\mu)\varphi^*$

Now we use $\varphi^* \partial_\mu D^\mu \varphi = \varphi^* D_\mu D^\mu \varphi + ie A_\mu \varphi^* D^\mu \varphi \Rightarrow$

$$\Rightarrow \partial_\mu j^\mu = -ie\left(\cancel{D_\mu\varphi^* D^\mu\varphi} - \cancel{D_\mu\varphi D^\mu\varphi^*} + \varphi^* D_\mu D^\mu\varphi - \varphi D_\mu D^\mu \varphi^*\right)$$

using e.o.m.: $\partial_\mu j^\mu = -ie\left(\varphi m^2 \varphi^* - \varphi^* m^2 \varphi\right) = 0$!!! $\boxed{\partial_\mu j^\mu = 0;}$

- Hence e.o.m. are consistent with each other.

⑩ Noether's theorem.

- **Statement:** If theory has a continious symmetry then there are corresponding conserved currents.
- In this lecture we consider two types of Noether currents:

  ⓐ Transformations of fields $\Rightarrow$ conserved currents j^a_μ;

  ⓑ Translations in space-time $\Rightarrow$ stress-energy tensor $T_{\mu\nu}$;
- Each of these currents have corresponding conserved charges if fields decay fast enough.

$$j^a_\mu \Rightarrow Q^a = \int d^3x\, j^a_0(x) \;;\quad T^{\mu\nu} \Rightarrow P^\mu = \int d^3x \cdot T^{0\mu}(x);$$

- **Setting:** Let Φ^I be the set of all fields in theory (for example in scalar electrodynamics it is A_μ, $\mathrm{Re}\varphi$, $\mathrm{Im}\varphi$)
- Consider Leagrangians containing only first derivatives of fields.

$$\mathcal{L}_0 = \mathcal{L}_0(\Phi^I, \partial_\mu \Phi^I);$$

then $\delta S = \delta \int d^4x\, \mathcal{L}_0(\Phi^I, \partial_\mu \Phi^I) = \int d^4x \left[\frac{\partial \mathcal{L}_0}{\partial \Phi^I} \delta \Phi^I + \frac{\partial \mathcal{L}_0}{\partial \Phi^I_{,\mu}} \partial_\mu \delta \Phi^I \right] =$

$$= \int d^4x \left[\frac{\partial \mathcal{L}_0}{\partial \Phi^I} - \partial_\mu \frac{\partial \mathcal{L}_0}{\partial \Phi^I_{,\mu}} \right] \delta \Phi^I \Rightarrow \text{Euler-Lagrange equation} \quad \frac{\partial \mathcal{L}_0}{\partial \Phi^I} = \partial_\mu \left(\frac{\partial \mathcal{L}_0}{\partial \Phi^I_{,\mu}} \right);$$

here $\Phi^I_{,\mu} \equiv \partial_\mu \Phi^I$

- **Noethers theorem:**
- Let's consider following field transformations:

$$\Phi^I \to \Phi'^I = (\delta^{IJ} + \varepsilon^a t_a^{IJ}) \Phi^J;$$

ε^a are infenetesemal parameters of transform not dependent on x.

- Due to the invariance of the Leagrangian:

$$\delta \mathcal{L}_0 = \mathcal{L}_0(\Phi + \delta\Phi, \Phi_{,\mu} + \delta \Phi_{,\mu}) - \mathcal{L}_0(\Phi, \Phi_{,\mu}) = 0;$$

where
$$\delta \Phi^I = \varepsilon^a t_a^{IJ} \Phi^J;$$
$$\delta \Phi^I_{,\mu} = \varepsilon^a t_a^{IJ} \Phi^J_{,\mu};$$

then we obtain

$$\delta \mathcal{L}_0 = \frac{\partial \mathcal{L}_0}{\partial \Phi^I} \varepsilon^a t_a^{IJ} \Phi^J + \frac{\partial \mathcal{L}_0}{\partial \Phi^I_{,\mu}} \varepsilon^a t_a^{IJ} \Phi^J_{,\mu} = 0$$

(11) • Now using Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}_e}{\partial \Phi^I} = \partial_\mu \frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\mu}} \Leftrightarrow \partial_\mu \left(\frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\mu}}\right) \varepsilon^a t_a^{IJ} \Phi^J + \frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\mu}} (\partial_\mu \Phi^J) \cdot \varepsilon^a t_a^{IJ} = 0 \Leftrightarrow$$

$$\Rightarrow \quad \varepsilon_a \cdot \partial_\mu \left(\frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\mu}} t_a^{IJ} \Phi^J\right) = 0$$ and as ε_a are arbitrary we obtain current conservation equation:

$$\boxed{\partial_\mu j^\mu = 0; \quad j^\mu_a = \frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\mu}} t_a^{IJ} \Phi^J;}$$

• Example: Complex scalar field.

Let's consider once again the Leagrangian

$$\mathcal{L}_e = \partial_\mu \varphi^* \partial^\mu \varphi - V(|\varphi|);$$

this Leagrangian is invariant under phase rotations:

$$\varphi \to e^{i\alpha}\varphi \qquad \varphi^* \to e^{-i\alpha}\varphi$$ wher $\alpha = \text{const}$.

this transformations in the infinetesemal form are

$$\varphi \to (1 + i\alpha)\varphi$$
$$\varphi^* \to (1 - i\alpha)\varphi^*$$

Hence to obtain conserved current we should put $\Phi^1 = \varphi$, $\Phi^2 = \varphi^*$, $\varepsilon = \alpha$, $t^{11} = i$; $t^{22} = -i$; $t^{12} = t^{21} = 0$; leading to the current

$$j^\mu = \frac{\partial \mathcal{L}_e}{\partial \varphi_{,\mu}} i\varphi + \frac{\partial \mathcal{L}_e}{\partial \varphi^*_{,\mu}} (-i)\varphi^* = i(\varphi \partial^\mu \varphi^* - \varphi^* \partial^\mu \varphi)$$ so

$$\boxed{j_\mu = i(\varphi \partial_\mu \varphi^* - \varphi^* \partial_\mu \varphi);}$$

this is the same current we observed before using Klein-Gordon equation.

• Translations.

Now let's consider translation in the space-time and their effect on the action. For the fields corresponding transformations are:

$$x^\mu \to x^\mu + \varepsilon^\mu$$
$$\Phi^I(x^\mu) \to \Phi'^I(x^\mu) = \Phi^I(x^\mu + \varepsilon^\mu) = \Phi^I(x^\mu) + \varepsilon^\mu \partial_\mu \Phi^I(x^\mu)$$

Then for the Leagrangian from one point of view changes according to:

(12) $$\mathcal{L}_e(\Phi', \Phi'_{,\mu}) = \mathcal{L}_e(x^\mu + \varepsilon^\mu) = \mathcal{L}_{e0} + \varepsilon^\mu \partial_\mu \mathcal{L}_e;$$

- At the same time:

$$\mathcal{L}_e(\Phi', \Phi'_{,\mu}) = \mathcal{L}_e(\Phi + \partial_\mu \Phi \cdot \varepsilon^\mu, \Phi_{,\mu} + \partial_\mu \partial_\nu \Phi \varepsilon^\nu) =$$

$$= \mathcal{L}_e(\Phi, \Phi_{,\mu}) + \frac{\partial \mathcal{L}_e}{\partial \Phi^I} \partial_\mu \Phi^I \cdot \varepsilon^\mu + \frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\mu}} \partial_\mu \partial_\nu \Phi^I \cdot \varepsilon^\nu$$

Now using Lagrange equations we get

$$\mathcal{L}_e(\Phi', \Phi'_{,\mu}) = \mathcal{L}_{e0} + \left(\partial_\nu \frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\nu}}\right) \partial_\mu \Phi^I \varepsilon^\mu + \frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\nu}} \partial_\nu \partial_\mu \Phi^I \varepsilon^\mu =$$

$$= \mathcal{L}_{e0} + \partial_\nu \left(\frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\nu}} \partial_\mu \Phi^I \right) \varepsilon^\mu$$

- Then comparing two results we obtain:

$$\partial_\mu \mathcal{L}_e = \partial_\nu \left(\frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\nu}} \partial_\mu \Phi^I \right) \Rightarrow \underline{\partial_\nu \left(\frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\nu}} \partial_\mu \Phi^I - \delta^\nu_\mu \mathcal{L}_e \right) = 0;}$$

- Hence our conserved current now is

$$\boxed{T^\mu{}_\nu = \frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\mu}} \cdot \Phi^I_{,\nu} - \delta_\mu{}^\nu \mathcal{L}_e;}$$ with the conservation law $\underline{\partial_\mu T^\mu{}_\nu = 0;}$

- Corresponding conserved charges are:

$$\begin{cases} E = \int T^0{}_0 \, d^3x = \int d^3x \left(\frac{\partial \mathcal{L}_e}{\partial \dot{\Phi}^I} \dot{\Phi}^I - \mathcal{L}_e \right); & \text{- field energy.} \\ P_i = \int d^3x \cdot T^0{}_i = \int d^3x \left(\frac{\partial \mathcal{L}_e}{\partial \dot{\Phi}^I} \partial_i \Phi^I \right); & \text{- field momentum.} \end{cases}$$

- <u>Inambiguity of Noether currents.</u>

Notice that if we have conserved current j^μ then the current $j'^\mu = j^\mu + \partial_\nu f^{\nu\mu}$ is also conserved if $f^{\mu\nu}$ is antisymmetric in $\mu\nu$ indices. In particular for the stress-energy tensor $\underline{T^{\mu\nu} \to T^{\mu\nu} + \partial_\alpha \Omega^{\alpha\mu\nu};}$ with $\Omega^{\alpha\mu\nu} = -\Omega^{\mu\alpha\nu};$

In general stress-energy tensor is not symmetric i.e. $T^{\mu\nu} \neq T^{\nu\mu}$ however it can be symmetrized with the proper choice of $\Omega^{\alpha\mu\nu}$.

- Direct way to obtain symmetric stress-energy tensor is by varying action w.r.t. the metric $g_{\mu\nu}$. i.e.:

$$\boxed{T_{\mu\nu} = \frac{\partial(\sqrt{-g}\,\mathcal{L}_e)}{\partial g^{\mu\nu}};} \qquad g = \det(g_{\mu\nu})$$

(13) Example: Real scalar field.

$$\mathcal{L}_\varphi = \frac{1}{2} g^{\mu\nu} \partial_\mu\varphi \partial_\nu\varphi - \frac{1}{2} m^2\varphi^2;$$

To variate this w.r.t metric we use the following equations:

- $\delta(g^{\mu\nu} g_{\nu\alpha}) = \delta(\delta^\mu{}_\alpha) = 0 \Rightarrow \delta g^{\mu\nu} \cdot g_{\nu\alpha} + g^{\mu\nu}\delta g_{\nu\alpha} = 0 \Rightarrow$

$$\Rightarrow \delta g^{\mu\nu} \cdot g_{\nu\alpha} \cdot g^{\alpha\beta} = -\delta g_{\nu\alpha} \cdot g^{\mu\nu} \cdot g^{\alpha\beta} \Rightarrow \delta g^{\mu\nu} \cdot \delta_\nu{}^\beta = -g^{\mu\nu} g^{\alpha\beta}\delta g_{\nu\alpha} \Rightarrow$$

$$\Rightarrow \quad \delta g^{\mu\beta} = -g^{\mu\nu} g^{\beta\alpha}\delta g_{\nu\alpha};$$

- $\log\det(g_{\mu\nu}) = \operatorname{tr}\log g_{\mu\nu} \Rightarrow \delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}$

- Let's now derive stress-energy tensor for the scalar field in 2 ways:

(1) Using variation w.r.t $g_{\mu\nu}$

$$\delta(\sqrt{-g}\,\mathcal{L}_\varphi) = (\delta\sqrt{-g})\,\mathcal{L}_\varphi + \sqrt{-g}\,\delta\mathcal{L}_\varphi = \sqrt{-g}\left(-\frac{1}{2} g_{\mu\nu}\mathcal{L}_\varphi + \frac{1}{2}\partial_\mu\varphi\cdot\partial_\nu\varphi\right)\delta g^{\mu\nu} =$$

$$= \sqrt{-g}\,\delta g^{\mu\nu}\left(-\frac{1}{2} g_{\mu\nu}\partial_\alpha\varphi\partial^\alpha\varphi + \partial_\mu\varphi\partial_\nu\varphi + \frac{1}{2} g_{\mu\nu} m^2\varphi^2\right)\cdot\frac{1}{2}$$

Hence: $\quad T_{\mu\nu} = \sqrt{-g}\left(\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2} g_{\mu\nu}\partial_\alpha\varphi\partial^\alpha\varphi + \frac{1}{2} g_{\mu\nu} m^2\varphi^2\right)$

Finally putting in flat metric we obtain:

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2} g_{\mu\nu}\partial_\alpha\varphi\partial^\alpha\varphi + \frac{1}{2} g_{\mu\nu} m^2\varphi^2;$$

(2) Using the expression:

$$T_{\mu\nu} = \frac{\partial\mathcal{L}_\varphi}{\partial\Phi^I{}_{,\mu}}\Phi^I{}_{,\nu} - g_{\mu\nu}\mathcal{L}_\varphi = \frac{\partial\mathcal{L}_\varphi}{\partial(\partial^\mu\varphi)}\partial(\partial_\nu\varphi) - g_{\mu\nu}\mathcal{L}_\varphi =$$

$$= \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\mathcal{L}_\varphi \Rightarrow \quad T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2} g_{\mu\nu}\partial_\alpha\varphi\partial^\alpha\varphi + \frac{1}{2} g_{\mu\nu} m^2\varphi^2;$$

- Notice that the energy of field is given by

$$E \equiv \int d^3x\, T^{00} = \int\left(\partial^0\varphi\,\partial^0\varphi - \frac{1}{2}\partial^0\varphi\partial^0\varphi + \frac{1}{2}\partial^i\varphi\partial^i\varphi + \frac{1}{2} m^2\varphi^2\right)d^3x \Rightarrow$$

$$\Rightarrow \quad E = \int d^3x\left(\frac{1}{2}(\partial_t\varphi)^2 + \frac{1}{2}(\vec\nabla\varphi)^2 + \frac{1}{2} m^2\varphi^2\right);$$ - the same expression we have obtained before!

① Lecture 1: Electromagnetic field theory (short reminder)

- In terms of electric ($\bar{E}$) and magnetic ($\bar{B}$) fields electromagnetic theory is not Lorentz invariant.

For Lorentz invariant formulation one introduces:

ⓐ Four-potential $A^\mu = (\varphi, \bar{A})$ where φ is electric potential and $\bar{A}$ is vector-potential so that:

$$\bar{E} = -\bar{\nabla}\phi - \frac{\partial \bar{A}}{\partial t};$$

$$\bar{H} = \bar{\nabla} \times \bar{A};$$

Notice that here and further in this course we use $\hbar = c = 1$ units.

ⓑ Electromagnetic (or field strength) tensor $F_{\mu\nu}$

$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, then

Notice: $F_{\mu\nu}$ is antisymmetric tensor $F_{\mu\nu} = -F_{\nu\mu}$;

- $$\left.\begin{aligned} F_{0i} &= \partial_0 A_i - \partial_i A_0 = E_i; \\ F_{ij} &= -\varepsilon_{ijk} H_k; \end{aligned}\right\} \Rightarrow F_{\underset{\text{line}}{\mu}\underset{\text{column}}{\nu}} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -H_3 & H_2 \\ -E_2 & H_3 & 0 & -H_1 \\ -E_3 & -H_2 & H & 0 \end{pmatrix}$$

Notice: we use the metric with the signature $\eta_{\mu\nu} = \mathrm{diag}(+1\,-1\,-1\,-1)$;

- The action of electromagnetic field is

$$S = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu};$$

- Varying S one obtains:

$$\delta S = -\frac{1}{2}\int d^4x \cdot F_{\mu\nu}\,\delta F^{\mu\nu} = -\int d^4x\, F_{\mu\nu}\,\partial^\mu \delta A^\nu = \int d^4x\, \partial_\mu F^{\mu\nu}\,\delta A_\nu$$

As result we obtain:

$$\boxed{\partial_\mu F^{\mu\nu} = 0} \quad \text{eqivalent to} \quad \left\{ \begin{aligned} &\bar{\nabla}\cdot\bar{E} = 0 \\ &\bar{\nabla}\times\bar{H} = \frac{\partial E}{\partial t} \end{aligned} \right. \rightarrow \text{second pair of Maxwell equations.}$$

② • One more pair of Maxwell equations comes from the Bianchi identity

$$\partial_{[\mu} F_{\nu\alpha]} = \frac{1}{6}\left(\partial_\mu F_{\nu\alpha} - \partial_\mu F_{\alpha\nu} + \partial_\nu F_{\alpha\mu} - \partial_\nu F_{\mu\alpha} + \partial_\alpha F_{\mu\nu} - \partial_\alpha F_{\nu\mu}\right) =$$

$$= \frac{1}{3}\left(\partial_\mu F_{\nu\alpha} + \partial_\alpha F_{\mu\nu} + \partial_\nu F_{\alpha\mu}\right) = 0$$

Notice: here we use notation $[\mu_1 \mu_2 \dots \mu_n] = \frac{1}{n!} \sum_{\sigma} (-1)^{P(\sigma)} P(\mu_1, \mu_2, \dots \mu_n)$

sign in front of the corresponding term

pemutations

This notation will be used in the future as well

• Bianchi identity can be also rewritten in the following form

$$\boxed{\varepsilon^{\mu\nu\alpha\beta} \partial_\nu F_{\alpha\beta} = \partial_\nu \tilde{F}^{\mu\nu} = 0}, \text{ where } \varepsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 \text{ if } \mu\nu\alpha\beta \text{ is even perm of } 0123 \\ -1 \text{ if } \mu\nu\alpha\beta \text{ is odd perm — " —} \end{cases}$$

and $\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$;

Bianchi identity is equivalent to $\begin{cases} \bar{\nabla} \cdot \bar{H} = 0; \\ \bar{\nabla} \times \bar{E} = -\frac{\partial \bar{H}}{\partial t}; \end{cases}$ first pair of Maxwell equations

• Gauge invariance

• Already when Maxwell formulated his equations it was noticed that the motion of the particles in external electromagnetic field is described by the interaction term in the action:

$$S_{int} = -e \int_{\mathcal{S}} A_\mu dx^\mu$$

$\mathcal{S}$ → path of the particle

which leads to the equations of motion:

$$\frac{d\bar{p}}{dt} = \underbrace{e(\bar{E} + \bar{v} \times \bar{H})}_{\text{Lorentz force}}$$

• Lorentz force depends only on $\bar{E}$ and $\bar{H}$ but not on the four-potential A^μ, hence A^μ is not physical, while $\bar{E}$ and $\bar{H}$ are!

③ Now notice that $\bar{H} = \bar{\nabla} \times \bar{A}$ is invariant under $\bar{A} \to \bar{A}' = \bar{A} + \bar{\nabla} f$, where $f(\bar{x})$ is some scalar function.

Then $\bar{E} = -\bar{\nabla}\varphi - \frac{\partial \bar{A}}{\partial t} \to \bar{E}' = -\bar{\nabla}\varphi' - \frac{\partial \bar{A}}{\partial t} - \bar{\nabla}\frac{\partial f}{\partial t}$ → this is invarian if $\varphi' = \varphi - \frac{\partial f}{\partial t}$;

- Hence gauge symmetry transformations are

$$\boxed{\begin{array}{l} \bar{A} \to \bar{A} + \bar{\nabla} f; \\ \varphi \to \varphi - \frac{\partial f}{\partial t}; \end{array} \Rightarrow A_\mu \to A_\mu - \partial_\mu f;}$$

- **Gauge symmetry means that physics stays invariant under gauge (i.e. coordinate-dependent) transformations!**

- Better way to test invariance of theory is to check the invariance of the action:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu - \cancel{\partial_\mu \partial_\nu f} - \partial_\nu A_\mu + \cancel{\partial_\nu \partial_\mu f} = F_{\mu\nu} \Rightarrow$$

$\Rightarrow$ $F_{\mu\nu}$ is gauge invariant!

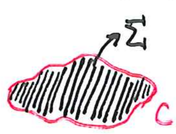
Hence the action $S = -\frac{1}{4}\int d^4x \, F_{\mu\nu} F^{\mu\nu}$ is also gauge invariant!!!

- One interesting gauge invariant quantity is given by

$$\oint_C A_\mu dx^\mu$$ → integration here is along some contour C.

If the space-time is simply connected this quantity can be expressed through $\bar{E}$ and $\bar{H}$. In particular due to the Stokes' theorem for the space-like contour C:

$$\oint_c A_i dx^i = -\int_\Sigma (\bar{\nabla} \times \bar{A}) d\bar{s} = -\int_\Sigma \bar{H} \cdot d\bar{s}$$



④ • However if space-time is not simply connected, the even if $\bar{E}=\bar{H}=0$ in all space we can have $\oint_C \bar{A}\,d\bar{x} \neq 0$

This quantity can be obtained in quantum mechanical systems:

Aharonov-Bohm experiment.

• In external vector-potential particles w.f. acquires the phase:

$$e^{ie\int_{x_0}^{x} d\bar{y}\cdot\bar{A}(y)}$$

Indeed the Hamiltonian of the charged particle in external electromagnetic field is

$$\hat{H} = \frac{1}{2m}(\hat{\bar{p}} - e\hat{\bar{A}})^2 + \underset{\substack{\uparrow \\ \text{includes } \varphi(\bar{x})}}{V(\bar{x})}$$

In the absence of field we have $\hat{H}_0 = \frac{1}{2m}\hat{\bar{p}}^2 + V(\bar{x})$

Assume we found the solution for it in the form of $\Psi_0(x,t)$:

$i\frac{\partial}{\partial t}\Psi_0 = \hat{H}_0\Psi_0$, now if we turn on e.m. field the solution would become $\Psi = \exp(ie\int_{x_0}^{x} d\bar{y}\cdot\bar{A}(y))\,\Psi_0$

as now $(\hat{\bar{p}} - e\hat{\bar{A}})\Psi = e\bar{A}(x)\Psi + \exp(ie\int^{x} d\bar{y}\bar{A}(\bar{y}))\,\bar{\nabla}\Psi_0(-i) -$

$- e\bar{A}\cdot\Psi = \exp(ie\int^{x} d\bar{y}\cdot\bar{A}(\bar{y}))(-i\bar{\nabla})\Psi_0$

Hence Schrödinger equation is rewritten in the form:

$$\exp(ie\int^{x} d\bar{y}\cdot\bar{A})\left\{-\frac{1}{2m}\Delta + V(\bar{x})\right\}\Psi_0 = \exp(ie\int^{x} d\bar{y}\cdot\bar{A})\cdot i\frac{\partial}{\partial t}\Psi_0;$$

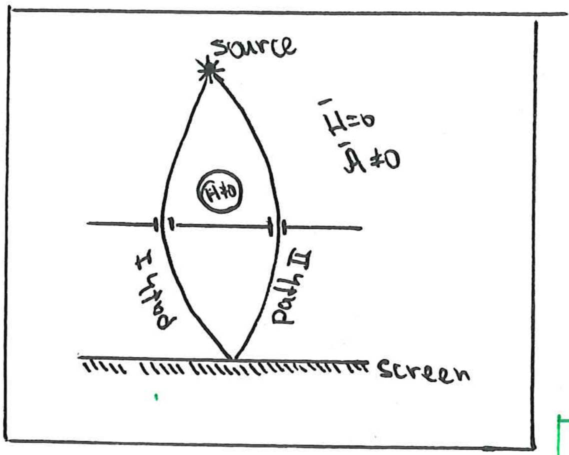
which is clearly satisfied.

Outline: Particles w.f. in magnetic field is

$$\Psi = \exp(i\int^{x} d\bar{y}\cdot\bar{A})\,\Psi_0(x,t);$$

where $\Psi_0(x,t)$ is the w.f. in the absence of field.

⑤ • This phase can be measured in the double-slit experiment



• There is solenoid between the slits. Magnetic field is nonzero only inside the solenoid.

• The difference of phases for the particles going along two pathes is given by

$$\Delta\varphi = \exp\left(ie\left(\int_{\text{path I}} - \int_{\text{path II}}\right) d\bar{x}\cdot\bar{A}(\bar{x})\right) =$$

$$= \exp\left(ie\oint d\bar{x}\,\bar{A}(\bar{x})\right) = e^{ie\Phi}$$

where Φ is the flux of the magnetic field through the solenoid.

• Hence though electrons do not experience magnetic field directly (walls of the solenoid can be made unpenetrable for electrons) they acquire measurable phase!

• General solutions for the Maxwell equations.

To solve Maxwell equations let's perform Fourier transform:

$$A_\mu = \int_{k^0 \geq 0} d^4k \left(a_\mu(k)\, e^{ik_\mu x^\mu} + \text{c.c.}\right),$$

here $a_\mu(k)$ is complex function of 4-vector k^μ. c.c. is complex conjugate.

In this case

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \int_{k^0\geq 0} d^4k \cdot i \cdot \left(\{k_\mu a_\nu - k_\nu a_\mu\} e^{ik_\mu x^\mu} + \text{c.c.}\right)$$

$$\partial^\mu F_{\mu\nu} = \int d^4k\,(-1)\left(\{k^2 a_\nu - k^\mu a_\mu \cdot k_\nu\} e^{ik_\alpha x^\alpha} + \text{c.c.}\right)$$

and Maxwell equations reads:

$$\boxed{k^2 a_\nu - k^\mu k_\nu a_\mu = 0;}$$

⑥ • Here we need to consider separately two cases:

① $k^2 \neq 0$: in this case we see that a_μ is collinear with k^μ ⇨ $a_\mu = c(k) k_\mu$; Substituting this solution back to the Maxwell equations we see that $c(k)$ is not defined by any conditions. It is arbitrary function.

② $k^2 = 0$: in this case we obtain orthogonality condition:

$k^\mu a_\mu = 0$;

In four dimensions there are 3 4-vectors transverse to k^μ. One of them is k^μ itself (as $k^2 = k^\mu k_\mu = 0$). We choose two remaining vectors to be $e_\mu^{(\alpha)}$ ($\alpha = 1, 2$) such that

- $e_0^{(\alpha)} = 0$ – pure space vectors (no time component)
- $e_i^{(\alpha)} \cdot k^i = 0$ – 3-vectors $\bar{e}^{(\alpha)}$ and $\bar{k}$ are orthogonal.
- $e_i^{(\alpha)} e^{i(\beta)} = \delta^{\alpha\beta}$ – 3-vectors $\bar{e}^{(\alpha)}$ $\bar{e}^{(\beta)}$ are orthogonal to each other.

So that the general solution for $k^2 = 0$ is

$a_\mu(k) = k_\mu \cdot c(\bar{k}) + e_\mu^{(\alpha)}(\bar{k})\, b_\alpha(\bar{k})$; where c, b_α are 3 arbitrary functions of $\bar{k}$;

• Uniting two cases we obtain:

Note: here and further we use notation $k \cdot x \doteq k_\mu x^\mu$;

$$\boxed{\begin{aligned} A_\mu(x) &= A_\mu^\perp(x) + A_\mu^{\parallel}(x); \\ A_\mu^\perp(x) &= \int d^3k \left[e^{ik\cdot x} \cdot e_\mu^{(\alpha)}(\bar{k}) \cdot b_\alpha(\bar{k}) + c.c. \right]\Big|_{k^0 = |\bar{k}|}; \\ A_\mu^{\parallel}(x) &= \int d^4k \left[e^{ikx} \cdot k_\mu \cdot c(k) + c.c. \right]; \end{aligned}}$$

Notice that $A_\mu^{\parallel}(x)$ is pure gauge as:

$$A_\mu^{\parallel}(x) = \partial_\mu \alpha(x), \text{ where } \alpha(x) = \int d^4x\, (-i) \left[c(k) e^{ik\cdot x} + c.c. \right];$$

• Non-trivial (i.e. not gauge) part of the solution is $A_\mu^\perp$ and it represents plane waves moving with the speed of light ($k^0 = |\bar{k}|$)

⑦ Gauge fixing.

Presence of gauge symmetry leads to the presence of disambiguity in the solution of Maxwell equations. This disambiguity is not physical and should be read off by implying some conditions on A_μ. This is cold gauge fixing.

Note: Important reason to fix the gauge is due to big problems it brings when one tries to quantize electromagnetic field.

Examples of gauge fixing.

① Coulomb gauge

$$\operatorname{div} \bar{A} \equiv \partial_i A^i = 0$$

Notice that this condition is not invariant under gauge transformations. However if A_μ satisfies $\partial_i A^i = 0$ then $A'_\mu = A_\mu + \partial_\mu \alpha$ satisfies the same condition only if:

$$\partial_i \partial^i \alpha = \Delta \alpha = 0;$$

This is the remaining gauge freedom.

This equation can be satisfied if α is constant or growing function. If the fields are decaying at infinity then gauge is fixed completely. Solution of Maxwell equations reads:

Coulomb gauge: $\bar{k}^2 c(k) = 0$, $A^{\parallel}_\mu(x) = 0$; $A_\mu = A^{\perp}_\mu(x)$; – solution of Maxwell equations.

② Lorentz gauge. $\partial_\mu A^\mu = 0;$

Remaining gauge symmetry $\partial_\mu \partial^\mu \alpha = 0;$

So the solution is up to the longitudal waves moving with the speed of light $\Rightarrow$ $c(k) = 0$ for $k^2 \neq 0$ but arbitrary for $k^2 = 0;$

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Gauge $A_0=0$;

Remaining gauge: $\partial_0 \alpha = 0$

General solution for Maxwell equation:

$$A_\mu(x) = A_\mu^\perp(x) + B_\mu(x) \text{, where } \begin{array}{l} B_0=0 \\ B_i=\partial_i \alpha(x) \end{array}$$

hence $c(k)\neq 0$ only when $k^0=0$;

Problem I

Show that equations $\partial_\mu F^{\mu\nu}$ are equivalent to the pair of equations

$$\bar{\nabla}\cdot\bar{E}=0$$
$$+\bar{\nabla}\times H = \frac{\partial \bar{E}}{\partial t}$$

while equations $\varepsilon^{\mu\nu\alpha\beta}\partial_\nu F_{\alpha\beta}=0$ corresponds to the pair of equations

$$\bar{\nabla}\cdot\bar{H}=0$$
$$\bar{\nabla}\times\bar{E} = -\frac{\partial \bar{H}}{\partial t};$$

Problem II

Find remaining gauge freedom and general solution of Maxwell equations for the axial gauge

$$\bar{n}\cdot\bar{A}=0$$

where $\bar{n}$ is some unity 3-vector.

Lecture 2: Scalar and vector fields. Noether's theorem.

• Action of the scalar field

Previous lecture – electromagnetic field, i.e. massless vector field. In nature we know following bosonic particles:

- photon γ: vector massless field.
- π^0, η mesons: real scalar fields.
- $\pi^\pm$ mesons: one complex scalar field.
- $W^\pm$ and Z bosons: massive vector fields.

Simpliest case is real scalar case.

We look for the action:

- Second order differential equations of motion ⇒ action is quadractic in derivatives
- Leagrangian is Lorentz invariant
- Equations of motion are linear ⇒ action is quadratic in fields.

⇓

$$\boxed{S = \int d^4x \left(\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2\right);} \quad \text{where } (\partial_\mu\varphi)^2 \equiv \partial_\mu\varphi \cdot \partial^\mu\varphi;$$

• Equations of motion:

$$\delta S = \int d^4x \left[2 \cdot \frac{1}{2}\partial^\mu\varphi \cdot \partial_\mu\delta\varphi - m^2\varphi\,\delta\varphi\right] = \int d^4x\left[-\partial_\mu\partial^\mu\varphi \cdot \delta\varphi - m^2\varphi\,\delta\varphi\right] +$$

$$+ \int_{\partial M} dS^\mu (\partial_\mu\varphi \cdot \delta\varphi);$$

Last term is over the boundary of space. We usually put these boundary terms to zero. Then the equations of motion are just Klein-Gordon equation (KG equation)

$$\boxed{\partial_\mu\partial^\mu\varphi + m^2\varphi = 0;}$$

• General solution of KG equation:

Fourier transforming: $\varphi(x) = \int_{k^0 \geq 0} d^4k\,[\tilde{\varphi}(k)e^{ikx} + c.c.];$

we get: $(k^2 - m^2)\tilde{\varphi}(k) = 0;$

Hence $\tilde{\varphi}(k)$ is arbitrary only if $k^2 = m^2$ ($k_0^2 = |\vec{k}|^2 + m^2$) and zero otherwise

② • Hence we have typical dispersion law of relativistic particle: $k^0 = \sqrt{|\bar{k}|^2 + m^2}$ with m being the mass of field.

$$\varphi(x) = \int d^3k \left[\tilde{\varphi}(\bar{k}) e^{ik\cdot x} + c.c. \right] \Big|_{k^0 = \sqrt{k^2+m^2}} ; \quad \varphi(\bar{k}) \text{ is arbitrary.}$$

• Energy of the scalar field:

Lagrangian is given by

$$L = \int d^3x \, \mathcal{L} = \int d^3x \left(\tfrac{1}{2}(\partial_\mu \varphi)^2 - \tfrac{m^2}{2}\varphi^2 \right) = \int d^3x \left(\tfrac{1}{2}\dot{\varphi}^2 - \tfrac{1}{2}(\bar{\nabla}\varphi)^2 - \tfrac{m^2}{2}\varphi^2 \right)$$

Notice that $\mathcal{L}$ is Lagrangian density rather then Lagrangian itself. However we sometimes refer it as the Lagrangian as well.

Then the energy of the scalar field can be derived using

$$E = \int \left(\frac{\delta L}{\delta \dot{\varphi}} \dot{\varphi} - L \right) d^3x \qquad \frac{\delta L}{\delta \dot{\varphi}(x)} = \dot{\varphi}(x); \text{ hence:}$$

$$\boxed{E = \int d^3x \left(\tfrac{1}{2}\dot{\varphi}^2 + \tfrac{1}{2}(\partial_i \varphi)^2 + \tfrac{1}{2} m^2 \varphi^2 \right);}$$

Comment on the choice of signs From the form of this expression for the energy of scalar field we can justify our choice of signs in the action.

• sign in front of the kinetic term $\partial_\mu \varphi \partial^\mu \varphi$ is chosen so that first two terms in the energy are positive definite and frequently oscillating fields have large positive energy.

• sign in front of the mass term is chosen so that the energy of large fields is positive and hence the energy is bounded from below.

③ Complex scalar field.

- $\varphi(x) = \mathrm{Re}\,\varphi(x) + i\,\mathrm{Im}\,\varphi(x)$ – another important example of the scalar field (it is very usefull for the description of charged scalar fields). Now on top of conditions we wanted to be satisfied before we also put reality condition. An appropriate action is then given by:

$$\boxed{S = \int d^4x \left(\partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi\right);}$$

- In order to find equations of motion corresponding to this action we should consider fields $\varphi(x)$ and $\varphi^*(x)$ as independent!

- varying w.r.t. $\varphi(x)$: $\delta S = \int d^4x \left(\partial_\mu \varphi^* \partial^\mu \delta\varphi - m^2 \varphi^* \delta\varphi\right) =$

$$= -\int d^4x \left(\partial_\mu \partial^\mu \varphi^* + m^2 \varphi^*\right)\delta\varphi + \begin{bmatrix}\text{Bondary}\\ \text{terms}\end{bmatrix} \Rightarrow \boxed{\partial_\mu \partial^\mu \varphi^* + m^2 \varphi^* = 0;}$$

- varying w.r.t. $\varphi^*(x)$: $\delta S = \int d^4x \left(\partial_\mu \delta\varphi^* \partial^\mu \varphi - m^2 \delta\varphi^* \cdot \varphi\right) =$

$$= -\int d^4x \left(\partial_\mu \partial^\mu \varphi + m^2 \varphi\right)\delta\varphi^* + \begin{bmatrix}\text{Boundary}\\ \text{terms}\end{bmatrix} \Rightarrow \boxed{\partial_\mu \partial^\mu \varphi + m^2 \varphi = 0;}$$

Hence we obtain two Klein-Gordon equations instead of one.

Comment: Instead of one complex scalar field we can introduce the pair of fields $\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$. Then the Leagrangian will become

$$\mathcal{L}_\varphi = \sum_a \left(\tfrac{1}{2} \partial_\mu \varphi^a \partial^\mu \varphi^a - \tfrac{1}{2} m^2 \varphi^a \cdot \varphi^a\right); \quad a = 1, 2;$$

and equations of motion turn to be $\partial_\mu \partial^\mu \varphi^a + m^2 \varphi^a = 0;$
This description is completely equivalent to the one considered above.

- Theory of complex scalar field also has conserved current

$$j_\mu = -i\left(\varphi^* \partial_\mu \varphi - \varphi\, \partial_\mu \varphi^*\right).$$

Indeed it is easy to show that $\partial_\mu j^\mu = 0$ using e.o.m.:

$$\partial_\mu j^\mu = -i\left(\partial_\mu \varphi^* \partial^\mu \varphi - \partial_\mu \varphi\, \partial^\mu \varphi^* + \varphi^* \partial_\mu \partial^\mu \varphi - \varphi\, \partial_\mu \partial^\mu \varphi^*\right) =$$

$$= -i\left(-m^2 \varphi^* \varphi + m^2 \varphi^* \varphi\right) = 0. \Rightarrow \text{charged is conserved!}$$

4 • Existence of conserved current $\Rightarrow$ conserved charge

$Q \equiv \int d^3x\, j_0$, then $\partial_0 Q = \int d^3x\, \partial_0 j_0 = -\int d^3x\, \partial_i j^i = -\int d\Sigma_i\, j_i$

Boundary of the space

If we take bdry of the space at the spatial infinity and field decay there fast enough, then the last integral is zero and we obtain $\underline{\partial_0 Q = 0} \Rightarrow$ conserved charge!

• Massive vector fields.

• Massive vector field should be described by the 4-vector $B_\mu(x)$ However just as in the case of e.m. field we can split vector field into really vector-like transverse component and longitudal gradient part:

$$B_\mu = B_\mu^\perp + \partial_\mu \chi; \text{ with } \partial_\mu B_\perp^\mu = 0;$$

• If the mass of field is nonzero then the expected dispersion relation is $k^0 = \sqrt{\vec{k}^2 + m^2}$; The easiest way to obtain this is to assume that every component of $B_\mu^\perp$ field satisfies KG equation:

$$(\partial_\mu \partial^\mu + m^2) B_\nu^\perp = 0 \rightarrow \text{goal!} + \partial_\mu B_\perp^\mu = 0; \quad (I)$$

An appropriate action fulfilling these conditions is:

$$\boxed{S = \int d^4x \left[-\tfrac{1}{4} B_{\mu\nu} B^{\mu\nu} + \tfrac{m^2}{2} B_\mu B^\mu \right];} \quad B_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu;$$

Corresponding equations of motion have the following form

$$\partial_\mu B^{\mu\nu} + m^2 B^\nu = 0 \qquad (II)$$

this term is obtained similarly to Maxwell eq's

term from $\delta\left(\frac{m^2}{2} B_\nu B^\nu\right) = m^2 B_\nu \delta B^\nu$;

• Let's differentiate (II) w.r.t x^ν: $\partial_\mu \partial_\nu B^{\mu\nu} + m^2 \partial_\nu B^\nu = 0$

due to antisymmetry of $B^{\mu\nu}$ $\partial_\mu \partial_\nu B^{\mu\nu} = 0$ and hence $\underline{\partial_\nu B^\nu = 0}$;

Now substituting this back into (II) we obtain:

⑤ $\partial_\mu \partial^\mu B^\nu - \underbrace{\partial_\mu \partial^\nu B^\mu}_{\partial^\nu \partial_\mu B^\mu = 0} + m^2 B^\nu = 0 \Rightarrow \boxed{\begin{matrix} \partial_\mu \partial^\mu B^\nu + m^2 B^\nu = 0; \\ + \\ \partial_\mu B^\mu = 0; \end{matrix}}$ Just what we want

• Interaction with external sources.

In electrodynamics interaction of e.m. fields with charge currents is constructed using current 4-vector $j^\mu = (\rho, \bar{j})$ (ρ – charge density, $\bar{j}$ – current density)

Corresponding action is:

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu\right)$$

Variation of this gives

$$\delta S = \int d^4x \left(\partial_\mu F^{\mu\nu} - j^\nu\right) \delta A_\nu = 0 \Rightarrow \underline{\partial_\mu F^{\mu\nu} = j^\nu} \rightarrow$$ Maxwells equations with the source

$$\bar{\nabla} \cdot \bar{E} = \rho;$$
$$\bar{\nabla} \times \bar{H} = \frac{\partial \bar{E}}{\partial t} + \bar{j};$$

• Notice that Maxwell equation implies current conservation. Indeed:

$$\partial_\mu F^{\mu\nu} = j^\nu \Rightarrow \underbrace{\partial_\mu \partial_\nu F^{\mu\nu}}_{\text{0 due to antisymm. of } F^{\mu\nu}} = \boxed{\partial_\nu j^\nu = 0}$$

• Conservation of the current leads in turn to the gauge invariance: $A'_\mu = A_\mu + \partial_\mu \alpha$

$$S[A'_\mu] = S[A_\mu] - \int d^4x\, j_\mu \partial^\mu \alpha = S[A_\mu] + \int d^4x \cdot \alpha \cdot \underbrace{\cancel{\partial_\mu j^\mu}}_{0} - \int d^4x\, \partial_\mu (\alpha j^\mu)$$

the last integral is integral over infinitely far surface (3d-surface)

$$\int d^4x\, \partial_\mu (j^\mu \alpha) = \int d\Sigma_\mu\, \alpha j^\mu = 0$$ if α decays at infinity fast enough.

Hence $S[A'_\mu] = S[A_\mu]$ and theory is gauge invariant.

• In analogy we can introduce source of the scalar field φ:

$$\boxed{S = \int d^4x \left[\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2\right] + \int d^4x\, \rho(x) \cdot \varphi(x);}$$

$\downarrow$ source is Lorentz scalar.

⑥ Interaction of fields. Scalar electrodynamics.

- Interaction terms in Leagrangians are terms with the powers of fields higher then two leading to the nonlinear terms in equations.
- In order for the action to be Lorentz invariant these interaction terms should also be Leorentz scalars.
- Simpliest example: interacting scalar field theory

$$S=\int d^4x\left[\frac{1}{2}(\partial_\mu\varphi)^2-V(\varphi)\right];\quad \text{where } V(\varphi)=\frac{1}{2}m^2\varphi^2+V_{int}(\varphi);$$

with $V_{int}(\varphi)$ containing terms like $\varphi^3, \varphi^4, \dots$ is polynomial of some finite degree.

Q: Do we still need $m^2>0$ condition or it should be modified somehow? How exactly?

Equations of motion obtained after varying $S[\varphi]$ are

$$\partial_\mu\partial^\mu\varphi+\frac{\partial V}{\partial\varphi}=0;$$

- Now let's ask ourselvs how to construct the action for the scalar field interacting with e.m. field?

Hard way (straight forward).

- We described sources in electrodynamics, which leads to the term $\mathcal{L}_e=-j_\mu A^\mu$ in the Leagrangian.
- We have also seen that complex scalar field has conserved current $e j^{(0)}_\mu=-i(\varphi^*\partial_\mu\varphi-\partial_\mu\varphi\cdot\varphi^*)e$ ← charge of the scalar

Hence it is naturall to assume that in order to couple scalar and e.m. fields we just write down the action:

$$S=\int d^4x\left[\partial_\mu\varphi^*\partial^\mu\varphi-m^2\varphi^*\varphi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-ej^{(0)}_\mu A^\mu\right];$$

This leads to the following equations of motion

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$$\partial_\mu F^{\mu\nu} = e j^{(0)\nu} \quad (\text{III})$$

$$\partial_\mu \partial^\mu \varphi + m^2 \varphi - 2ie\, \partial_\mu \varphi A^\mu - ie\, \varphi \partial_\mu A^\mu = 0 \quad (\text{IV})$$

$$\partial_\mu \partial^\mu \varphi^* + m^2 \varphi^* + 2ie\, \partial_\mu \varphi^* A^\mu + ie\, \varphi^* \partial_\mu A^\mu = 0 \quad (\text{V})$$

Q: Derive these equations.

However using (IV) and (V) we find:

$$\partial_\mu j^{\mu(0)} = -i(\varphi^* \partial_\mu \partial^\mu \varphi - \varphi \partial_\mu \partial^\mu \varphi^*) = -i \cdot (2ie\, \varphi^* \partial_\mu \varphi A^\mu +$$
$$+ ie\, \varphi^* \varphi \partial_\mu A^\mu + 2ie\, \varphi \partial_\mu \varphi^* A^\mu + ie\, \varphi^* \varphi \partial_\mu A^\mu) =$$
$$= 2e\, \partial_\mu(\varphi^* \varphi A^\mu) \Rightarrow \boxed{\partial_\mu j^\mu_{(0)} = 2e \cdot \partial_\mu(\varphi^* \varphi A^\mu);} \rightarrow \text{the}$$

curren is not conserved any more! This contradicts (III)

- Now to fix this problem we modify the current as follows: $j_\mu = j_\mu^{(0)} + c \cdot \varphi^* \varphi A_\mu$ then (IV) and (V) become:

$$\partial_\mu \partial^\mu \varphi + m^2 \varphi - 2ie\, \partial_\mu \varphi A^\mu - ie\, \varphi \partial_\mu A^\mu + ec \cdot \varphi A_\mu A^\mu = 0;$$
$$\partial_\mu \partial^\mu \varphi^* + m^2 \varphi^* + 2ie\, \partial_\mu \varphi^* A^\mu + ie\, \varphi^* \partial_\mu A^\mu + e \cdot c \cdot \varphi^* A_\mu A^\mu = 0;$$

- Then still $\partial_\mu j^\mu_{(0)} = -i(\varphi^* \partial_\mu \partial^\mu \varphi - \varphi \partial_\mu \partial^\mu \varphi^*) = 2e \cdot \partial_\mu(\varphi^* \varphi A^\mu)$
- However now eq (III) turns into

$$\partial_\mu F^{\mu\nu} = e j^{(0)\nu} + 2e \cdot c\, \varphi^* \varphi A^\nu \Rightarrow 0 = \partial_\mu \partial_\nu F^{\mu\nu} = e\, \partial_\nu j^\nu_{(0)} +$$
$$+ 2e \cdot c \cdot \partial_\nu(\varphi^* \varphi A^\nu) = 0 \quad \text{if} \quad \underline{c = e}!$$

- Finally we can write the action:

$$\boxed{S = \int d^4x \left[\partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + ie(\varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^*) A^\mu - e^2 \varphi^* \varphi A_\mu A^\mu \right];}$$

- We would also like our action to be <u>gauge invariant</u>

The action above is gauge inv. indeed. This can be checked by the direct calculation if we assume the following form of gauge transformations:

(8)

$$\boxed{\begin{aligned} &\varphi(x) \to \varphi'(x) = e^{i\alpha(x)}\varphi(x); \\ &\varphi^*(x) \to \varphi'^*(x) = e^{-i\alpha(x)}\varphi^*(x); \\ &A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x); \end{aligned}}$$

- Notice that transformations of φ, φ^* are symmetries of the free (no interaction) scalar field action only if $\alpha(x) = \alpha = const$ (global symmetry)

- This gives us recipie of introducing gauge invariant actions.

(1) Take the global symmetry for scalar fields (or fermions) and gauge it (make it x-dependent)

(2) Now notice that $\partial_\mu\varphi\partial^\mu\varphi^*$ term is not inv. under this local transformation anymore, because

$$\partial_\mu\varphi'(x) = e^{i\alpha(x)}\left[\partial_\mu\varphi(x) + i\partial_\mu\alpha(x)\cdot\varphi(x)\right];$$

To fix this introduce another derivative (covariant derivative) satisfying $(D_\mu\varphi)' = e^{i\alpha(x)}D_\mu\varphi$;

(3) In order for this equation to work we need to compensate $i\partial_\mu\alpha\cdot\varphi$ term in derivative which can be done by $\boxed{D_\mu \equiv \partial_\mu - ieA_\mu}$

Then indeed $(D_\mu\varphi') = (\partial_\mu - ieA'_\mu)e^{i\alpha(x)}\varphi = ((\partial_\mu - ieA_\mu)\varphi)\cdot e^{i\alpha(x)} +$
$+ (i\partial_\mu\alpha\cdot\varphi - ie\cdot\frac{1}{e}\partial_\mu\alpha\cdot\varphi)e^{i\alpha(x)} = e^{i\alpha(x)}D_\mu\varphi \Rightarrow$ just as we want.

(4) Then we directly observe the action with the gauge invariant interaction:

$$\boxed{S = \int d^4x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^* D_\mu\varphi - m^2\varphi^*\varphi\right];}$$

This is exactly the same action as the one derived before

⑨ Equations of Motion.

- Varying w.r.t A_μ:

$$\delta\left(\int d^4x \left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)\right) = \int d^4x\, \partial_\mu F^{\mu\nu}\delta A_\nu \text{ as usually}$$

$$\delta\left(\int d^4x (D_\mu\varphi)^* D^\mu\varphi\right) = \delta\int d^4x\, (\partial_\mu + ieA_\mu)\varphi^* \cdot (\partial^\mu - ieA^\mu)\varphi =$$

$$= ie\int d^4x\, \delta A_\mu\, (\varphi^* D^\mu\varphi - \varphi D^\mu\varphi^*).$$

Hence we obtain the Maxwell equation

$$\boxed{\partial_\mu F^{\mu\nu} = j^\nu \quad \text{with } j^\nu = -ie(\varphi^* D^\nu\varphi - \varphi D^\nu\varphi^*);}$$

- Varying w.r.t. φ^*:

$$\delta\int d^4x (\partial_\mu + ieA_\mu)\varphi^* (\partial^\mu - ieA^\mu)\varphi = -\int d^4x\, \delta\varphi^* (\partial^\mu - ieA^\mu)(\partial_\mu - ieA_\mu)\varphi =$$

$= -\int d^4x \cdot \delta\varphi^* D_\mu D^\mu\varphi$; leading to the equations of motion:

$$\boxed{\begin{aligned} &D_\mu D^\mu\varphi + m^2\varphi = 0; \\ &(D_\mu D^\mu\varphi)^* + m^2\varphi^* = 0; \end{aligned}}$$

and in analogy:

- Notice that the current j^μ is conserved:

$$\partial_\mu j^\mu = -ie\left(\partial_\mu\varphi^* D^\mu\varphi - \partial_\mu\varphi \cdot D^\mu\varphi^* + \varphi^*\partial_\mu D^\mu\varphi - \varphi\partial_\mu D^\mu\varphi^*\right)$$

for simplicity we use $D_\mu\varphi^* = (\partial_\mu + ieA_\mu)\varphi^*$

Now we use $\varphi^*\partial_\mu D^\mu\varphi = \varphi^* D_\mu D^\mu\varphi + ie A_\mu\varphi^* D^\mu\varphi \Rightarrow$

$$\Rightarrow \partial_\mu j^\mu = -ie\left(\cancel{D_\mu\varphi^* D^\mu\varphi} - \cancel{D_\mu\varphi D^\mu\varphi^*} + \varphi^* D_\mu D^\mu\varphi - \varphi D_\mu D^\mu\varphi^*\right)$$

using e.o.m.: $\partial_\mu j^\mu = -ie(\varphi m^2\varphi^* - \varphi^* m^2\varphi) = 0$!!! $\boxed{\partial_\mu j^\mu = 0;}$

- Hence e.o.m. are consistent with each other.

(10) Noether's theorem.

- **Statement**: If theory has a continious symmetry then there are corresponding conserved currents.
- In this lecture we consider two types of Noether currents:

  (a) Transformations of fields $\Rightarrow$ conserved currents j^a_μ;

  (b) Translations in space-time $\Rightarrow$ stress-energy tensor $T_{\mu\nu}$;
- Each of these currents have corresponding conserved charges if fields decay fast enough.

$$j^a_\mu \Rightarrow Q^a = \int d^3x\, j^a_0(x) \;;\quad T^{\mu\nu} \Rightarrow P^\mu = \int d^3x \cdot T^{0\mu}(x);$$

- **Setting**: Let Φ^I be the set of all fields in theory (for example in scalar electrodynamics it is A_μ, $\mathrm{Re}\varphi$, $\mathrm{Im}\varphi$)
- Consider Leagrangians containing only first derivatives of fields.

$$\mathcal{L}_0 = \mathcal{L}_0(\Phi^I, \partial_\mu \Phi^I);$$

then $\delta S = \delta \int d^4x\, \mathcal{L}_0(\Phi^I, \partial_\mu\Phi^I) = \int d^4x \left[\frac{\partial \mathcal{L}_0}{\partial \Phi^I}\delta\Phi^I + \frac{\partial \mathcal{L}_0}{\partial \Phi^I_{,\mu}} \partial_\mu \delta \Phi^I \right] =$

$= \int d^4x \left[\frac{\partial \mathcal{L}_0}{\partial \Phi^I} - \partial_\mu \frac{\partial \mathcal{L}_0}{\partial \Phi^I_{,\mu}} \right] \delta\Phi^I \Rightarrow$ Euler-Lagrange equation $\frac{\partial \mathcal{L}_0}{\partial \Phi^I} = \partial_\mu \left(\frac{\partial \mathcal{L}_0}{\partial \Phi^I_{,\mu}} \right);$

here $\Phi^I_{,\mu} \equiv \partial_\mu \Phi^I$

- **Noethers theorem**:
- Let's consider following field transformations:

$$\Phi^I \to \Phi'^I = (\delta^{IJ} + \varepsilon^a t_a^{IJ})\Phi^J;$$

ε^a are infenetesemal parameters of transform not dependent on x.

- Due to the invariance of the Leagrangian:

$$\delta\mathcal{L}_0 = \mathcal{L}_0(\Phi + \delta\Phi, \Phi_{,\mu} + \delta\Phi_{,\mu}) - \mathcal{L}_0(\Phi, \Phi_{,\mu}) = 0;$$

where $\delta\Phi^I = \varepsilon^a t_a^{IJ} \Phi^J;$

$\delta\Phi^I_{,\mu} = \varepsilon^a t_a^{IJ} \Phi^J_{,\mu};$

then we obtain

$$\delta\mathcal{L}_0 = \frac{\partial\mathcal{L}_0}{\partial\Phi^I}\varepsilon^a t_a^{IJ}\Phi^J + \frac{\partial\mathcal{L}_0}{\partial\Phi^I_{,\mu}}\varepsilon^a t_a^{IJ}\Phi^J_{,\mu} = 0$$

(11) • Now using Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \Phi^I} = \partial_\mu \frac{\partial \mathcal{L}}{\partial \Phi^I_{,\mu}} \Leftrightarrow \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \Phi^I_{,\mu}}\right)\varepsilon^a t_a^{IJ}\Phi^J + \frac{\partial \mathcal{L}}{\partial \Phi^I_{,\mu}}(\partial_\mu \Phi^J)\cdot \varepsilon^a t_a^{IJ} = 0 \Leftrightarrow$$

$$\Rightarrow \quad \varepsilon_a \cdot \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \Phi^I_{,\mu}} t_a^{IJ}\Phi^J\right) = 0$$ and as ε_a are arbitrary we obtain current conservation equation:

$$\boxed{\partial_\mu j^\mu = 0; \quad j^\mu_a = \frac{\partial \mathcal{L}}{\partial \Phi^I_{,\mu}} t^{IJ}_a \Phi^J;}$$

• Example: Complex scalar field.

Let's consider once again the Leagrangian

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - V(|\varphi|);$$

this Leagrangian is invariant under phase rotations:

$$\varphi \to e^{i\alpha}\varphi \qquad \varphi^* \to e^{-i\alpha}\varphi$$ wher $\alpha = \text{const}$.

this transformations in the infinetesemal form are

$$\varphi \to (1+i\alpha)\varphi$$
$$\varphi^* \to (1-i\alpha)\varphi^*$$

Hence to obtain conserved current we should put $\Phi^1 = \varphi$, $\Phi^2 = \varphi^*$, $\varepsilon = \alpha$, $t^{11} = i$; $t^{22} = -i$; $t^{12} = t^{21} = 0$; leading to the current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \varphi_{,\mu}} i\varphi + \frac{\partial \mathcal{L}}{\partial \varphi^*_{,\mu}}(-i)\varphi^* = i(\varphi \partial^\mu \varphi^* - \varphi^* \partial^\mu \varphi)$$ so

$$\boxed{j_\mu = i(\varphi \partial_\mu \varphi^* - \varphi^* \partial_\mu \varphi);}$$

this is the same current we observed before using Klein-Gordon equation.

• Translations.

Now let's consider translation in the space-time and their effect on the action. For the fields corresponding transformations are:

$$x^\mu \to x^\mu + \varepsilon^\mu$$
$$\Phi^I(x^\mu) \to \Phi'^I(x^\mu) = \Phi^I(x^\mu + \varepsilon^\mu) = \Phi^I(x^\mu) + \varepsilon^\mu \partial_\mu \Phi^I(x^\mu)$$

Then for the Leagrangian from one point of view changes according to:

(12) $$\mathcal{L}_e(\Phi', \Phi'_{,\mu}) = \mathcal{L}_e(x^\mu + \varepsilon^\mu) = \mathcal{L}_{e0} + \varepsilon^\mu \partial_\mu \mathcal{L}_e;$$

- At the same time:

$$\mathcal{L}_e(\Phi', \Phi'_{,\mu}) = \mathcal{L}_e(\Phi + \partial_\mu \Phi \cdot \varepsilon^\mu, \Phi_{,\mu} + \partial_\mu \partial_\nu \Phi \varepsilon^\nu) =$$

$$= \mathcal{L}_e(\Phi, \Phi_{,\mu}) + \frac{\partial \mathcal{L}_e}{\partial \Phi^I} \partial_\mu \Phi^I \cdot \varepsilon^\mu + \frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\mu}} \partial_\mu \partial_\nu \Phi^I \cdot \varepsilon^\nu$$

Now using Lagrange equations we get

$$\mathcal{L}_e(\Phi', \Phi'_{,\mu}) = \mathcal{L}_{e0} + \left(\partial_\nu \frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\nu}}\right) \partial_\mu \Phi^I \varepsilon^\mu + \frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\nu}} \partial_\nu \partial_\mu \Phi^I \varepsilon^\mu =$$

$$= \mathcal{L}_{e0} + \partial_\nu \left(\frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\nu}} \partial_\mu \Phi^I \right) \varepsilon^\mu$$

- Then comparing two results we obtain:

$$\partial_\mu \mathcal{L}_e = \partial_\nu \left(\frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\nu}} \partial_\mu \Phi^I \right) \Rightarrow \underline{\partial_\nu \left(\frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\nu}} \partial_\mu \Phi^I - \delta^\nu_\mu \mathcal{L}_e \right) = 0;}$$

- Hence our conserved current now is

$$\boxed{T^\mu{}_\nu = \frac{\partial \mathcal{L}_e}{\partial \Phi^I_{,\mu}} \cdot \Phi^I_{,\nu} - \delta_\mu{}^\nu \mathcal{L}_e;}$$ with the conservation law $\underline{\partial_\mu T^\mu{}_\nu = 0;}$

- Corresponding conserved charges are:

$$E = \int T^0{}_0 \, d^3x = \int d^3x \left(\frac{\partial \mathcal{L}_e}{\partial \dot{\Phi}^I} \dot{\Phi}^I - \mathcal{L}_e \right);$$ – field energy.

$$P_i = \int d^3x \cdot T^0{}_i = \int d^3x \left(\frac{\partial \mathcal{L}_e}{\partial \dot{\Phi}^I} \partial_i \Phi^I \right);$$ – field momentum.

- <u>Inambiguity of Noether currents.</u>

Notice that if we have conserved current j^μ then the current $j'^\mu = j^\mu + \partial_\nu f^{\nu\mu}$ is also conserved if $f^{\mu\nu}$ is antisymmetric in $\mu\nu$ indices. In particular for the stress-energy tensor $\underline{T^{\mu\nu} \to T^{\mu\nu} + \partial_\alpha \Omega^{\alpha\mu\nu};}$ with $\Omega^{\alpha\mu\nu} = -\Omega^{\mu\alpha\nu};$

In general stress-energy tensor is not symmetric i.e. $T^{\mu\nu} \neq T^{\nu\mu}$ however it can be symmetrized with the proper choice of $\Omega^{\alpha\mu\nu}$.

- Direct way to obtain symmetric stress-energy tensor is by varying action w.r.t. the metric $g_{\mu\nu}$. i.e.:

$$\boxed{T_{\mu\nu} = \frac{\partial(\sqrt{-g}\, \mathcal{L}_e)}{\partial g^{\mu\nu}};} \qquad g = \det(g_{\mu\nu})$$

(13) Example: Real scalar field.

$$\mathcal{L}_\varphi = \frac{1}{2} g^{\mu\nu} \partial_\mu\varphi \partial_\nu\varphi - \frac{1}{2} m^2 \varphi^2;$$

To variate this w.r.t metric we use the following equations:

- $\delta(g^{\mu\nu} g_{\nu\alpha}) = \delta(\delta^\mu{}_\alpha) = 0 \Rightarrow \delta g^{\mu\nu} \cdot g_{\nu\alpha} + g^{\mu\nu}\delta g_{\nu\alpha} = 0 \Rightarrow$

$\Rightarrow \delta g^{\mu\nu} \cdot g_{\nu\alpha} \cdot g^{\alpha\beta} = -\delta g_{\nu\alpha} \cdot g^{\mu\nu} \cdot g^{\alpha\beta} \Rightarrow \delta g^{\mu\nu} \cdot \delta_\nu{}^\beta = -g^{\mu\nu} g^{\alpha\beta} \delta g_{\nu\alpha} \Rightarrow$

$\Rightarrow \quad \delta g^{\mu\beta} = -g^{\mu\nu} g^{\beta\alpha} \delta g_{\nu\alpha};$

- $\log\det(g_{\mu\nu}) = \operatorname{tr} \log g_{\mu\nu} \Rightarrow \delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}$

- Let's now derive stress-energy tensor for the scalar field in 2 ways:

(1) Using variation w.r.t $g_{\mu\nu}$

$$\delta(\sqrt{-g}\,\mathcal{L}_\varphi) = (\delta\sqrt{-g})\,\mathcal{L}_\varphi + \sqrt{-g}\,\delta\mathcal{L}_\varphi = \sqrt{-g}\left(-\frac{1}{2} g_{\mu\nu}\mathcal{L}_\varphi + \frac{1}{2}\partial_\mu\varphi\cdot\partial_\nu\varphi\right)\delta g^{\mu\nu} =$$

$$= \sqrt{-g}\,\delta g^{\mu\nu}\left(-\frac{1}{2} g_{\mu\nu}\partial_\alpha\varphi\partial^\alpha\varphi + \partial_\mu\varphi\partial_\nu\varphi + \frac{1}{2} g_{\mu\nu} m^2\varphi^2\right)\cdot\frac{1}{2}$$

Hence: $\quad T_{\mu\nu} = \sqrt{-g}\left(\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2} g_{\mu\nu}\partial_\alpha\varphi\partial^\alpha\varphi + \frac{1}{2} g_{\mu\nu} m^2\varphi^2\right)$

Finally putting in flat metric we obtain:

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2} g_{\mu\nu}\partial_\alpha\varphi\partial^\alpha\varphi + \frac{1}{2} g_{\mu\nu} m^2\varphi^2;$$

(2) Using the expression:

$$T_{\mu\nu} = \frac{\partial\mathcal{L}_\varphi}{\partial\Phi^I{}_{,\mu}}\Phi^I{}_{,\nu} - g_{\mu\nu}\mathcal{L}_\varphi = \frac{\partial\mathcal{L}_\varphi}{\partial(\partial^\mu\varphi)}\partial(\partial_\nu\varphi) - g_{\mu\nu}\mathcal{L}_\varphi =$$

$$= \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\mathcal{L}_\varphi \Rightarrow T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2} g_{\mu\nu}\partial_\alpha\varphi\partial^\alpha\varphi + \frac{1}{2} g_{\mu\nu} m^2\varphi^2;$$

- Notice that the energy of field is given by

$$E \equiv \int d^3x\, T^{00} = \int\left(\partial^0\varphi\,\partial^0\varphi - \frac{1}{2}\partial^0\varphi\,\partial^0\varphi + \frac{1}{2}\partial^i\varphi\,\partial^i\varphi + \frac{1}{2}m^2\varphi^2\right)d^3x \Rightarrow$$

$\Rightarrow \quad E = \int d^3x\left(\frac{1}{2}(\partial_t\varphi)^2 + \frac{1}{2}(\vec\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2\right);$ – the same expression we have obtained before!

Lecture 10: Magnetic monopoles.

- Soliton in the model with $SU(2)$ gauge group.

- Simpliest model containing monopole solution is Georgi-Glashow model: gauge theory with $SU(2)$ group and triplet of real Higgs fields φ^a, $a=1,2,3$.

$$\mathcal{L}_\varphi = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2}(\mathcal{D}_\mu \varphi)^a (\mathcal{D}^\mu \varphi)^a - \frac{\lambda}{4}(\varphi^a\varphi^a - v^2)^2;$$

where $\mu, \nu = 0,1,2,3$ (four-dimensional space-time)

$$\begin{cases} (\mathcal{D}_\mu \varphi)^a = \partial_\mu \varphi^a + g\varepsilon^{abc} A^b_\mu \varphi^c; & A_\mu = -ig\frac{\tau^a}{2} A^a_\mu; \\ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon^{abc} A^b_\mu A^c_\nu; & \varphi = -i\frac{\tau^a}{2}\varphi^a; \end{cases} \quad A_\mu, \varphi \in SU(2);$$

- Ground state is chosen: $\varphi^1_0 = \varphi^2_0 = 0$, $\varphi^3_0 = v$;

Notice that generators wich annihilate this vacuum are given by: $f^{abc}\varphi^c_0 = 0$ As the only non-zero element of φ_0 is φ^3_0 then corresponding generator should be $T^3 = f^{3ab}$, hence there is only one generator annihilating ground state and hence only $U(1)$ symmetry is unbroken by this ground state.

- Gauge field corresponding to this $U(1)$ symmetry is $A_\mu \equiv A^3_\mu$. It stays massless.

- Two remaining components A^1_μ, A^2_μ become massive with the mass $m_v = gv$.

- There is also Higgs field of the mass $m_H = \sqrt{2\lambda}\, v$;
- All these spectrum can be derived in the unitary gauge $\varphi^1 = \varphi^2 = 0$; $\varphi^3 = v + \eta(x)$;

- It is convinient also to introduce "W"-Bosons:

$$W^\pm_\mu = \frac{1}{\sqrt{2}}(A^1_\mu \pm i A^2_\mu);$$ wich have $\pm g$ charges under unbroken $U(1)$ subgroup.

(2) • Let's now find static soliton solution in this gauge theory. As field configuration should be static we take $A_0^a = 0$ and $A_i^a = A_i^a(\bar{x}), \varphi^a = \varphi^a(\bar{x})$;

• Energy functional is given by:

$$E = \int d^3x \left[\frac{1}{4} F_{ij}^a \cdot F_{ij}^a + \frac{1}{2} (\mathcal{D}_i \varphi)^a (\mathcal{D}_i \varphi)^a + \frac{\lambda}{4} (\varphi^a \cdot \varphi^a - v^2)^2 \right];$$

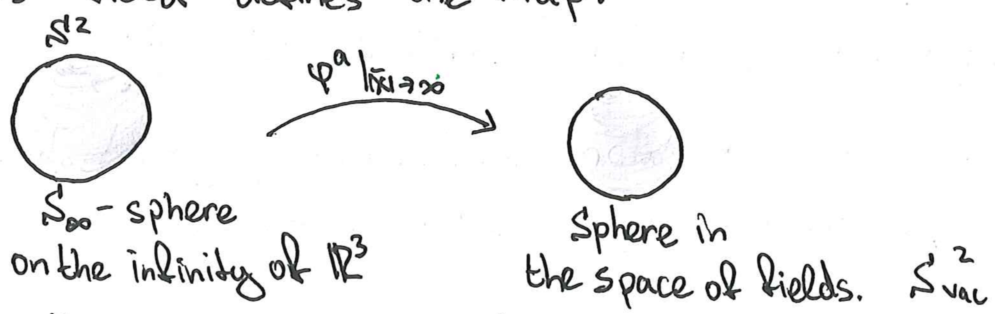
In order for the energy to be finite we demand

$$\varphi^a \varphi^a = v^2 \quad \text{as } |\bar{x}| \to \infty;$$

However the value of the field can "rotate" on the sphere defined by $\varphi^a \varphi^a = v^2$ condition, while we change position $\bar{x}$ on infinitely distinct S^2 (picture similar to the vortex).

$$\varphi^a|_{|\bar{x}| \to \infty} = \varphi^a(\bar{n}); \quad \bar{n} = \frac{\bar{x}}{r}; \text{ - unity length vector.}$$

This field defines the map:



• As the map is $S^2 \to S^2$ we can build homotopy classes as usually and characterize them by $\pi_2(S^2) = \mathbb{Z}$;

• So we can characterize maps with integer numbers $n = 0, \pm 1, \pm 2, \ldots$
Smooth trasformations (deformations) of the field $\varphi^a(\bar{n})$ doesn't lead to the change of topological number n, and hence field configuration always stays in the same topological sector.

• $n=0$ sector - vacuum sector, energy is minimized when field equals vacuum one everywhere.

• $n=1$ sector - contains soliton solution minimizing the energy. Let's find solution in this sector:

③ The most appropriate ansatz we can come up with is

$$\varphi^a(\bar{n}) = n^a \cdot v \quad \text{as } r \to \infty$$

This asymptotics is invariant under the combination of simultaneous rotations in real space and $SU(2)$ transformations of the fields, which can be described by

$$(\Lambda^{-1})^a{}_b \varphi^b(\Lambda^i_j n^j) = \varphi^a(n^i) \quad \Lambda \text{ is } SO(3) \text{ rotation matrix}$$

and $SO(3) \simeq SU(2)/\mathbb{Z}_2$;

- In order to have the finite energy of the soliton we aslo want $\mathcal{D}_i\varphi^a$ to decrease faster then $\frac{1}{r}$ at spatial infinity.
- We know at the same time that $\partial_i\varphi^a = v \cdot \partial_i n^a = \frac{1}{r} v(\delta^{ai} - n^i \cdot n^a) \to$ $\to$ not falling fast enough $\to$ should be compensated by the gauge field. An appropriate one is:

$$A_i^a = \frac{1}{gr}\varepsilon^{aij} n^j \quad \text{as } r \to \infty;$$

Then $A_i^b \varphi^c \cdot g\varepsilon^{abc} = g\varepsilon^{abc}\frac{1}{gr}\varepsilon^{bij} n^j \cdot n^c \cdot v = \frac{v}{r}(-\delta^{ai}\cdot\delta^{cj} + \delta^{aj}\cdot\delta^{ci}) n^j \cdot n^c =$

$= \frac{v}{r}(-\delta^{ai} + n^a n^i)$ so that the covariant derivative

$$(\mathcal{D}_i\varphi)^a \equiv \partial_i\varphi^a + g\varepsilon^{abc} A_i^b \varphi^c = 0, \text{ as } r \to \infty;$$

- Now we want to find ansatz for the fields that have the same symmetry as asymptote:

$$\boxed{\begin{aligned} \varphi^a &= n^a \cdot v(1 - H(r)); \\ A_i^a &= \frac{1}{gr}\varepsilon^{aij} n^j (1 - F(r)); \end{aligned}}$$

is the most general ansatz preserving rotations in space and fields + the symmetry $\varphi'(x) = -\varphi(-x)$, $A_i'(x) = -A_i(-x)$ of the energy functional.

Comment: If we don't impose the last condition (odd parity of fields) most general ansatz for gauge field is

$$A_i^a(\bar{x}) = n^i \cdot n^a \cdot a(r) + (\delta^{ai} - n^a n^i) f_1(r) + \varepsilon^{aij} \cdot n^j \cdot f_2(r);$$

④ But first two terms are even under $x \to -x$.

- Boundary conditions are

  - at $r \to \infty$: $F(r) = H(r) = 0$ as asymptotics should be as above
  - at $r \to 0$: $\left.\begin{array}{l} H(r) = 1 - O(r); \\ F(r) = 1 - O(r^2); \end{array}\right\}$ in order to have smooth functions at $r=0$.

- Without proof: Equations of motion reduce to ordinary differential equations for $F(r)$ and $H(r)$.

- This solutions will have the following structure similar to the one for the vortex:

$r = \infty$

$$F_\infty(r) \sim e^{-m_V r} \cdot C_F$$
$$H_\infty(r) \sim e^{-m_H r} \cdot C_H$$

$\uparrow$ two-parametric family of solutions.

$$\boxed{\begin{array}{l} F_0(r_0) = F_\infty(r_0) \\ F_0'(r_0) = F_\infty'(r_0) \\ H_0(r_0) = H_\infty(r_0) \\ H_0'(r_0) = H_\infty'(r_0) \end{array}}$$

$\uparrow$ 4 equations in some intermed. point fix 4 parameters.

$r = 0$

$$F_0(r) = 1 + \alpha_H r + \beta_H r^3 + \dots$$
$$H_0(r) = 1 + \alpha_F \cdot r^2 + \beta_F \cdot r^4 + \dots$$

$\uparrow$ two-parametric family of solutions. (β_H, β_F can be expressed through α_H, α_F).

- Solution can not be found analytically. Only numerically.

- Mass and size of the soliton.

To find the mass and the size of the soliton without finding the solution let's use dimensional analysis.

- We want to present everything in terms of dimensionless parameters as much as we can.

① Start with φ^a and potential term $V(\varphi) = \frac{\lambda}{4}(\varphi^a\varphi^a - v^2)^2$ it is obvious that we should introduce $f^a = v\varphi^a$ so that

$$V(f) = \frac{\lambda v^4}{4}(f^a f^a - 1)^2;$$

② Now we want F^a_{ij} and $\partial_i \varphi^a$ to transform

⑤ similarly in order to put common factor outside the Lagrangian, Let's assume:

$$\left.\begin{aligned} y^i &= \alpha \cdot x^i \\ B_i^a(y) &= c \cdot A_i^a(x) \end{aligned}\right\} \begin{aligned} &\text{new coordinates} \\ &\text{and gauge field} \end{aligned}$$

- As we wish $F_{ij}^a \sim \partial_i A_j^a + \dots$ to have the same factor as $\partial_i \varphi^a \sim v \partial_i f^a$ we conclude that $c^{-1} = v$ so that $A_i^a(x) = v B_i^a(y)$
- Finally we want covariant derivative not to contain any coupling which otherwise would define the scale of the corresponding term: $(\mathcal{D}_i \varphi)^a \to (\mathcal{D}_i f)^a \cdot v \equiv v \cdot (\partial_i f^a + g \varepsilon^{abc} {}^b{}_i f^c) =$
$= v(\alpha \cdot \frac{\partial}{\partial y^i} f^a + g v \varepsilon^{abc} B_i^b f^c) = g v^2 \underbrace{(\partial_{y^i} f^a + \varepsilon^{abc} B_i^b f^c)}_{(\tilde{\mathcal{D}}_i f)^a} = g v^2 \tilde{\mathcal{D}}_i f^a$

We see that in order to obtain uniform transformation of this term we need $\alpha = g v \Rightarrow y^i = g v \cdot x^i$;

- To summ up for the energy functional we get:

$E = \int d^3x \{\frac{1}{4}(F_{ij})^2 + \frac{1}{2}(\mathcal{D}_i \varphi)^a (\mathcal{D}_i \varphi)^a + \frac{\lambda}{4}(\varphi^a \varphi^a - v^2)^2\}$ goes to:

$E = \frac{v}{g} \int d^3y \{\frac{1}{4}(B_{ij}^a)^2 + \frac{1}{2}(\tilde{\mathcal{D}}_i f^a)^2 + \frac{\lambda}{4g^2}(f^a f^a - 1)^2\}$; $B_{ij}^a = \partial_i B_j^a - \partial_j B_i^a + \varepsilon^{abc} B_i^b B_j^c$;

- The only parameter here is $\frac{\lambda}{4g^2} = \frac{m_H^2}{8 m_v^2}$ let's assume it is of order 1

Then $f^a \sim 1$ and $B_i^a \sim 1$,

so that the energy of the soliton is $E \sim \frac{v}{g} \sim \frac{m_v}{g^2}$;

and the size of the soliton is of order 1 in terms of y_i or of order $\frac{1}{gv} \sim m_v^{-1}$ in terms of x $r_0 \sim m_v^{-1}$

- Notice that mass of the soliton is $M \sim \frac{m_v}{g^2}$ and Compton length is $\lambda \sim \frac{g^2}{m_v}$ so that $\frac{\lambda}{r_0} \sim g^2 \Rightarrow$ $\boxed{\text{Soliton is classical in the weak coupling } g^2 \ll 1}$

⑥ Magnetic charge.

- Let's introduce $\mathcal{F}_{\mu\nu} = -\frac{2}{v}\mathrm{tr}(F_{\mu\nu}\varphi) = \frac{1}{v}F^a_{\mu\nu}\varphi^a$ which is obviously gauge invariant.

In unitary gauge where $\varphi^a=\delta^{a3}\cdot v$ and A^3_μ fluctuations around zero correspond to the electromagnetic field (unbroken subgroup of $SU(2)$), $\mathcal{F}_{\mu\nu}=F^3_{\mu\nu}=F^{e.m.}_{\mu\nu}$ - electromagnetic field strength tensor.

- Now $\mathcal{F}_{\mu\nu}$ can be used in any gauge as it is gauge invariant. To give right expression for electromagnetic field though we need to read off massive gauge fields. Fortunatelly at $r\to\infty$ they are exponentially suppressed ($F(r)\sim e^{-m_V r}$ as $r\to\infty$) so at $r\to\infty$ limit only e.m. field contributes to $\mathcal{F}_{\mu\nu}$;

- Asymptotically at $r\to\infty$ $A^a_i = \frac{1}{gr}\varepsilon^{aij}n_j$;

Let's introduce magnetic field $\mathcal{H}_i \equiv -\frac{1}{2}\varepsilon_{ijk}\mathcal{F}_{jk} = \frac{1}{v}H^a_i\varphi^a$

where $H^a_i \equiv -\frac{1}{2}\varepsilon_{ijk}F^a_{jk}$;

Substituting asymptote into this expression we obtain:

$$H^a_i = -\frac{1}{2}\varepsilon_{ijk}2\left(\frac{1}{g}\partial_j\left(\varepsilon^{ake}n_e\cdot\frac{1}{r}\right)+\frac{1\cancel{g}}{2g^2}\varepsilon^{abc}\left(\varepsilon^{bje}\frac{n_e}{r}\right)\left(\varepsilon^{ckm}\frac{n_m}{r}\right)\right)$$

- first term: $-\frac{1}{g}\varepsilon_{ijk}\varepsilon^{ake}\cdot\left(-\frac{n_jn_e}{r^2}+\frac{1}{r^2}(\delta_{je}-n_jn_e)\right)=$

$$= +\frac{1}{g}(\delta_{ia}\delta_{je}-\delta_{ie}\delta_{ja})\cdot\frac{1}{r^2}(\delta_{je}-2n_jn_e)=\frac{1}{gr^2}(3\delta_{ia}-2\delta_{ia}-\delta_{ia}+2n_in_a)=\frac{2n_in_a}{gr^2}$$

- second term: $-\frac{1}{2g}\varepsilon_{ijk}\varepsilon^{abc}\varepsilon^{bje}\varepsilon^{ckm}\frac{n_en_m}{r^2}=+\frac{1}{2g}\frac{n_en_m}{r^2}\varepsilon^{abc}\varepsilon^{bje}\times$

$$\times(\delta_{ic}\delta_{jm}-\delta_{im}\delta_{jc})=\frac{1}{2gr^2}\Big(\underbrace{\varepsilon^{abi}\varepsilon^{bme}n_en_m}_{\text{0 due to symmetries}}-n_in_e\varepsilon^{abj}\varepsilon^{bje}\Big)=$$

$$=-\frac{1}{2gr^2}n_in_e(\delta^{ae}\delta^{bb}-\delta^{ab}\delta^{eb})=-\frac{1}{gr^2}n_an_i$$ so that $\boxed{H^a_i=\frac{n_in_a}{gr^2}}$;

(7) • Notice that

•• H_i^a is directed along n_i - radial direction in the real space.

•• H_i^a is directed along n^a or equivalently along φ^a in the space of fields. Notice that if we consider unitary gauge $\varphi^a = v\delta^{a3}$ only H_i^3 will survive wich corresponds to the unbroken electromagnetic subgroup, so that H_i^a really describes only magnetic field (no W-bosons components)

• Then finally $\mathcal{H}_i = \frac{1}{v} H_i^a \varphi^a = \frac{1}{gr^2} n_i n_a n_a = \frac{n_i}{gr^2}$

$$\boxed{\mathcal{H}_i = \frac{n_i}{gr^2} \text{ - magnetic field of the magnetic charge } g^{-1}}$$

• Another approach.

• Let's try to put our system into unitary gauge in order to make expressions for electromagnetic field more visible.

• There is no way to put theory into unitary gauge with non-singular transformation on all S^2_∞. This is because $\pi_2(SU(2)) = 0$ so that any gauge transformation is homotopic to identity transformation $\omega(\bar{n}) = 1$.

• Then gauge transformation reducing fields to unitary gauge exists only on the part of the sphere. For example everywhere except small region around the south pole (SP). Let's call it $\omega_N(\bar{n})$.

• There is anothe r such gauge transformation working everywhere except north pole ($\omega_S(\bar{n})$)

• Let's denote gauge transformed fields as φ^ω

$$(\varphi^{\omega_N})^a = (\varphi^{\omega_S})^a = v\delta^{a3} \equiv (\varphi_{vac})^a \quad \text{or} \quad (\varphi_{vac}^{\omega_N^{-1}})^a = (\varphi_{vac}^{\omega_S^{-1}})^a = n^a \cdot v.$$

everywhere except north and south pole

⑧ • Then $\varphi_{vac}^{\omega_S \omega_N^{-1}} = \varphi_{vac}$, so that $\Omega(\bar{n}) \equiv \omega_S \cdot \omega_N^{-1}$ is group transformation under which vacuum is symmetric

$\Rightarrow$ $\Omega(\bar{n}) \in U(1)_{e.m.}$

• Consider $\Omega(\bar{n})$ on the equator of S^2_∞

then $\Omega(\bar{n}) : \underset{\text{equator}}{S^1} \to U(1)_{e.m.}$

• As $\pi_1(U(1)) = \mathbb{Z}$ we have homotopic classes in this case.

• Assume $\Omega(\bar{n})$ is in trivial class. Then it can be continiously extended to the north sphere becoming $\tilde{\Omega}(\bar{n})$

Then:

$$\omega(\bar{n}) = \begin{cases} \omega_S(\bar{n}) & \text{south hemisph} \\ \tilde{\Omega}(\bar{n})\,\omega_N(\bar{n}) & \text{north. hemisph.} \end{cases}$$

continious on S^2_∞ and transforms $\varphi^a = n^a v \to \varphi^a = \delta^{a3} \cdot v$.

As this is not possible $\Rightarrow$ $\Omega(\bar{n})$ is in nontrivial homotopy class. (nontrivial winding number)

• Let's define on the equator

$$\Omega = e^{i f(\varphi) \tau^3}; \quad \varphi \text{ is angle on equator.}$$

then $f(2\pi) - f(0) = 2\pi n$, n winding number

it is actually the same as the degree of the map $S^2_\infty \to S^2_{vac}$;

• Let's go to the unitary gauge every where except south pole:

$$\hat{A}_i^N = \omega_N A_i \omega_N^{-1} + \omega_N \partial_i \omega_N^{-1};$$

On spatial infinity only e.m. field components survive

$$\hat{A}_i^N = \frac{g\tau^3}{2i} A_i^N$$

A_i^N e.m. field components.

• Similarly for $S^2_\infty \setminus \{\text{north pole}\}$:

$$\hat{A}_i^S = \omega_S A_i \omega_S^{-1} + \omega_S \partial_i \omega_S^{-1}; \quad \text{and} \quad \hat{A}_i^S = \frac{g\tau^3}{2i} A_i^S \text{ at } r \to \infty;$$

• A_i^S and A_i^N are related by $U(1)_{em.}$ gauge transformation

$$\hat{A}_i^S = \Omega \hat{A}_i^N \Omega + \Omega \partial_i \Omega^{-1} \Rightarrow \frac{g\tau^3}{2i} A_i^S = \frac{g}{2i} e^{if\tau^3} \cdot \tau^3 e^{-if\tau^3} A_i^N +$$

$$+ e^{if\tau^3} \cdot i \partial_i f \cdot \tau^3 e^{-if\tau^3} \Rightarrow \quad A_i^S = A_i^N + \frac{2}{g} \partial_i f;$$

(9) • Finally let's calculate magnetic flux through S^2_∞:

• magnetic field

$$\bar{\mathcal{H}} = \begin{cases} -\mathrm{rot}\,\bar{A}^N & \text{for north sphere} \\ -\mathrm{rot}\,\bar{A}^S & \text{south sphere} \end{cases}$$

$\mathrm{rot}\,\bar{A}^N = \mathrm{rot}\,\bar{A}^S$ everywhere except south and north poles.

• magnetic flux:

$$\Phi = \int_{S^2_\infty} \bar{\mathcal{H}} d\bar{s} = -\int_{\underset{\text{north hemisphere}}{\uparrow}{S^2_N}} \mathrm{rot}\,\bar{A}^N - \int_{\underset{\text{south hemisphere}}{\uparrow}{S^2_S}} \mathrm{rot}\,\bar{A}^S = -\int_{\partial S^2_N} \bar{A}^N \cdot d\bar{\ell} - \int_{\partial S^2_S} \bar{A}^S d\bar{\ell}$$

$\partial S^2_N = S^1$ - equator; $\partial S^2_S = -S^1$ - equator with the opposite orientation

then $\Phi = \int_{S^1} d\bar{\ell}\,(\bar{A}^S - \bar{A}^N) = \frac{2}{g}\int_{S^1} d\bar{\ell}\cdot\bar{\nabla}\cdot f(\varphi) =$

$$= \frac{2}{g}\int_0^{2\pi} d\varphi\, f'(\varphi) = \frac{2}{g}(f(2\pi) - f(0)) = 2\pi n \cdot \frac{2}{g} = \frac{4\pi n}{g} \Rightarrow \Phi = \frac{4\pi n}{g};$$

• So that we obtain magnetic monopole with the charge $\boxed{g_M = \frac{n}{g}}$ where n is degree of the map $S^2_\infty \to S^2_{vac}$

Generalizations for other models.

• Let's assume we have some theory with semisimple compact group G.

• Assume theory includes Higgs field φ transforming in the representation $T(g)$ of the gauge group G. The ground state is invariant under subgroup H: $T(h)\varphi_{vac} = \varphi_{vac}\ \forall h \in H$,

• Assume M_{vac} is the set of all vacua. G acts transitively on M_{vac} and H is the stationary subgroup of M_{vac}, hence:

$$M_{vac} = G/H;$$

• Static configurations have field φ equal to the vacuum values but can depend on the direction in space: $\varphi = \varphi_v(\bar{n})$;

• $\varphi_v(\bar{n})$ is the map between S^2_∞ (infinitely remote 2-sphere in three dimensions) to $M_{vac} = G/H$. These maps can be

(10) Broken into homotopic classes, that are characterized by the homotopic groups $\pi_2(M_{vac}) = \pi_2(G/H) = \pi_1(H)$

For the last line we use $\pi_1(G)=0$ for any Lie group G and $\pi_2(G)=0$ which is true for any compact group. except $U(1)$ and $SO(2)$ this is true

- If $\pi_1(H) \neq 0$ we can break configuration space of fields into non-intersecting topological sectors. Configuration with the minimal energy in each sector corresponds to the soliton.
- Hence monopoles always exist in theories with compact semi-simple gauge groups broken down to non simply-connected subgroups by the Higgs mechanism.
- As nature has electromagnetic gauge theory $U(1)_{e.m.}$ then any Grand Unification Theory with semi-simple gauge group.

BPS limit.

- Let's consider Georgi-Glashow theory in the Bogomol'nyi-Prasad-Sommerfield (BPS) limit when $m_H \ll m_V$ where $m_H = \sqrt{2\lambda}\,v$, $m_V = gv$; are masses of Higgs and vector bosons.

This happens when $\lambda \to 0$.

- As $\lambda \to 0$ we can just neglect potential term. Hence monopole should be just the minimum of the following energy functional:

$$E = \int d^3x \left[\frac{1}{2}(H_i^a)^2 + \frac{1}{2}(\mathcal{D}_i\varphi)^a(\mathcal{D}_i\varphi)^a\right]; \text{ here } H_i^a \equiv -\frac{1}{2}\varepsilon_{ijk}F_{jk}^a;$$

- Bogomol'nyi trick:

notice that: $\int d^3x \cdot \frac{1}{2}(H_i^a - \mathcal{D}_i\varphi^a)(H_i^a - \mathcal{D}_i\varphi^a) \geq 0$;

Equality works when $H_i^a = \mathcal{D}_i\varphi^a$;

- Then $$\underbrace{\int d^3x \left(\frac{1}{2}(H_i^a)^2 + \frac{1}{2}(\mathcal{D}_i\varphi)^a(\mathcal{D}_i\varphi)^a\right)}_{E} - \int d^3x \cdot H_i^a \mathcal{D}_i\varphi^a \geq 0$$

- Let's consider last term in details:

(11) $\int d^3x \cdot H_i^a D_i\varphi^a = \int d^3x \cdot H_i^a (\partial_i\varphi^a + g\varepsilon^{abc} A_i^b \cdot \varphi^c) = \int d^3x\, \partial_i(H_i^a \varphi^a) -$
$- \int d^3x \cdot \varphi^a (\partial_i H_i^a + g\,\varepsilon^{abc} A_i^b H_i^c) = \int_{S^2_\infty} d\Sigma^i \cdot H_i^a\varphi^a - \int d^3x\, \varphi^a (\partial_i H_i^a + g\varepsilon^{abc} A_i^b \cdot H_i^c)$

using stokes theorem we reduced to integration over infinitely remote sphere.

- Using Bianchi identity

$\varepsilon^{\mu\nu\alpha\beta} D_\nu F_{\alpha\beta}^a = 0 \Rightarrow$ for $\mu=0$ $\quad \varepsilon^{ijk}(\partial_i F_{jk}^a + g\,\varepsilon^{abc} A_i^b F_{jk}^c) = 0 \Rightarrow$

$\Rightarrow \partial_i H_i^a + g\varepsilon^{abc} A_i^b H_i^c = 0;$ $\rightarrow$ thus second term in $\int H_i^a D_i \varphi^a$ is zero

and $\int d^3x\, H_i^a D_i\varphi^a = \int_{S^2_\infty} H_i^a \varphi^a d\Sigma^i = v \int_{S^2_\infty} \mathcal{H}_i\, d\Sigma^i = \frac{4\pi v \cdot n}{g}$

- Finally $\boxed{E \geqslant \frac{4\pi n v}{g};}$

- Monopole corresponds to the minimum of energy so that in order to find this configuration we should solve:

$H_i^a = D_i\varphi^a$

We use the ansatz $\quad \varphi^a = v n^a h(r);$
$\quad A_i^a = \varepsilon^{aij} n_j \frac{1}{gr}(1 - F(r));$

Then $F_{jk}^a = \partial_j A_k^a - \partial_k A_j^a + \varepsilon^{abc} A_j^b A_k^c$ and $\varepsilon^{ijk} F_{jk}^a = 2\varepsilon^{ijk}(\partial_j A_k^a +$
$+ \frac{g}{2}\varepsilon^{abc} A_j^b A_k^c)$, then $H_i^a = -\frac{1}{2}\varepsilon_{ijk} F_{jk}^a = -\frac{\delta_{ia} - n_i n_a}{gr} F' + \frac{n_i n_a}{gr^2}(1 - F^2);$
$D_i\varphi^a = v\,\frac{\delta_{ia} - n_i n_a}{r} F h + v n_i n_a h';$

Then $H_i^a = D_i\varphi^a$ if $\quad F' = -gvhF;$ $\quad h' = \frac{1}{gvr^2}(1 - F^2);$ + Bound. cond. $F(0)=1,\ h(0)=0,$ $F(r\to\infty)=0,\ h(r\to\infty)=1,$

- Solutions of these equations and bdry conditions are

$$\boxed{F = \frac{\rho}{\sinh\rho},\quad h = \coth\rho - \rho^{-1}; \qquad \rho = gvr = \frac{r}{r_0},\quad r_0 = m_v^{-1};}$$

at $r \to \infty$: $\quad F(r) \sim e^{-r/r_0}$

$h(r) \sim 1 - \frac{c r_0}{r}$

slow decay (no e^{-r/r_0}) due to the zero Higgs mass.

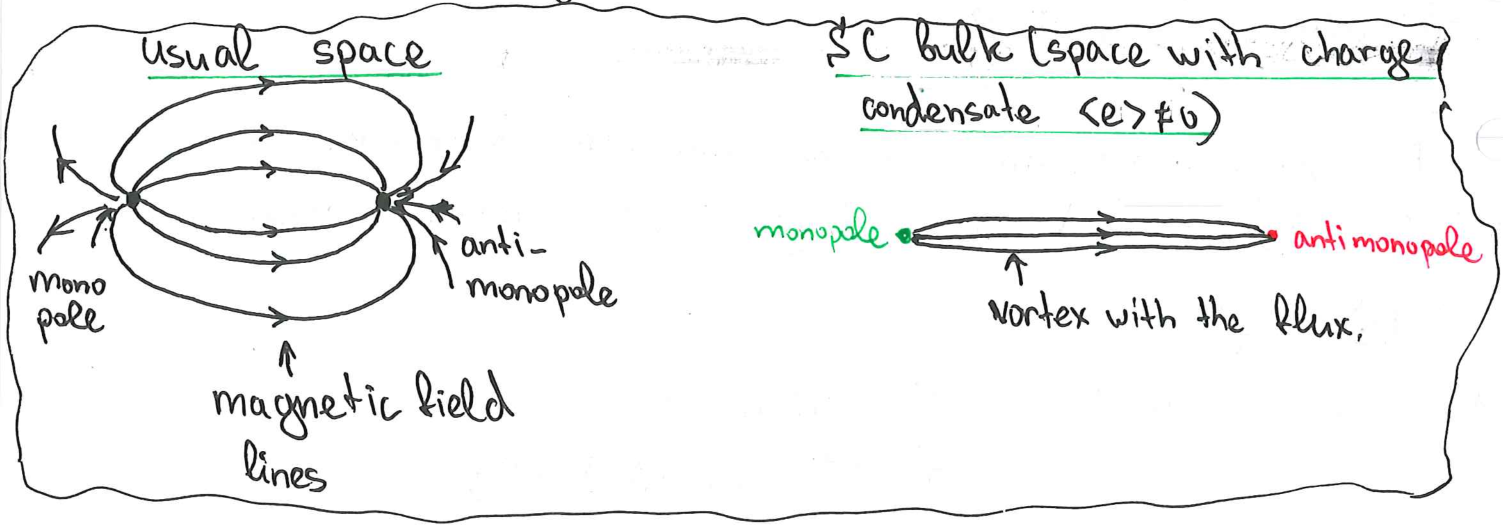
- Mass of the monopole is $\quad M = E = \frac{4\pi v}{g} = \frac{4\pi m_v}{g^2}$ – the same expression as in dimensional analysis.

(12) Dual superconducting model of the confinement.

• Usual superconductors.

• Can have monopoles inside. But as there can be no magnetic field in the bulk of SC monopoles should come in pairs and be connected by the vortex containing all magnetic flux coming from monopoles.

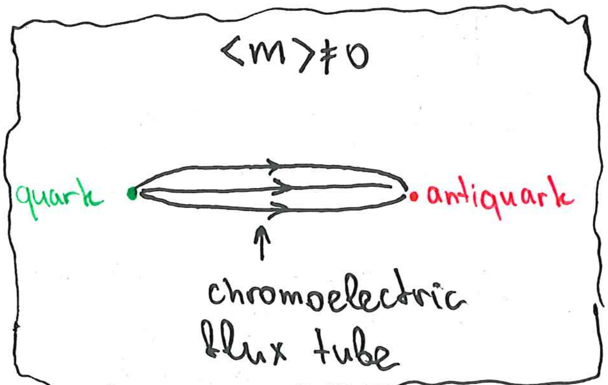
• In the bulk of SC there is condensate of Cooper pairs wich are charged hence:



• Notice that energy of the vortex is proportional to its length $E \sim L$ (see lecture 8) $\Rightarrow$ potential of monopoles interaction is also proportional to the distance between them

$$\boxed{V(r) = \sigma \cdot r \quad \Rightarrow \text{ confinement potential}}$$

• Now if we have monopole condensation instead of charge condensation roles of electric and magnetic fields are switched and we obtain charge confinement



Lecture 11: Instantons in gauge theories.

- Instantons are solitonic solutions presented in the euclidian gauge theories. So first of all we need to define gauge theories in euclidian space.

- Euclidian gauge theories

In Minkowski space-time we have theory that has the Leagrangian containing terms like $-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$ and $\frac{1}{2}|\mathcal{D}_\mu\varphi|^2$ for example. Here $F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+g f^{abc}A^b_\mu A^c_\nu$;

- Now we rotate the time direction $t=-i\tau$;

- If we do nothing with the fields at the same time then we would run into problem because $F^a_{0i}\equiv\partial_0 A^a_i-\partial_i A^a_0+g f^{abc}A^b_0A^c_i = i\partial_\tau A^a_i-\partial_i A^a_0+g f^{abc}A^b_0 A^c_i \rightarrow$ becomes complex $\Rightarrow$

$\Rightarrow$ equations of motion are also complex. Instead we would like to have something similar to the free scalar theory:

$$S=\int d^dx\left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-V(\varphi)\right)=\int d^{d-1}x\,dt\left(\frac{1}{2}(\partial_0\varphi)^2-\frac{1}{2}(\vec\nabla\varphi)^2-V(\varphi)\right)=$$

d-dimensional space-time

$$=\int d^{d-1}x\,d\tau\cdot(-i)\cdot(-1)\left[\frac{1}{2}(\partial_0\varphi)^2+\frac{1}{2}(\vec\nabla\varphi)^2+V(\varphi)\right]=$$

$$=i\int d^dx\left(\frac{1}{2}\underbrace{\partial_\mu\varphi\cdot\partial^\mu\varphi}_{\text{summation with euclidian metric}}+V(\varphi)\right)\equiv iS_E;$$

Where $S_E\equiv -iS$ is euclidian action.

- In order to have something similar in gauge theory it is reasonable to make the following change of variables:

$t=-i\tau$, $A^a_0\rightarrow iA^a_0$, $A^a_i\rightarrow A^a_i$, $\varphi\rightarrow\varphi$;

- Then $F_{0i}=i\partial_\tau A^a_i-i\partial_i A^a_0+igf^{abc}A^b_0A^c_i$ so that the action is

$$S\equiv\int d^{d-1}x\cdot dt\left(\frac{1}{2}F^a_{0i}\cdot F^a_{0i}-\frac{1}{4}(F^a_{ij})^2\right)\longrightarrow iS_E \text{ where } S_E=\int d^dx\,\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu};$$

- For the scalar field part

$$S_E=\int d^dx\left[(\mathcal{D}_\mu\varphi)^\dagger\mathcal{D}^\mu\varphi+V(\varphi^\dagger,\varphi)\right];\text{ where } \mathcal{D}_\mu\varphi=(\partial_\mu-ig\underbrace{T^a}_{\text{hermitian.}}A^a_\mu)\varphi$$

② Instanton in Yang-Mills theory

- We consider non-Abelian gauge theory (no scalar fields coupled) with euclidian action:

$$S_E = -\frac{1}{2g^2}\int \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\, d^4x$$

→ 4 dimensional space-time.

- We write gauge fields as $A_\mu = -ig\, t^a A_\mu^a$, where t^a are generators of Lie algebra normalized by $\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$.
- We want to consider field configurations with the finite energy ⇒ A_μ should decrease fast at the spatial infinity $r \to \infty$..
- Then it is reasonable to take A_μ as the pure gauge

$$A_\mu = \omega \partial_\mu \omega^{-1} \text{ as } |\vec{x}| \to \infty; \text{ where } \omega(x) \in G$$

- Let's consider infinitely remote sphere S^3. Then any function $\omega(x)$ is the map:

$$\omega(x): S^3 \to G;$$

- We know that $\pi_3(G) = \mathbb{Z}$ for any simple Lie group. (We have shown it for $G = SU(n)$, which is physically most important example, but this is true for any simple group.)
- Hence gauge functions (and corresponding field configurations) can be broken into non-intersecting classes characterized by integer number Q;

- We can assume that $\omega(x)$ depends only on the angles on S^3 but not it's radius. Let's assume that it depends on r: i.e. we have $\omega(r, n_\mu)$. Then we can built $\Omega(r, n_\mu) = \omega(R, n_\mu)\omega^{-1}(r, n_\mu)$ where R is some radius of fixed remote sphere.

$\Omega(r, n_\mu)$ is homotopic to the identity element of G, because $\Omega(R, n_\mu) = \mathbb{1}$ and we change r continiously. Hence if we start with some gauge field $A_\mu(\vec{x})$ and perform gauge transformation $\Omega(r, n_\mu)\, A_\mu(x) = A'_\mu(\vec{x})$, then the new field will belong to the same homotopic class as the original one.

③ Hence $\tilde{\omega}(\bar{x}) = \Omega(r,n_\mu)\cdot\omega(r,n_\mu) = \omega(R,n_\mu)$ is homotopic to $\omega(r,n_\mu)$ and we can always just fix $\omega(r,n_\mu)$ at some value of r.

- Construction described above leads to the topological classification of field configurations. Configuration minimizing the energy in each sector should satisfy Yang-Mills equation.
- Let's consider such solutions in $Q=1$ sector (instanton) and $Q=-1$ (antiinstanton) for the theory with $SU(2)$ gauge symmetry.
  - The expression for the topological charge is

$$Q = \frac{1}{24\pi^2}\underbrace{\int d\sigma_\mu}_{\text{integration over } S^3} \varepsilon^{\mu\nu\lambda\rho}\cdot \operatorname{tr}(\omega\partial_\nu\omega^{-1}\cdot\omega\partial_\lambda\omega^{-1}\cdot\omega\partial_\rho\omega^{-1});$$

- The fact that this is right topological charge can be checked by the direct calculation. In particular we can take group element to be

$$\omega(\bar{n}) = v_\alpha(\bar{n})\cdot\sigma_\alpha \text{ , where } \alpha=0,1,2,3,\ \sigma_0=1,\ \underbrace{\sigma_i=-i\tau_i}_{\text{Pauli matrices.}};$$

- If we substitute this into definition above we would get

$$Q = \underbrace{\int d^3\theta}_{\text{integral over the angles}}\ \underbrace{J(\theta)}_{\text{Jacobian of } v_\alpha(\bar{n}) \text{ maps.}}$$

This is exactly the formula we have derived before (see Lecture 9) for the degree of $S^n \to S^n$ mapping.

- The way to see that expression above is topological charge is to evaluate variation of Q under infinitesemal variation of $\omega(x)$. In particular let's substitute: $\omega'(x) = \omega(x)\cdot(1+i\varepsilon^a\tau^a)$;

$\varepsilon^a(x)$ is small

Then $$Q' = \frac{1}{24\pi^2}\int d\sigma_\mu \varepsilon^{\mu\nu\lambda\rho}\cdot\operatorname{tr}(\omega'\partial_\nu\omega'^{-1}\cdot\omega'\partial_\lambda\omega'^{-1}\cdot\omega'\partial_\rho\omega'^{-1})$$

Now notice that $\omega'\partial_\nu\omega'^{-1} = \omega(1+i\varepsilon^a\tau^a)\partial_\nu((1-i\varepsilon^b\tau^b)\omega^{-1}) =$

$= \omega\partial_\nu\omega^{-1} - i\omega\partial_\nu\varepsilon^a\tau^a\omega^{-1}$, and then:

④ $Q' = \frac{1}{24\pi^2}\int d\sigma_\mu \cdot \varepsilon^{\mu\nu\lambda\rho}\, tr(\omega\, \partial_\nu \omega^{-1}\, \omega\, \partial_\lambda \omega^{-1} (\omega\, \partial_\rho \omega^{-1} - 3i\, \omega\, \partial_\rho \varepsilon^a \tau^a \omega^{-1})) =$

$= Q - \frac{1}{8\pi^2}\int d\sigma_\mu\, \varepsilon^{\mu\nu\lambda\rho}\, tr(\cancel{\omega}\, \partial_\nu \omega^{-1}\, \omega\, \partial_\lambda \omega^{-1} \cdot \omega i \tau^a \partial_\rho \varepsilon^a \cancel{\omega^{-1}})$ → from contributions of 3 derivatives $(\nu\lambda\rho)$

Notice that

$\varepsilon^{\mu\nu\lambda\rho}\, tr\, \partial_\rho(\partial_\nu \omega^{-1}\, \omega\, \partial_\lambda \omega^{-1} \cdot \omega\, i\tau^a \varepsilon^a) = \varepsilon^{\mu\nu\lambda\rho}\, tr\{\overbrace{\partial_\rho \partial_\nu}^{\rho\leftrightarrow\nu\ \text{symm} \to \text{zero contrib}} \omega^{-1}\, \omega\, \partial_\lambda \omega^{-1}\, \omega\, i\tau^a \varepsilon^a +$

$+ \partial_\nu \omega^{-1} \cdot \underbrace{\omega\, \partial_\rho \omega^{-1}\, \omega}_{\partial_\rho \omega}\, \partial_\lambda \omega^{-1}\, \omega\, i\tau^a \varepsilon^a + \partial_\nu \omega^{-1}\, \omega\, \partial_\lambda \omega^{-1}\, \underbrace{\omega\, \partial_\rho \omega^{-1}\, \omega}_{\partial_\rho \omega}\, i\tau^a \varepsilon^a +$

this two terms cancell each other due to symmetry $\lambda \leftrightarrow \rho$

$+ \partial_\nu \omega^{-1}\, \omega\, \underbrace{\partial_\lambda \partial_\rho}_{\substack{\rho\leftrightarrow\nu\ \text{symmetric} \Rightarrow \\ \Rightarrow \text{doesn't contribute}}} \omega^{-1}\, \omega\, i\tau^a \varepsilon^a + \partial_\nu \omega^{-1}\, \omega\, \partial_\lambda \omega^{-1}\, \omega\, i\tau^a \partial_\rho \varepsilon^a\}$ → this is integrant in previous equation

hence $\quad Q' = Q - \frac{1}{8\pi^2}\int d\sigma_\mu\, \varepsilon^{\mu\nu\lambda\rho}\, \underbrace{\partial_\rho tr(\partial_\nu \omega^{-1}\, \omega\, \partial_\lambda \omega^{-1}\, \omega\, i\tau^a \varepsilon^a)}_{\text{full derivative}} = Q$

Thus Q does not change under small variations of $\omega(x)$. ⇒

⇒ Q is topological.

- Usefull examples of $S^3 \to S^3$ maps:

① $\omega^{(0)}(x) = 1$ identity map $Q=0$;

② $\omega^{(1)}(x) = \frac{1}{r}(x_0 + i x^i \tau^i)$, $Q=1$; $\quad x_0^2 + x^i x^i = 1 \quad i=1,2,3$

↑ equation of S^3 with unity radius.

③ $\omega^{(n)}(x) = [\omega^{(1)}(x)]^n$, $Q=n$;

- The formula for Q was explained in the cause of the gauge group $G = SU(2)$ but can be used for any simple group.

- Let's now notice the following. Assume we have term of the form $\quad \varepsilon^{\mu\nu\lambda\rho}\, tr(F_{\mu\nu} F_{\lambda\rho}) = 4\varepsilon^{\mu\nu\lambda\rho}\, tr((\partial_\mu A_\nu + A_\mu A_\nu)(\partial_\lambda A_\rho + A_\lambda A_\rho))$

Now notice

* $\varepsilon^{\mu\nu\lambda\rho}\, tr(\partial_\mu A_\nu\, \partial_\lambda A_\rho) = \varepsilon^{\mu\nu\lambda\rho}\, \partial_\mu tr(A_\nu \partial_\lambda A_\rho)$; (Because $\varepsilon^{\mu\nu\lambda\rho}\, \partial_\mu \partial_\lambda A_\rho = 0$)
* $\varepsilon^{\mu\nu\lambda\rho}\, tr(\partial_\mu A_\nu \cdot A_\lambda A_\rho) = \frac{1}{3}\varepsilon^{\mu\nu\lambda\rho}\, \partial_\mu tr(A_\nu A_\lambda A_\rho)$;
* $\varepsilon^{\mu\nu\lambda\rho}\, tr(A_\mu A_\nu \partial_\lambda A_\rho) = \frac{1}{3}\varepsilon^{\mu\nu\lambda\rho}\, \partial_\lambda tr(A_\mu A_\nu A_\rho) = \frac{1}{3}\varepsilon^{\mu\nu\lambda\rho}\, \partial_\mu tr(A_\nu A_\lambda A_\rho)$;
* $\varepsilon^{\mu\nu\lambda\rho}\, tr(A_\mu A_\nu A_\lambda A_\rho) = \binom{\text{using trace}}{\text{cyclisity}} = \varepsilon^{\mu\nu\lambda\rho}\, tr(A_\rho A_\mu A_\nu A_\lambda) = -\varepsilon^{\mu\nu\lambda\rho}\, tr(A_\mu A_\nu A_\lambda A_\rho)$ //0

⑤ • Hence $\mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}(F_{\mu\nu}F_{\alpha\beta}) =$

$= \varepsilon^{\mu\nu\alpha\beta}\cdot 2\,\mathrm{tr}\left(\partial_\mu\left(A_\nu\partial_\alpha A_\beta + \frac{2}{3}A_\nu A_\alpha A_\beta\right)\right) = \partial_\mu K^\mu$, where

$K^\mu = \varepsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\left(A_\nu F_{\alpha\beta} - \frac{2}{3}A_\nu A_\alpha A_\beta\right);$

• Comment: That is why we have never been using $\mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})$ terms in the Lagrangian → they are equivalent to full derivative and hence classicaly don't contribute to the equations of motion.

• Now let's consider integral

$$\underset{\substack{\uparrow\\\text{integral over}\\\text{infinetely remote sphere}}}{\int d\sigma_\mu}\, K^\mu = \int d\sigma_\mu\cdot\varepsilon^{\mu\nu\alpha\beta}\cdot\mathrm{tr}\left(F_{\nu\alpha}\cdot A_\beta - \frac{2}{3}A_\nu A_\alpha A_\beta\right)$$

• Notice that $F_{\nu\alpha}$ decrease faster then r^{-2} at $r\to\infty$, while $A_\mu \sim r^{-1}$ so we can neglect first term in the integral and write down:

$$\int d\sigma_\mu K^\mu = -\frac{2}{3}\int d\sigma_\mu\,\varepsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}(\omega\partial_\nu\omega^{-1}\cdot\omega\partial_\alpha\omega^{-1}\cdot\omega\partial_\beta\omega^{-1}) = -16\pi^2 Q$$

where we have used $A_\mu = \omega\partial_\mu\omega^{-1}$.

• At the same time $\int d\sigma_\mu K^\mu = \int d^4x\,\partial_\mu K^\mu = \int d^4x\,\mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})$;

So that we finally express topological charge as

$$\boxed{Q = -\frac{1}{16\pi^2}\int d^4x\cdot\mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu});}$$

• Now notice that $-\int d^4x\cdot\mathrm{tr}(F_{\mu\nu} \qquad)^2 \geq 0$ equality works if $F_{\mu\nu} = \tilde{F}_{\mu\nu}$;

• We also know that $F_{\mu\nu}F^{\mu\nu} = \tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$ then inequality is given by

$$-\int d^4x\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\cdot 2 + 2\int d^4x\,\mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu}) \geq 0$$

so that $\quad 2g^2 S - 16\pi^2 Q \geq 0 \Rightarrow S \geq \frac{8\pi^2}{g^2}Q$

• If Q is negative we should use $-\int d^4x\,\mathrm{tr}(F_{\mu\nu} + \tilde{F}_{\mu\nu})^2 \geq 0$ so that $S \geq \frac{8\pi^2}{g^2}(-Q)$ and we obtain $\boxed{S \geq \frac{8\pi^2}{g^2}|Q|;}$

⑥ • Minimum of the action is obtained for the field configurations satisfying

$$\boxed{\begin{array}{l} F_{\mu\nu}=\tilde{F}_{\mu\nu}\,, \text{ if } Q>0 \text{ - self-duality equation.} \\ F_{\mu\nu}=-\tilde{F}_{\mu\nu} \text{ if } Q<0 \text{ - anti self-duality equation.} \end{array}}$$

• Notice that due to the Bianchi identity

$\varepsilon^{\mu\nu\alpha\beta}D_\nu F_{\alpha\beta}=0 \Leftrightarrow D_\nu \tilde{F}^{\mu\nu}=0$ and for the self-dual (or anti self-dual) field configuration this leads to $D_\nu F^{\mu\nu}=0$ i.e. Yang-Mills equation is automatically satisfied for (anti-) self-dual configurations! (Other way around is not true)

• Let's now find particular instanton configuration. First of all we build asymptotics $r\to\infty$. It is given by pure gauge: $A_\mu=\omega\partial_\mu\omega^{-1}$; where $\omega(n^\mu)$ is the map belonging to the first nontrivial homotopy class of the map $S^3\to SU(2)$. Simpliest choise is as usually

$$v_\alpha(n_\mu)=n_\alpha; \quad \text{so that} \quad \omega=n_\alpha\cdot\sigma_\alpha;$$

• Then the gauge field is $A_\mu(r\to\infty)=\omega\partial_\mu\omega^{-1}=\sigma_\alpha\sigma_\beta^+ n_\alpha \frac{\delta_{\mu\beta}-n_\mu n_\beta}{r}$;

Then we can use $\sigma_\alpha\sigma_\beta^+=\delta_{\alpha\beta}+i\eta_{\alpha\beta a}\tau^a$, here $\eta_{\alpha\beta a}$ are called 't Hooft symbols and can be calculated by the direct substitution: $\sigma_\alpha=(1,-i\tau_i)$, $\sigma_\alpha^+=(1,i\tau_i)$;

$\alpha=0,\ \beta=i:\ \sigma_0\sigma_\beta^+=1 i\tau_i=i\tau_i \Rightarrow \eta_{0ia}=\delta_{ia}$;

$\alpha=i,\ \beta=0:\ \sigma_\alpha\sigma_0^+=-i\tau_i \Rightarrow \qquad \eta_{i0a}=-\delta_{ia}$;

$\alpha=i,\ \beta=j:\ \sigma_i\sigma_j^+=\tau_i\tau_j=\delta_{ij}+i\varepsilon_{ijk}\tau^k \Rightarrow \eta_{ija}=\varepsilon_{ija}$;

$\alpha=0,\ \beta=0\ :\ \sigma_0\sigma_0^+=1$;

So we conclude that 't Hooft symbols have the following values:

$$-\eta_{i0a}=\eta_{0ia}=\delta_{ia};\quad \eta_{ija}=\varepsilon_{ija};$$

• It can also be checked that $\eta_{\alpha\beta a}$ is self-dual under first two indices: $\frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}\eta_{\gamma\delta a}=\eta_{\alpha\beta a}$

(7) • This can be checked by direct substitution of the components we have evaluated previously. For example

$$\frac{1}{2}\varepsilon^{0ijk}\eta_{jka} = \frac{1}{2}\varepsilon^{ijk}\varepsilon_{jka} = \frac{1}{2}\cdot 2\,\delta^{ia} = \eta_{0ia};$$

$$\frac{1}{2}\varepsilon^{ij\alpha\beta}\eta_{\alpha\beta a} = \frac{1}{2}(\varepsilon^{ijk0}\eta_{k0a} + \varepsilon^{ij0k}\eta_{0ka}) = \varepsilon^{ijk0}\eta_{k0a} = -\varepsilon^{ijk}\cdot(-\delta_{ka}) = \varepsilon^{ija} = \eta^{ija};$$

So checking by components we see that these self-duality relations work properly.

• Then finally:

$$A_\mu = -i\eta_{\mu\alpha a}\frac{n_\alpha}{r}\tau_a,\ r\to\infty;$$

• Let's derive this expression: $A_\mu = (\delta_{\alpha\beta} + i\eta_{\alpha\beta a}\tau^a)n_\alpha\frac{\delta_{\mu\beta}-n_\mu n_\beta}{r}$

first of all $\delta_{\alpha\beta}n_\alpha(\delta_{\mu\beta}-n_\mu n_\beta) = n_\mu - n_\mu\cdot \underbrace{n^2}_{\to 1} = 0$

then also $\eta_{\alpha\beta a}n_\alpha n_\beta n_\mu = 0$ due to antisymmetry of $\eta_{\alpha\beta a}$ in first two indicies.

$i\eta_{\alpha\beta a}\tau^a n_\alpha\delta_{\mu\beta}\cdot\frac{1}{r} = \frac{i}{r}\eta_{\alpha\mu a}\tau^a n_\alpha = -i\eta_{\mu\alpha a}\tau^a n_\alpha;$

So that we arrive to the expression written above.

• Now using asymptotic behavior derived above we can make ansatz for the whole space

$$A_\mu = f(r)\cdot\omega\partial_\mu\omega^{-1} = -i\eta_{\mu\alpha a}\frac{n_\alpha}{r}f(r)\tau_a;$$

Then $\partial_\mu A_\nu - \partial_\nu A_\mu = -i\eta_{\nu\alpha a}\left(\frac{1}{r^2}(\delta_{\alpha\mu}-n_\alpha n_\mu) - \frac{n_\alpha n_\mu}{r^2}\right)f(r)\tau_a +$

$+ i\eta_{\mu\alpha a}\frac{1}{r^2}(\delta_{\alpha\nu}-2n_\alpha n_\nu)f(r)\tau_a - i\eta_{\nu\alpha a}\frac{n_\alpha n_\mu}{r}f'(r)\tau_a + i\eta_{\mu\alpha a}\frac{n_\alpha n_\nu}{r}f'(r)\tau_a =$

$= 2i\eta_{\mu\nu a}\frac{1}{r^2}f\tau_a + i\left(2\frac{f}{r^2} - \frac{f'}{r}\right)(\eta_{\nu\alpha a}n_\mu n_\alpha - \eta_{\mu\alpha a}n_\nu n_\alpha)\tau_a;$

and $[A_\mu, A_\nu] = -2i\eta_{\mu\nu a}\frac{f^2}{r^2}\tau_a - 2i\frac{f^2}{r^2}(n_\mu\eta_{\nu\alpha a}n_\alpha - n_\nu\eta_{\mu\alpha a}n_\alpha)\tau_a \to$ this should be checked by components. Then:

$$F_{\mu\nu} = 2i\eta_{\mu\nu a}\frac{f(1-f)}{r^2}\tau_a + i\left(2\frac{f(1-f)}{r^2} - \frac{f'}{r}\right)(n_\mu\eta_{\nu\alpha a}n_\alpha - n_\nu\eta_{\mu\alpha a}n_\alpha)\tau_a;$$

Then we can reduce self-duality equation to the nice form:

• first term $2i\eta_{\mu\nu a}\frac{f(1-f)}{r^2}\tau_a$ is self-dual due to 't Hooft symbol self-duality

⑧ • However second term is not self-dual in gene and thus we just put it to zero by assuming:

$$f' = \frac{2}{r} f(1-f);$$

• On top of this ODE we add boundary conditions

$$\begin{cases} f(r) \to 1 \text{ as } r \to \infty \text{ (from asymptotes)} \\ f(r) \to 0 \text{ as } r \to 0 \text{ (field should be regular at the origin)} \end{cases}$$

• Integrating this equation we get

$$2\int \frac{dr}{r} = \int \frac{df}{f(1-f)} \Rightarrow \log\left(\frac{r}{r_0}\right)^2 = \int \frac{df}{f} + \int \frac{df}{1-f} = \log \frac{f}{1-f} \Rightarrow \frac{f}{1-f} = \left(\frac{r}{r_0}\right)^2$$

r_0 is integration constant

• Then finally $f(r) = \frac{r^2}{r^2+r_0^2}$; r_0 plays role of instanton size and it is arbitrary.

• Then finally instanton solution looks like

$$\boxed{\begin{aligned} A_\mu^{inst} &= -i\eta_{\mu\nu a} x_\nu \tau_a \frac{1}{r^2+r_0^2}; \\ F_{\mu\nu}^{inst} &= 2i\eta_{\mu\nu a} \frac{f(1-f)}{r^2}\tau_a = 2i\eta_{\mu\nu a}\frac{r_0^2}{r^2+r_0^2}\tau_a; \end{aligned}}$$

• Instanton action is $S_I = \frac{8\pi^2}{g^2}$; $\to$ from Bogomol'nyi arguments but can also be derived from

$$S = -\frac{1}{2g^2}\int d^4x \, tr(F_{\mu\nu}^{inst})^2;$$

• Solution for anti-instanton is similar:

Now we should use gauge function $\tilde{\omega} = \omega^{-1} = \sigma_\alpha^+ n_\alpha$ to obtain

$$A_\mu^{anti.} = -i\bar{\eta}_{\mu\nu a} x_\nu \tau_a \frac{1}{r^2+r_0^2};$$

where $\bar{\eta}_{\mu\nu a}$ is anti-self-dual 't Hooft symbol:

$$\bar{\eta}_{0ia} = -\bar{\eta}_{i0a} = -\delta_{ia};$$

$$\bar{\eta}_{ija} = \varepsilon_{ija}$$

• Action for anti-instanton is still $S = \frac{8\pi^2}{g^2}$ by the Bogomol'nyi argument.

⑨ • θ-vacua.

• From now on let's try to give physical interpretation of instantons. For this we will go to the static gauge $A_0=0$. This gauge is usefull here because we don't need to do any operations on A_μ in order to perform Wick rotation.

• General expression for the instanton solution is

$$A_\mu^{inst} = -i\eta_{\mu\nu a} x^\nu \tau_a \frac{1}{r^2+r_0^2}$$ where r_0 is the size of instanton

• Let's assume there is gauge transformation Ω:

$$A_\mu = \Omega A_\mu^{inst.} \Omega^{-1} + \Omega \partial_\mu \Omega^{-1} = \Omega(A_\mu^{inst.} - \Omega^{-1}\partial_\mu \Omega)\Omega^{-1};$$

So that A_μ is in the static gauge. As we wish to have $A_0=0$ Ω should satisfy $\Omega\partial_0\Omega^{-1} = A_0^{inst} = -i x_a \tau_a \frac{1}{\bar{x}^2+x_0^2+r_0^2}$;

• This equation defines gauge function $\Omega(x_0,\bar{x})$ upto gauge transformation. independent of time. As $F_{ij}^{inst.} \to 0$ as $x^0=\tau\to\pm\infty$ we can use this remaining gauge symmetry to put:

$$A_i(\bar{x},\tau)\to 0 \text{ as } \tau\to-\infty;$$

If we adress our instanton solution

$$A_i^{inst.} = \left[i\delta_{ia}\frac{x_0}{\bar{x}^2+x_0^2+r_0^2} - i\varepsilon_{ija}\frac{x_j}{\bar{x}^2+x_0^2+r_0^2}\right]\tau_a;$$ notice that $A_i^{inst} \to 0$ as $x_0\to-\infty$

Hence $\partial_i\Omega(\bar{x},\tau\to-\infty)=0$ and we can assume $\Omega(\bar{x},\tau\to-\infty)=1$;

• Then we can choose $\Omega(\bar{x},\tau) = e^{-i\tau^a\hat{x}^a\cdot F(|\bar{x}|,\tau)}$, where $\hat{x}^a = \frac{x^a}{|\bar{x}|}$; and

$$F(|\bar{x}|,\tau) = \frac{|\bar{x}|}{\sqrt{|\bar{x}|^2+r_0^2}}\left(\operatorname{arctg}\frac{\tau}{\sqrt{|\bar{x}|^2+\rho^2}} + \frac{\pi}{2}\right);$$

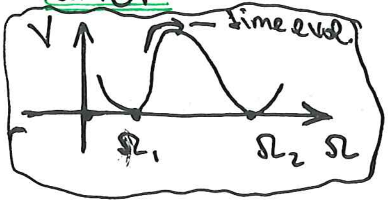
• Now let's find the asymptotic at $\tau\to+\infty$. As $A_i^{inst}\to 0$ as $\tau\to+\infty$ we conclude $A_i(\bar{x},\tau\to+\infty) = \Omega_1\partial_i\Omega_1^{-1}$; We again choose

$$\Omega_1(\bar{x}) = e^{-i\tau^a\hat{x}^a F_1(|\bar{x}|)} = \Omega(\bar{x},\tau\to+\infty)$$ and here $F_1(\bar{x}) = \pi\frac{|\bar{x}|}{\sqrt{|\bar{x}|^2+r_0^2}}$;

(10) • Hence instantons describe the transition between different vacua of the form $A_i = \tilde{\Omega}\partial_i\tilde{\Omega}^{-1}$ where $\tilde{\Omega} = \tilde{\Omega}(\bar{x})$ depends only on spatial coordinates.

• In order not to have divergent energy during the transition A_i shouldn't change at infinity, because otherwise $\int d^3x\, (\partial_0 A_i^a)^2$ will be divergent. So the transition are only between vacua with the same asymptote of Ω which we can take to be $\tilde{\Omega}(|\bar{x}| \to \infty) = 1$ for all vacua.

• Sometimes different vacua are separated by potential barier



But sometimes they are not separated. In this case we can connect different vacua by transforming continiously gauge function $\tilde{\Omega}(x)$. In this case there exists path in the set of classical vacua, connecting two of them.

Comment: By the potential barrier we mean hear the static energy of the field configuration $\mathcal{E}_{stat.} = -\frac{1}{2g^2}\int d^3x\, tr(F_{ij}F_{ij})$;

• Notice that we consider gauge transformations $\Omega(x)$ that at each fixed slice of time satisfy $\tilde{\Omega}(|\bar{x}| \to \infty) = 1$. Hence at each moment of time we can identify all the points at the infinity thus space can be considered as S^3 and $\Omega(\bar{x})$ are the maps $S^3 \to G$ which can be classified into homotopic classes according to $\pi_3(G) = \mathbb{Z}$ (classes are characterized by the integer numbers)

• Notice that if two vacua are in the same homotopic class we can connect them by continious family of gauge transformations hence they are not separated by the barriers. However homotopically inequivalent vacua are separated by the barriers.

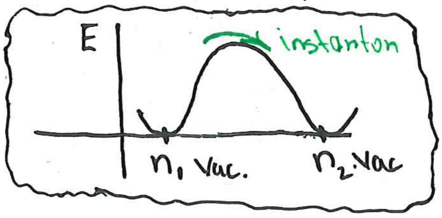
- Instanton = configuration interpolating between $A=0$ vacuum (0 topological charge) and vacuum with topological charge 1.
- In general top. charge of the vacuum can be written as

$$n(\tilde{\Omega})=\frac{1}{24\pi^2}\int d^3x\, \varepsilon^{ijk}\, \mathrm{tr}\left(\tilde{\Omega}\partial_i\tilde{\Omega}^{-1}\cdot\tilde{\Omega}\partial_j\tilde{\Omega}^{-1}\cdot\tilde{\Omega}\partial_k\tilde{\Omega}^{-1}\right)$$

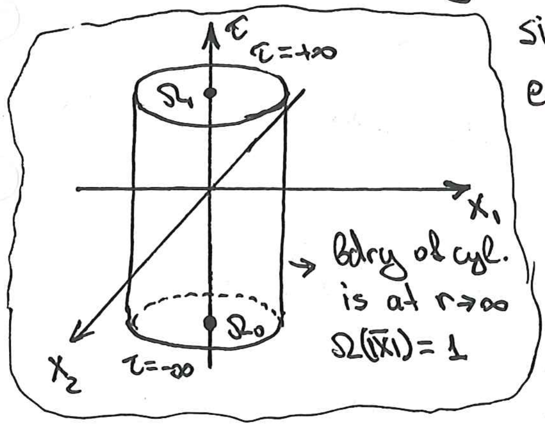
If we put $\tilde{\Omega}=\Omega_1=e^{i\tau^a\hat{x}^a F_1(|\bar{x}|)}$ with $F_1(|\bar{x}|)=\pi\frac{|\bar{x}|}{\sqrt{|\bar{x}|^2+r_0^2}}$;

we directly obtain $n(\Omega_1)=1$;

- Comment: Notice that top. number of the vacuum and top. charge of euclidian configuration are different (though related) objects.
- General statement: Any field config. with finite energy and top. charge $Q=-\frac{1}{16\pi^2}\int d^4x\, \mathrm{tr}(F_{\mu\nu}\tilde{F}^{\mu\nu})$, in the static gauge $A_0=0$ interpolates between two vacua with top. numbers n_1 and n_2: $Q=n_2-n_1$;



- To show that this statement is true let's consider the cylinder, with the side surface at infinity $|\bar{x}|\to\infty$ at each moment of time



At this side surface A_i does not change because $\Omega(\bar{x})=1$ and we can put $A_i=0$ on it so that only integral over top and botom where

$$\begin{cases} A_i(\bar{x},\tau\to-\infty)=\Omega_0\partial_i\Omega_0^{-1} & \text{- initial vacuum} \\ A_i(\bar{x},\tau\to+\infty)=\Omega_1\partial_i\Omega_1^{-1} & \text{- final vacuum.} \end{cases}$$

Then topologigal charge is

$$Q=\frac{1}{24\pi^2}\int_{\text{cylinder}} d\sigma_\mu\, \varepsilon^{\mu\nu\lambda\rho}\,\mathrm{tr}(A_\nu A_\lambda A_\rho)=\frac{1}{24\pi^2}\int_{\text{top}} d^3x\, \varepsilon^{ijk}\,\mathrm{tr}\left(\Omega_1\partial_i\Omega_1^{-1}\cdot\Omega_1\partial_j\Omega_1^{-1}\cdot\Omega_1\partial_k\Omega_1^{-1}\right)-$$

(12) $-\frac{1}{24\pi^2}\int_{\text{Bottom}} d^3x\, \varepsilon^{ijk}\, tr\,(\Omega_0 \partial_i \Omega_0^{-1}\, \Omega_0 \partial_j \Omega_0^{-1}\, \Omega_0 \partial_k \Omega_0^{-1}) \equiv n(\Omega_1) - n(\Omega_0)$

So we have shown that $\underline{Q = n(\Omega_1) - n(\Omega_0)}$;

- We have thus many vacua with different topological numbers n: $\underline{A_i^{(n)}(x) = \Omega_n(x)\partial_i \Omega_n^{-1}(\bar{x})}$;

Transitions between two vacua correspond to instanton ($n \to n+1$) or antiinstanton ($n \to n-1$) configurations.

- Let's introduce operator of gauge transformation $\hat{T} = \hat{T}(\Omega_1)$

$$\hat{T}^{-1} A_i \hat{T} \equiv \Omega_1 A_i \Omega_1^{-1} + \Omega_1 \partial_i \Omega_1^{-1}$$

As theory is gauge invariant T commutes with the Hamiltonian of the system $[\hat{T}, \hat{H}] = 0$, hence we can diagonalize i simultaneously with the Hamiltonian:

$$\hat{T}|\Psi_\theta\rangle = e^{i\theta}|\Psi_\theta\rangle;$$

- Let's introduce $|n\rangle$ - states with wave-functions localized around vacuum with top. number n. Then:

$$|\theta\rangle = \sum_{n=-\infty}^{+\infty} e^{-i\theta n}|n\rangle \text{ is the eigenstate of } T:$$

$$T|\theta\rangle = \sum_{n=-\infty}^{+\infty} e^{-i\theta n} \underbrace{T|n\rangle}_{|n+1\rangle} = \sum_{n=-\infty}^{+\infty} e^{-i\theta(n+1)} \cdot e^{i\theta} \cdot |n+1\rangle = e^{i\theta}|\theta\rangle;$$

- If we act with gauge-inv operator $\hat{O}$ on θ-vacuum $|\theta\rangle$ we will obtain the state $|\Psi_\theta\rangle$ with the same eigenvalue of $\hat{T}$:

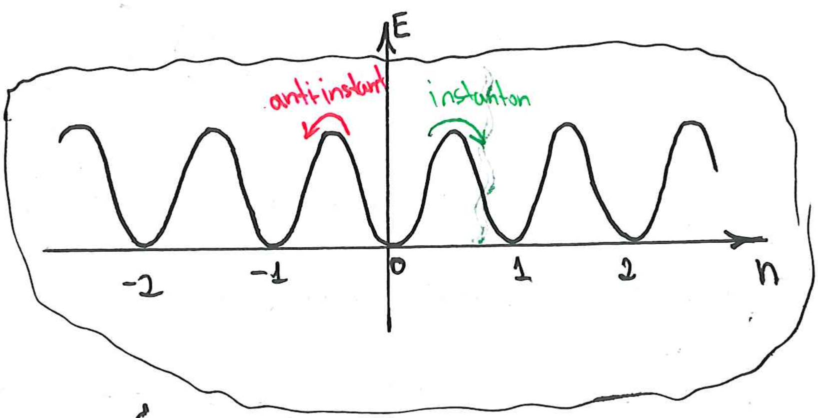
$$\hat{T}|\Psi_\theta\rangle \equiv \hat{T}\hat{O}|\theta\rangle = \hat{O}\hat{T}|\theta\rangle = e^{i\theta}|\Psi_\theta\rangle$$

- Hence θ is integral of motion and can be considered as one more constant of coupling.

- θ-dependence of physical quantities comes from expectation values

$$\langle\theta|\hat{O}|\theta\rangle \sim ... + e^{i\theta} \underbrace{\langle n|\hat{O}|n+1\rangle}$$

these kind of matrix elements are proportional to $\langle n|n+1\rangle \sim e^{-S_{inst}}$



(13) • On the level of the Lagrangian θ parameter is introduced by adding extra term

$$S_{inst} = -\frac{\theta}{16\pi^2}\int d^4x \cdot tr(F_{\mu\nu}\tilde{F}^{\mu\nu});$$

• As this term can be written as full derivative it does not contribute into equations of motion. However on the instanton configuration this term is nonzero and creates θ-dependence of observables.

• Notice that S_{inst} term is odd under CP-symmetry. However we know from experiment that $\theta < 10^{-9}$ and QCD does not seem to break CP-symmetry. One of the ways to solve this problem is to introduce axions.

Lecture 12 Quantization of gauge fields

- Scheme for quantization of the scalar field (reminder)

① Start with KG action

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 \Rightarrow \partial_\mu\partial^\mu\varphi + m^2\varphi = 0$$

② Write down the general solution of the form:

$$\varphi(x) = \underbrace{\int\frac{d^3k}{(2\pi)^3\,2\omega}}_{\text{Lorent inv. volume element.}}\left[a(\bar{k})e^{-ikx} + a^*(\bar{k})e^{ikx}\right] \quad \text{where } k^0 = \omega = \sqrt{|\bar{k}|^2 + m^2};$$

Or inverting $\quad a(\bar{k}) = i\int d^3x\cdot e^{+ikx}\overleftrightarrow{\partial_0}\varphi(x)$ where

$$f\overleftrightarrow{\partial_\mu}g = f\partial_\mu g - \partial_\mu f\cdot g;$$

③ Consider your field as the coordinate and derive canonical momentum together with the Hamiltonian.

$$\pi(x) = \frac{\partial\mathcal{L}_0}{\partial\dot{\varphi}} = \dot{\varphi}(x);\quad \mathcal{H} = \pi\dot{\varphi} - \mathcal{L}_0 = \frac{1}{2}\pi^2 + \frac{1}{2}(\bar{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2;$$

④ Make fields and canonical momentum an operators with the commutation relations:

$$\begin{cases} [\varphi(x),\varphi(x')]\big|_{t=t'} = 0;\quad [\pi(\bar{x},t),\pi(\bar{x}',t)] = 0; \\ [\varphi(x,t),\pi(x',t)] = i\delta^{(3)}(\bar{x}-\bar{x}'); \end{cases}$$

⑤ Then for $a(\bar{k})$ we get: $\mathcal{H} = \frac{1}{2}\omega\left(a^*(\bar{k})a(\bar{k}) + a(\bar{k})a^*(\bar{k})\right)$

usual harmonic oscillator.

and $[a(\bar{k}),a(\bar{k}')] = [a^*(\bar{k}),a^*(\bar{k}')] = 0$, $[a(\bar{k}),a^+(\bar{k}')] = (2\pi)^3\cdot 2\omega\cdot\delta^{(3)}(\bar{k}-\bar{k}')$;

Hence we have usual h.o. problem.

⑥ Using LSZ reduction formula reduce scattering amplitude to the correlation function:

- initial state $|i\rangle = \lim\limits_{t\to-\infty} a_1^+(t)\,a_2^+(t)\ldots|0\rangle \qquad a_i^+ \equiv \int d^3k\, f_i(\bar{k})\,a^+(\bar{k})$

some wave packet function $f_i(\bar{k}) \sim \exp\left(-\frac{(\bar{k}-\bar{k}_i)^2}{4\sigma}\right)$

- final state $|f\rangle = \lim\limits_{t\to+\infty} a_1(t)\,a_2(t)\ldots|0\rangle$

(6) • Then it can be shown:

$$\boxed{\underset{n'\text{ part.}\quad n\text{ part.}}{\langle f|i\rangle} = i^{n+n'}\int dx_1\, e^{ik_1x_1}(-\partial_1^2+m^2)\dots\dots dx_i'\, e^{-ik_i'x_i'}(-\partial_{1'}^2+m^2)\times \times\langle 0|\underset{\text{time ordering}}{T}\varphi(x_1)\dots\varphi(x_1')\dots|0\rangle;}$$

(7) Using path integral derive propogator:

$$\langle 0|T\varphi(x_1)\cdot\varphi(x_2)|0\rangle = \frac{1}{i}\cdot\frac{\delta}{\delta J(x_1)}\cdot\frac{1}{i}\cdot\frac{\delta}{\delta J(x_2)}\,Z_0(J)\Big|_{J=0}$$

where $Z_0(J)=\int D\varphi\, e^{iS+i\int J\varphi}$;

And using Wick's theorem we reduce higher correlators to the product's of propogators.

(8) In interacting theory expand in coupling constant

$$Z(J)=\int D\varphi\, e^{iS+igS_{int}+i\int J\varphi}=\int D\varphi\,\left(1+igS_{int}+\frac{1}{2}g^2S_{int}^2+\dots\right)e^{iS+i\int J\varphi}$$

So that correlators become

$$\langle 0|T\varphi(x_1)\varphi(x_2)\dots\varphi(x_n)|0\rangle=\frac{1}{i}\frac{\delta}{\delta J(x_1)}\cdot\frac{1}{i}\frac{\delta}{\delta J(x_2)}\dots e^{ig\int d^4x\,\mathcal{L}_{int}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)}\,Z_0[J];$$

• Now let's try to go through the similar procedure for gauge fields, in particular we start with electrodynamics and consider

$$\mathcal{L}_0=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\qquad\text{action.}$$

• However this action does not contain $\partial_0 A_0$ term and hence A_0 field doesn't have the dynamics and no canonically conjugated momentum hence $[A_0,\Pi^0]=i\delta(\bar{x}-\bar{y})$ cannot be satisfied.

• To solve this problem let's fix the gauge (for example Coulomb gauge $\bar{\nabla}\cdot\bar{A}=0$)

Then the Leagrangian takes the form:

$$\mathcal{L}_0=+\frac{1}{2}F_{0i}F_{0i}-\frac{1}{4}F_{ij}F_{ij}=\frac{1}{2}(\partial_0A_i-\partial_iA_0)(\partial_0A_i-\partial_iA_0)-$$

③ $-\frac{1}{4}(\partial_i A_j - \partial_j A_i)(\partial_i A_j - \partial_j A_i) = \frac{1}{2}\dot{A}_i \dot{A}_i - \frac{1}{2}\partial_i A_j \partial_i A_j + \frac{1}{2}\partial_i A_j \partial_j A_i +$
$+\frac{1}{2}\partial_i A_0 \partial_i A_0 - \partial_0 A_i \partial_i A_0 = $ (integrating by parts) $=$
$= \frac{1}{2}\dot{A}_i \dot{A}_i + \frac{1}{2}\partial_i A_0 \partial_i A_0 - \frac{1}{2}\partial_i A_j \partial_i A_j - \frac{1}{2}\partial_i \underbrace{\partial_j A_j}_{0} \cdot A_i + \partial_0 \underbrace{\partial_i A_i}_{0} A_0$

So the field A_0 satisfies Laplace equation
$\partial_i^2 A_0 = 0$ and in the absence of sources can be just putted to the constant (c-number)

- Conjugate momenta

$\pi^i = \frac{\partial \mathcal{L}_e}{\partial \dot{A}_i} = \dot{A}_i$ and the Hamiltonian is given by

$\mathcal{H} = \pi^i \dot{A}_i - \mathcal{L}_e = \frac{1}{2}\pi^i \pi^i + \frac{1}{2}\partial_i A_j \partial_i A_j = \frac{1}{2}(\bar{E}^2 + \bar{B}^2);$

- The only nontrivial commutator in the canonical quantization is

$[\pi^i(\bar{x},t), A_j(\bar{x}',t)] = [\dot{A}_i, A_j] = ?$ let's suppose that this commutator is $-i\,\delta_{ij}\,\delta^{(3)}(\bar{x}-\bar{x}')$

- Unfortunatelly it can't be the case becaus $\partial_i \dot{A}_i = \partial_i E_i = 0$ due to the Maxwell equations.

Hence $\frac{\partial}{\partial x^i}[\pi^i(x,t), A_j(\bar{x}',t)] = 0$, but

$-i\,\delta_{ij}\frac{\partial}{\partial x^i}\delta^{(3)}(\bar{x}-\bar{x}') = \int \frac{d^3k}{(2\pi)^3} k_j e^{i\bar{k}(\bar{x}-\bar{x}')} \neq 0$ and is divergent at $\bar{x}=\bar{x}'$

- To remove divergence we modify commutation relations by adding on the r.h.s. term proportional to k_j:

$$\delta_{ij}\delta^{(3)}(\bar{x}-\bar{x}') \to \tilde{\delta}_{ij}(\bar{x}-\bar{x}') \equiv \int \frac{d^3k}{(2\pi)^3} e^{i\bar{k}(\bar{x}-\bar{x}')}\left(\delta_{ij} - \frac{k_i k_j}{\bar{k}^2}\right)$$

Notice that now $\partial_i \tilde{\delta}_{ij}(\bar{x}-\bar{x}') = \int \frac{d^3k}{(2\pi)^3} e^{i\bar{k}(\bar{x}-\bar{x}')}(k_j - k_j) = 0$ and commutation relations are consistent with Maxwell equations

- Notice that $\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)$ is projector on the Coulomb gauge

Indeed if we take $A_i(\bar{x})$ and Fourier transform

④ then $\nabla_i \Pi_{ij} A_j = \int \frac{d^3\bar{k}}{(2\pi)^3} e^{i\bar{k}\bar{x}} \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) A_j \cdot k_i = 0$

In the real space we can write this projector as

$$\Pi_{ij} = \left(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta}\right)$$

- As we discussed the general solution of Maxwell equation

$$\bar{A}(x,t) = \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^3 2\omega} \left(\bar{\varepsilon}^*_\lambda a_\lambda(\bar{k}) e^{-ikx} + \bar{\varepsilon}_\lambda(\bar{k}) \cdot a^+_\lambda(\bar{k}) e^{+ikx}\right);$$

Where $\varepsilon_\pm(\bar{k})$ are two polarization vectors that should be orthogonal to $\bar{k}$ to satisfy Coulomb gauge condition, and also $\bar{\varepsilon}_\lambda$ should be orthogonal and we choose them to be normalized to one:

$$\bar{k} \cdot \bar{\varepsilon}_\lambda(\bar{k}) = 0; \quad \varepsilon_{\lambda'}(\bar{k}) \cdot \varepsilon^*_\lambda(\bar{k}) = \delta_{\lambda'\lambda}; \quad \sum_{\lambda=\pm} \varepsilon^*_{i\lambda}(\bar{k}) \cdot \varepsilon_{j\lambda}(\bar{k}) = \delta_{ij} - \frac{k_i k_j}{k^2};$$

Hence these two vectors form orthonormal set, together with unit vector in $\bar{k}$ direction this set is complete.

- Now if we substitute this solution into commutation relation and Hamiltonian we obtain:

only nontrivial commutator:

$$[a_\lambda(\bar{k}), a^+_{\lambda'}(\bar{k}')] = (2\pi)^3 2\omega\, \delta^{(3)}(\bar{k}-\bar{k}') \cdot \delta_{\lambda\lambda'};$$

Hamiltonian:

$$H = \sum_{\lambda=\pm} \int \frac{d^3k}{(2\pi)^3} \omega\, a^+_\lambda(\bar{k}) a_\lambda(\bar{k}) + \dots$$

So we have usual story similar with scalar fields but now more components.

- LSZ relation

Inverting solution: $a^+_\lambda(\bar{k}) = -i\, \varepsilon^*_\lambda(\bar{k}) \int d^3x\, e^{ikx} \overleftrightarrow{\partial_0} \bar{A}(x);$

$a_\lambda(\bar{k}) = i\, \varepsilon_\lambda(\bar{k}) \cdot \int d^3x \cdot e^{-ikx} \overleftrightarrow{\partial_0} \bar{A}(x);$

then
$$\begin{cases} a^+_\lambda(\bar{k})_{in} \to i\, \varepsilon^{*\mu}_\lambda(\bar{k}) \int d^4x\, e^{-ikx} (-\partial^2) A_\mu; \\ a_\lambda(\bar{k})_{out} \to i\, \varepsilon^\mu_\lambda(\bar{k}) \int d^4x\, e^{ikx} (-\partial^2) A_\mu; \end{cases}$$

⑤ • Finally calculation of the propogator results in:

$$\Delta^{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x'-x)}}{k^2+i\varepsilon} \underbrace{\sum_{a=1} \varepsilon^{\mu}_{a}(\bar{k})\, \varepsilon^{\nu}_{a}(\bar{k})}_{-g_{\mu\nu}+\eta_\nu\eta_\mu-\hat{k}_\mu\hat{k}_\nu} \qquad \hat{k}_\mu = \frac{k_\mu-(k\cdot\eta)\eta_\mu}{\sqrt{(k\cdot\eta)^2-k^2}};$$

where $\eta_\nu = (1,0,0,0)$

• It appears that usually parts of the propogator containing momentum components k_μ are not relevant for physical observables because the correspond to the terms $\partial_\mu J^\mu$ (J^μ is the current), and as currents are conserve this is just zero. To see this notice that in the partition function we always have

$$Z_0(J) = \exp\left(\frac{i}{2}\int d^4x\, d^4y\, J_\mu(x)\, \Delta^{\mu\nu} J_\nu(y)\right)$$

then if $\Delta^{\mu\nu}(x-y) \sim \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} k^\mu k^\nu$ and

$$\int d^4y\, \Delta^{\mu\nu}(x-y) J_\nu(y) \infty \int d^4y \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} k^\mu k^\nu \int \frac{d^4p}{(2\pi)^4} J_\nu(p) e^{ipy} =$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{ikx} k^\mu k^\nu J_\nu(k) \underset{\uparrow}{\sim} \partial_\nu J^\nu(y) = 0$$ due to the current conservation

performing inverse Fourier transform.

• Finally terms containing $\eta_\mu\eta_\nu$ is given by

$$-\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x'-x)}}{(k\cdot\eta)^2-k^2} \eta_\nu\eta_\mu = -\frac{g_{\nu 0}\, g_{\mu 0}\, \delta(t-t')}{4\pi|\bar{x}'-\bar{x}|}$$

where we used $\int_{-\infty}^{+\infty} \frac{dk^0}{2\pi} e^{-ik^0(x'^0-x^0)} = \delta(x'^0-x^0)$ and $\int \frac{d^3\bar{k}}{(2\pi)^3} \frac{e^{i\bar{k}(x'-x)}}{|\bar{k}|^2} = \frac{1}{4\pi|\bar{x}-\bar{y}|}$;

• This terms correspond to the Coulomb interaction and will be cancelled by the following term in the Lagrangian:

If we include charge density $\rho(\bar{x})$ then A^0 should satisfy

$$\Delta A^0 = -\rho \Leftrightarrow A^0 = \int d^3y \frac{\rho(\bar{y},t)}{4\pi|\bar{x}-\bar{y}|};$$

Substituting this back to the action we obtain:

$$\mathcal{L}_e = \ldots \tfrac{1}{2}(\bar{\nabla}A^0)^2 + A_\mu J^\mu = \ldots + \tfrac{1}{2}A^0\rho$$ so that additional term is

↳ two comes from $\frac{1}{2}(\bar{\nabla}A^0)^2 = -\frac{1}{2}A^0\Delta A^0 = -\frac{1}{2}A^0\cdot\rho$

⑥ $\delta S = S_{Coul.} = \frac{1}{2}\int d^4x\, d^4y\, \delta(x^0-y^0)\,\frac{\rho(x)\rho(y)}{4\pi|\vec{x}-\vec{y}|}$

which exactly cancell $\eta_\mu\eta_\nu$ term from $\frac{1}{2}\iint d^4x\, d^4y\, J_\mu(x)\,\Delta^{\mu\nu}(x-y)\,J_\nu(y)$

- To summarize we are left with the propogator

$$\Delta^{\mu\nu}(k) = -\frac{g^{\mu\nu}}{k^2-i\varepsilon};$$

One more disadventage is that $\vec{\nabla}\vec{A}=0$ gauge is not Lorentz-invariant

- However this derivation is long and not so nice! Instead it is better to proceed directly with the path integral

Let's consider path integral

$$Z = \int \mathcal{D}A\, e^{iS[A]};$$

where we take an action $S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) =$

$$= -\frac{1}{4}\int d^4x\left((\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\right) = \frac{1}{2}\int d^4x\left(A_\mu(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu\right) =$$

$$= (\text{Fourier transforming}) = \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}A_\mu(k)\left(-k^2g^{\mu\nu}+k^\mu k^\nu\right)A_\nu(-k);$$

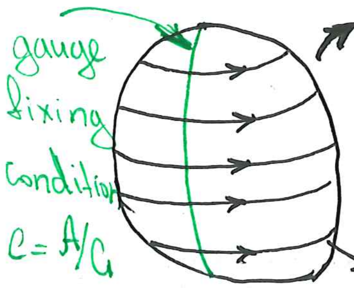
There is problem with this expression which take place whenever $A_\mu(k) = k_\mu\alpha(k)$ (pure gauge) and $S=0$; $\Rightarrow$ there is no gaussian supression and path integral is divergent.

- Another point of view revealing this problem comes from the equation defining Green function of the Maxwell equation:

$(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)D^{\nu\rho}(x-y) = i\delta_\mu^{\rho}\,\delta^{(4)}(x-y)$; fourier transforming we obtain $(-k^2 g_{\mu\nu}+k_\mu k_\nu)D_F^{\nu\rho}(k) = i\delta_\mu^{\rho}$;

But the matrix $(-k^2g_{\mu\nu}+k_\mu k_\nu)$ can not be inversed because it has zero eignvalues.

- Physical picture of the problem:



A - space of all possible gauge fields $A_\mu(x)$

We see that if we take just an integral over all possible $A_\mu(x)$ we overcount field configurations which results in the end in divergence of path integral

gauge orbits of gauge fields (i.e. subspaces of gauge fields $A_\mu(x)$ equivalent to each other)

⑦ To reduce this overcounting we should just fix some gauge which we denote by $F[A]=0$, for example

- Lorentz gauge (Landau or Feynman gauge also)
  $\partial_\mu A^\mu = 0 \Rightarrow F[A] = \partial_\mu A^\mu$;
- Coulomb gauge: $\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow F[A] = \vec{\nabla}\vec{A}$;

and so on....

- Let's now try to restrict path integration to the quotient space $\mathcal{A}/\mathcal{G}$. In particular let's take some A such that $F[A]\neq 0$. We can always find group element g_0 such that
  $A^\mu_{g_0} \equiv g_0(A^\mu + \partial_\mu)g_0^{-1}$ satisfies gauge fixing condition $F[A_{g_0}]=0$;
- Let's denote all such elemens $g_0(A)$ ($g_0 : A \to A_{g_0} : F[A_{g_0}]=0$).
- Now let's insert the following identity into path integral:
  $$1 = \int \mathcal{D}[g]\cdot\delta(g - g_0(A))$$
- As for the usual function $f_i(x)$ with zero's $\bar{x}_0 : \bar{f}(\bar{x}_0)=0$ we have $\delta(\bar{x}-\bar{x}_0) = \delta(f(x))\left|\frac{\partial f_i}{\partial x_j}\right|_{\bar{x}=\bar{x}_0}$
- In analog with this equation we write:
  $$1 = \int \mathcal{D}g\, \delta(F[A_g])\cdot\left|\mathrm{Det}\left[\frac{\partial F[A_g]}{\partial g}\right]\right|$$
- Inserting this identity into path integral we get:
  $$Z = \int \mathcal{D}A\,\mathcal{D}g\, \delta(F[A_g])\left|\mathrm{Det}\left[\frac{\delta F[A_g]}{\delta g}\right]\right| e^{-S[A]}$$
- Now we can interchange integration over the gauge group and over the gauge fields A.
- We can also rename the variables $A_g \to A$. In principle it is not obvious that we can always do it without obtaining extra complications. For this we need path integral measure $\mathcal{D}A$ be gauge invariant. In case of Abelian theory $A^\mu_g = A^\mu + \partial^\mu\alpha$, where $g = e^{i\alpha(x)}$ it is just the shift so that invariance is obvious if

(8) we think about path integral measure as

$$\mathcal{D}A = \underbrace{\prod_x}_{\substack{\text{product over all space points} \\ \text{in particular it is better to discuss discretization} \\ \text{of space time and considering} \\ \text{values of fields in lattice vertices } x_i}} \underbrace{\prod_{\mu=1}^{4}}_{\text{product over components of gauge field.}} \overbrace{\prod_{a=1}^{\dim G}}^{\text{product over components of gauge field in the groupspace}} dA_\mu^a$$

- For Abelian symmetry corresponding to the shif of A_μ it is obviously invariant. $A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha(x) \Leftrightarrow dA'_\mu = dA_\mu$
- For the case of non-Abelian symmetry

$$(A^g)^a_\mu t^a = e^{i\alpha^a t^a}(A^b_\mu t^b + \frac{i}{g}\partial_\mu)e^{-i\alpha^c t^c}$$

in infenetesemal form $(A^g_\mu)^a t^a = (1 + i\alpha^a t^a)(A^b_\mu t^b + \frac{1}{g} t^c \partial_\mu \alpha^c)(1 - i\alpha^c t^c) =$

$$= A^a_\mu t^a + \alpha^a A^b_\mu [t^a, t^b] + \frac{1}{g} t^a \partial_\mu \alpha^a = (A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c) t^a$$

So that finally $\underline{(A^g_\mu)^a = A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c}$

These transformations are in principle more complicated, but they are just combination of shift with unitary rotations of A^a_μ components, which also preserve the measure.

- So we finally write (we here assume that det >0, i.e. no Gribov copies)

$$\boxed{Z = \int \mathcal{D}g \int \mathcal{D}A\, \delta(F[A]) \det\left(\frac{\delta F(A_g)}{\delta g}\right)\bigg|_{A_g = A} e^{iS[A]};}$$

where we also used invariance of the action $\underline{S[A_g] = S[A]}$;

- Notice that integrant does not depend on g and hence $\int \mathcal{D}g = \mathrm{Vol}_g$. is overall normalization factor and we just omit it in the
- Using this expression for the partition function we can now find the propogator.

⑨ • **Propogator of electromagnetic field**

• Let's start with the simpliest case of Abelian gauge symmetry. In this case let's take gauge fixing condition to be $F[A] = \partial^\mu A_\mu - \omega(x)$ where $\omega(x)$ is just some function.

• Then $\frac{\delta F[A_g]}{\delta g} = \frac{\delta F[A_\alpha]}{\delta \alpha} = \frac{1}{e}\partial^2$ where we used simple parametrization of gauge group element $g = e^{i\alpha}$ so that $A_g \equiv A^\mu_\alpha = A^\mu + \frac{1}{e}\partial^\mu \alpha$; and $\delta g = i\delta\alpha$;

• Then $\det\left(\frac{\delta F[A_g]}{\delta g}\right) = \det\left(\frac{1}{e}\partial^2\right)$ does not depend on your gauge fields and can also be factored out.

• Then we obtain:

$$Z = \int \mathcal{D}A\, e^{iS[A]} = \det\left(\tfrac{1}{e}\partial^2\right)\cdot \mathrm{Vol}(U(1)) \int \mathcal{D}A\, e^{iS[A]}\, \delta(\partial_\mu A^\mu - \omega(x))$$

This equation is valid for any $\omega(x)$ so we can substitute any properly normalized linear combination of expressions with different $\omega(x)$. In particular let's integrate over all possible $\omega(x)$ with Gaussian weight:

$$Z = \underset{\substack{\uparrow \\ \text{norm.} \\ \text{factor}}}{N(\xi)} \int \mathcal{D}\omega \exp\left(-i\int d^4x \frac{\omega^2}{2\xi}\right) \det\left(\tfrac{1}{e}\partial^2\right) \int \mathcal{D}\alpha \int \mathcal{D}A\, e^{iS[A]}\, \delta(\partial_\mu A^\mu - \omega(x)) =$$

$$= N(\xi)\cdot \det\left(\tfrac{1}{e}\partial^2\right) \int \mathcal{D}\alpha \int \mathcal{D}A\, e^{iS[A]} \cdot \exp\left(-i\int d^4x \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right);$$

So we get effective partition function

$$Z = \text{const.} \int \mathcal{D}A\, e^{iS[A]} \cdot \exp\left(-i\int d^4x \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right);$$

Now our effective action is given by the Lagrangian:

$$\mathcal{L}_{eff} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2;$$

10. Corresponding equations of motion are

$$\partial_\mu F^{\mu\nu} + \frac{1}{\xi}\partial_\mu \partial^\nu A^\mu = 0$$

Then propogator can be found through the Green function:

$$\left(\partial_\mu \partial^\mu g^{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial^\mu \partial^\nu\right) A_\mu = 0 \Rightarrow \quad \underline{\left(-k^2 g_{\mu\nu} + \left(1-\frac{1}{\xi}\right)k_\mu k_\nu\right) G^{\nu\rho}(k) = \delta_\mu^\rho}$$

Fourier transforming and writing Green function equation

So that inverting: $\boxed{G^{\mu\nu}(k) = \frac{-1}{k^2+i\varepsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right);}$

· This is called ξ-gauge. Two most usefull values of ξ are

- $\xi = 0 \rightarrow$ Landau gauge.
- $\xi = 1 \rightarrow$ Feynman gauge. $\rightarrow$ in this case $G^{\mu\nu}(k) = -\frac{g^{\mu\nu}}{k^2+i\varepsilon}$

and this is result we have obtained previously!

Non-Abelian symmetry.

· Let's now choose gauge fixing condition to be

$$F^a(A) = \partial^\mu A^a_\mu - \omega^a(x);$$

The story with propogator is exactly the same as before because introducing Gaussian weight for $\omega^a(x)$ we obtain:

$$Z = N(\xi)\int \mathcal{D}g \int \mathcal{D}\omega^a \exp\left(-i\int d^4x \frac{\omega^2}{2\xi}\right)\int \mathcal{D}A\, e^{iS[A]} \cdot \det\left(\frac{\delta F^a[A]}{\delta g^b}\right)\delta(\partial_\mu A^{a\mu} - \omega^a)$$

So that $$\underline{Z = N(\xi)\,\mathrm{Vol}_g \int \mathcal{D}A\, e^{iS[A] - i\int d^4x \frac{(\partial^\mu A^a_\mu)^2}{2\xi}} \cdot \det\left(\frac{\delta F^a[A]}{\delta g^b}\right)}$$

· Before writing out Feynman rules for this theory let's calculate $\underline{\det\left(\frac{\delta F^a}{\delta g^b}\right)}$

Performing infenetesemal gauge transformation we do the following:

(11)

$$g' = 1 + \delta g = 1 + \underbrace{\alpha^a t^a}_{\delta g^a \cdot t^a};$$

$$A^a_\mu \to A^a_\mu + \frac{1}{g}\partial_\mu \alpha^a + f^{abc} A^b_\mu \alpha^c = A^a_\mu + \underbrace{\frac{1}{g} D_\mu \alpha^a}_{\delta A^a_\mu}$$

Hence $$\det \frac{\delta F^a}{\delta g} = \det \left(\frac{1}{g}\partial_\mu D^\mu\right)_{ab} = \det\left(\frac{1}{g}(\delta_{ab}\partial_\mu \partial^\mu - f^{abc}\partial_\mu A^\mu_c)\right);$$

- Now this determinant depends on A-field and hence cannot be factored out.
- In order to compute it Faddeev and Popov came up with the idea to introduce fermionic fields. As we remember for Grassmann fields

$$\int d\theta d\bar\theta \, e^{-\bar\theta B \theta} = \det B$$ hence we can write

$$\det\left(\frac{\delta F^a}{\delta g^b}\right) = \det\left(\frac{1}{g}\partial^\mu D_\mu\right) = \int Dc D\bar c \, e^{i\int d^4x \cdot \bar c(-\partial^\mu D_\mu)c}$$

- Inserting this expression into our action we finally get:

$$\boxed{\mathcal{L}_\xi = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 - \bar c^a(\partial^2\delta^{ac} + g\partial^\mu f^{abc} A^b_\mu)c^c;}$$

- Fields c^a and $\bar c^a$ are called Faddeev-Popov ghosts. They are:
  - Fermions in statistics
  - Scalars in Lorentz structure
  - Transform in adjoint representation.
- Feynman rules.
  - Gluon propogator:

comes from the terms $$\mathcal{L}_{gluon} = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2$$

→ the same as in electrodynamics so we have → extra term!

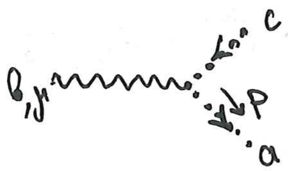
$$\langle A^a_\mu(x) A^b_\nu(y)\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{-i}{k^2+i\varepsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab} = x^{a}_{\mu} \sim\sim\sim\sim y^{b}_{\nu}$$

  - Ghost propogator:

Comes from $\mathcal{L}_{ghost} = \bar c^a(-\partial^2\delta^{ab})c^b$ and leads to the massless scalar propogator

$$\langle c^a(x)\bar{c}^b(y)\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\delta^{ab}e^{-ik(x-y)} =$$

- Ghost – gluon interaction



$= -g f^{abc} p^\mu$

Comes from the term $\mathcal{L}_{gh-g} = \bar{c}^a(-g\partial^\mu f^{abc} A^b_\mu c^c) =$

$= g\partial^\mu \bar{c}^a f^{abc} A^b_\mu c^c$

↑

ip^μ momenta of outgoing ghost

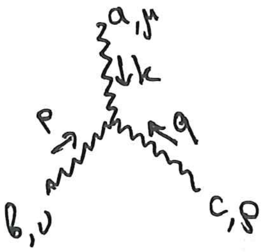
- Gluon vertices

Comes from

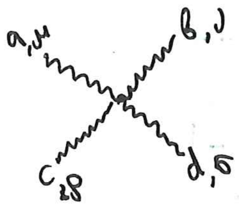
momentum of photon

↑

$$i\left(-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}\right) \sim i\left(-2\frac{1}{2}g f^{abc} A^b_\mu A^c_\nu \partial^\mu A^{a\nu} - \frac{1}{4}g^2 f^{abc} f^{alm} A^b_\mu A^c_\nu A^{l\mu} A^{m\nu}\right)$$



$$= g f^{abc}\left(g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right)$$



$$= -ig^2\left(f^{abe}f^{cde}(g^{\rho\mu}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\rho\nu}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\right)$$

Lectures 4 and 5: Non-abelian gauge fields.

1 Non-Abelian global symmetries.

We have seen that complex scalar field that is described by the action:

$$\mathcal{L}_\varphi = \partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi - V(\varphi^*\varphi);$$

has $U(1)$ symmetry $\varphi(x) \to g\varphi(x)$ where $g = e^{i\alpha} \in U(1)$

- Goal: generalize global $U(1)$ symmetry to other symmetries (non-Abelian)
- Simpliest possible model:
  - introduce N scalar complex fields: $\varphi_i(x)$ $i = 1, \dots, N$;
  - Choose the action just as the sum of single scalar field actions:

$$\mathcal{L}_\varphi = \partial_\mu\varphi_i^*\partial^\mu\varphi_i - m^2\varphi_i^*\varphi_i - V(\varphi_i^*\varphi_i);$$

Symmetries:
- $U(1)$ symmetry $\varphi_i \to e^{i\alpha}\varphi_i$
- $SU(N)$ symmetry $\varphi_i \to \omega_{ij}\varphi_j$, $\omega \in SU(N)$;

to see $SU(N)$ invariance of the action notice that:

$$\varphi_i^*\varphi_i \to \varphi_j^*\omega_{ij}^*\omega_{il}\varphi_l = \varphi_j^* \underbrace{(\omega^+\omega)_{jl}}_{\delta_{jl} \text{ as } \omega \in SU(N)} \varphi_l = \varphi_k^*\varphi_k;$$

- Similar chain of equations is valid for kinetic term because transformation is global.
- Notice that in order for the action to be invariant we need:
  - Masses of all scalars to be the same, i.e the mass term is $m^2\varphi_i^*\varphi_i$ not $\sum_i m_i^2\varphi_i^*\varphi_i$;
  - Potential term should also depend on the "length" of the φ_i vector, i.e. on $\varphi_i^*\varphi_i$;
- If we introduce the column

$$\varphi \equiv \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{pmatrix}, \varphi^+ = (\varphi_1^*, \varphi_2^*, \dots, \varphi_N^*);$$

then the Lagrangian takes the form

$$\boxed{\mathcal{L}_\varphi = \partial_\mu\varphi^+\partial^\mu\varphi - m^2\varphi^+\varphi - V(\varphi^+\varphi);}$$

- $\varphi(x)$ is vector field in the corresponding vector space of columns transforming in fundamental representation of $SU(N)$;

$\varphi(x) \to \omega\,\varphi(x)$

② • Let's generalize this construction for other gauge groups:

$\begin{cases} G - \text{gauge group;} \\ T(G) - \text{unitary representation (meaning operator } T \text{ is unitary)} \end{cases}$

(notice that any representation of the compact group is eqivalent to unitary)

• Then Leagrangian

$$\mathcal{L}_\varphi = \partial_\mu \varphi \partial^\mu \varphi^+ - V(\varphi^+\varphi)$$

is invariant under the transformation $\begin{cases} \varphi \to T(\omega).\varphi ; \\ \varphi^+ \to \varphi^+ . T^+(\omega); \end{cases}$

The invariance is valid due to the condition $T^+(\omega) T(\omega) = 1;$ (because representation is unitary)

Examples:

(I) Assume that we have 2 sets of N complex scalar fields:
$\varphi_i(x), \chi_i(x); \; i = 1, \dots, N;$

then we can build the following inv. Leagrangian:

$$\mathcal{L}_\varphi = \partial_\mu \varphi^+ \cdot \partial^\mu \varphi + \partial_\mu \chi^+ \cdot \partial^\mu \chi - m_\varphi^2 \varphi^+ \varphi - m_\chi^2 \chi^+ \cdot \chi - V(\varphi^+\varphi, \chi^+\chi, \varphi^+\chi, \chi^+\varphi)$$

which is invariant under $\begin{matrix} \varphi \to \omega\varphi; \\ \chi \to \omega\chi; \end{matrix} \quad \omega \in SU(N)$

Notice that if we want Leagrangian to be invariant we should choose potential V dependent on the invariant products:

$\varphi^+\varphi; \chi^+\chi, \varphi^+\chi$ or $\chi^+\varphi;$

example of the invariant potential: $V = \lambda_1 (\varphi^+\varphi)^2 + \lambda_2 [(\varphi^+\chi)^2 + (\chi^+\varphi)^2] + \lambda_3 (\chi^+\chi)^2;$

• Composite field (φ, χ) can be considered as the field trasforming in $F \otimes F$ representation of $SU(N)$

F – fundamental representation.

(II) Let's consider $\varphi(x)$ doublet transforming in the fundamental representation of $SU(2)$ i.e. it is column $\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}$; $\varphi \to \omega\varphi$, $\omega \in SU(2)$.

We also introduce triplet $\chi^a(x)$ $a = 1,2,3$ transforming in the adjoint repr. of $SU(2)$ $\Rightarrow$ $\chi(x) = \chi^a \cdot \tau^a$, where τ^a are pauli matricies. (generators of $SU(2)$) so that $\chi(x) \in ASU(2)$ (algebra) and hence $\chi(x) \to \omega \chi(x) \omega^{-1}$;

③ Notice also that $\chi^2 = \chi^a \chi^b \cdot \tau^a \tau^b$ hence

$$\mathrm{tr}\,\chi^2 = \chi^a \chi^b \left(\delta^{ab}\,\mathrm{tr}\,\mathbb{1} + i\varepsilon^{abc}\,\underbrace{\cancel{\mathrm{tr}\,\tau^c}}_{0}\right) = \chi^a\chi^a \cdot 2 \Rightarrow \chi^a\chi^a = \frac{1}{2}\mathrm{tr}\,\chi^2;$$

- Invariant quadratic terms are given by

$\varphi^+\varphi \to \varphi^+\omega^+\omega\varphi = \varphi^+\varphi$ - invariant !!!

using cyclisity of trace.

$$\chi^a\chi^a = \frac{1}{2}\mathrm{tr}\,\chi\cdot\chi \to \frac{1}{2}\mathrm{tr}\,(\omega\chi\omega^{-1}\cdot\omega\chi\omega^{-1}) = \frac{1}{2}\mathrm{tr}(\omega\chi^2\omega^{-1}) = \frac{1}{2}\mathrm{tr}\,\chi^2 = \chi^a\chi^a \to \text{inv. !!!}$$

- Invariant cubic terms:

the only inv. cubic term for self-interaction is

$\mathrm{tr}\,\chi^3$ (in general any term of the form $\mathrm{tr}\,\chi^n$ is inv. under adjoint transform. $\chi \to \omega\chi\omega^{-1}$)

one more term describing interaction between φ and χ is

$$\varphi^+\chi\varphi \to \varphi^+\underbrace{\omega^+\omega}_{1 \text{ as } \omega\in SU(N)}\chi\underbrace{\omega^{-1}\omega}_{1}\varphi = \varphi^+\chi\varphi \text{ - invariant !!!}$$

- Invariant quartic terms:

$$\mathrm{tr}\,\chi^4,\quad (\mathrm{tr}\,\chi^2)^2 = (\chi^a\chi^a)^2,\quad (\varphi^+\varphi)^2,\quad \varphi^+\chi^2\varphi$$

- Physical interpretation:

Already in 30's it was noticed that some nucleons and mesons can be assigned internal d.o.f. so called isospin.

In particular:

proton $I_3 = +\frac{1}{2}$

neutron $I_3 = -\frac{1}{2}$

"z"-component of isospin

pions: π^+ : $I_3 = +1$;

π^0 : $I_3 = 0$;

π^- : $I_3 = -1$;

So that the Leagrangian we write for these particles should be invariant under isospin rotations, transforming one particles into other. Corresponding isosymmetry group is $SU(2)$ and we can group particles as the following

nucleons doublet : $\varphi = \begin{pmatrix} p \\ n \end{pmatrix}$;

transform in the fundamental repr. of $SU(2)$

pions triplet : $\chi = \begin{pmatrix} \pi^1 \\ \pi^2 \\ \pi^3 \end{pmatrix}$

$\pi^1 = \frac{1}{\sqrt{2}}(\pi^+ + \pi^-)$;

$\pi^2 = \frac{1}{\sqrt{2}}(\pi^+ - \pi^-)$;

$\pi^3 = \pi^0$

transforms in the adjoint representation.

④ III $\varphi_{i\alpha}(x)$ – "matrix" of $m \times n$ complex scalar fields
$i = 1, \dots, n$; $\alpha = 1, \dots, m$; Let's say it transforms in the representation which is the direct product of the fundamental representations of $SU(n)$ and $SU(m)$:

$$\varphi_{i\alpha} \to \omega_{ij} \Omega_{\alpha\beta} \varphi_{j\beta} \qquad \omega \in SU(n); \quad \Omega \in SU(m);$$

• Invariant Lagrangian is:

$$\mathcal{L}_\varphi = \partial_\mu \varphi^*_{i\alpha} \partial^\mu \varphi_{i\alpha} - m^2 \varphi^*_{i\alpha}\varphi_{i\alpha} - \lambda(\varphi^*_{i\alpha}\varphi_{i\alpha})^2;$$

in the future we will omit indicies so that transform. takes the form: $\varphi \to \omega\Omega\varphi$; and Lagrangian is

$$\mathcal{L}_\varphi = \partial_\mu \varphi^+ \partial^\mu \varphi - m^2 \varphi^+ \varphi - \lambda(\varphi^+\varphi)^2;$$

Physical picture:

In QCD fields of quarks transform in exactly the same way: • in the fund. repr. of the gauge group $SU_c(3)$;
• and simultaneously in the fund. repr. of the flavor group $SU_f(3)$

IV Let's build Lagrangian inv. under $SU(2) \times U(1)$ global symmetry with the following fields content:

$\varphi, \chi \to$ doublets of $SU(2)$, charged under $U(1)$:

$$\varphi \to \omega\varphi; \qquad \varphi \to e^{iq_\varphi \alpha}\varphi;$$
$$\chi \to \omega\chi; \qquad \chi \to e^{iq_\chi \alpha}\chi;$$

$\xi \to$ singlet of $SU(2)$, charged under $U(1)$:

$$\xi \to \xi \qquad \xi \to e^{iq_\xi \alpha}\xi;$$

• Kinetic terms are the same as before, as well as mass terms:

$$\mathcal{L}_{e,1} = \partial_\mu \varphi^+ \partial^\mu \varphi + \partial_\mu \chi^+ \partial^\mu \chi + \partial_\mu \xi^* \partial^\mu \xi - m_\varphi^2 \varphi^+\varphi - m_\chi^2 \chi^+\chi - m_\xi^2 \xi^*\xi;$$

• Possible interaction terms to lowest order:

$$\boxed{\mathcal{L}_{int} = \lambda\left[(\varphi^+ \xi)\chi + h.c.\right]}$$

indeed it is inv. under $SU(2)$: $\varphi^+\xi\chi \to \varphi^+\xi\omega^+\omega\chi = \varphi^+\xi\chi \to$ inv!
and inv. under $U(1)$ if $-q_\varphi + q_\xi + q_\chi = 0$ because
under $U(1)$: $\varphi^+\xi\chi \to \exp(i(-q_\varphi + q_\xi + q_\chi)\alpha)\varphi^+\xi\chi$;

⑤ Notice that other terms are not possible:

ⓐ to be inv. under $SU(2)$ term should have at least two doublet fields (but not 3). Hence the alternative is $\varphi^+\xi\varphi$ or $\chi^+\xi\chi$

ⓑ Both terms above are not inv under $U(1)$:

$$\varphi^+\xi\varphi \to \exp(i(q_\varphi - q_\varphi + q_\xi)\alpha)\varphi^+\xi\varphi = e^{iq_\xi\alpha}\varphi^+\xi\varphi;$$

similarly for $\chi^+\xi\chi$; $\Rightarrow$ inv only if $q_\xi = 0$! (trivial case)

Physical picture.

The model built above corresponds to the electro weak sector of the Standard Model.

- φ - doublet of left leptons: $\varphi = \begin{pmatrix} e^-_L \\ \nu_e \end{pmatrix}$ for example.
- ξ - rigt-handed lepton: e^-_R for example.
- χ - doublet of Higgs fields: $H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix}$;

q_φ, q_χ, q_ξ - Weak hypercharges of corresponding particles.

- Final notice: here we considered only scalar fields.

However all considerations can be generalized to vector and spinor fields as internal and Lorentz structures don't feel each other, we just should be carefull vith the Lorentz inv. as well in this case!

Non-Abelian gauge theory: $SU(2)$.

- Now the goal is to generalize the construction of scalar electrodynamics to non-Abelian gauge groups. We can start with the simpliest group $SU(2)$.
- We start with the doublet of complex scalar fields:

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix};$$ transforming in the fundamental repr. of $SU(2)$ (global)

$$\varphi \to \omega\varphi \quad, \omega \in SU(2);$$

- Lagrangian is then given by:

$$\mathcal{L}_0 = \partial_\mu\varphi^+\partial^\mu\varphi - m^2\varphi^+\varphi - V(\varphi^+\varphi);$$

(6) • Nexst step is gauging transformation so that

$$\varphi \to \omega(x)\varphi(x), \ \omega(x) \in SU(2);$$

• terms of the form $\varphi^+\varphi$ remain the same, however after gauging symmetry kinetic terms become non-invariant under the local $SU(2)$ symmetry; because:

$$\partial_\mu \varphi(x) \to \partial_\mu \varphi'(x) = \omega(x)\cdot\partial_\mu\varphi(x) + \partial_\mu\omega(x)\cdot\varphi(x);$$

• In order to fix this we act in the same way as we did for the scalar electrodynamics, i.e we construct covariant derivative D_μ such that:

$$D_\mu\varphi \to (D_\mu\varphi)' = \omega D_\mu\varphi \text{ under } SU(2) \text{ transformation.}$$

• To find D_μ let's make shift similar to scalar e.d. case:

$$D_\mu = \partial_\mu + A_\mu;$$

• Next question to answer is what is appropriate transform. of A_μ?

Let's look on $(D_\mu\varphi)' = \partial_\mu\varphi' + A'_\mu\varphi' = \omega\partial_\mu\varphi + \partial_\mu\omega\cdot\varphi + A'_\mu\cdot\omega\varphi \overset{?}{=}$ $\overset{?}{=} \omega\partial_\mu\varphi + \omega A_\mu\varphi$. In order for the last equation to work we demand: $A'_\mu\cdot\omega = \omega A_\mu - \partial_\mu\omega \Leftrightarrow A'_\mu = \omega A_\mu\omega^{-1} + \omega\partial_\mu\omega^{-1};$

where we have used: $\partial_\mu(\omega\omega^{-1}) = \partial_\mu\omega\cdot\omega^{-1} + \omega\partial_\mu\omega^{-1} = 0;$

• Notice that $\omega\partial_\mu\omega^{-1}$ is the element of $SU(2)$ algebra. Indeed if we consider $\omega = 1 + \varepsilon(x)$ then

$$\omega\partial_\mu\omega^{-1} = (1+\varepsilon)(-\partial_\mu\varepsilon) = -\partial_\mu\varepsilon + O(\varepsilon^2) \in ASU(2)!$$

If we take the product of two element $g_1, g_2 \in SU(2)$ then corresponding term is

$$g_1 g_2 \partial_\mu(g_2^{-1}g_1^{-1}) = g_1(g_2\partial_\mu g_2^{-1})g_1^{-1} + g_1 g_2 g_2^{-1}\partial_\mu g_1^{-1} =$$
$$= \underbrace{g_1(g_2\partial_\mu g_2^{-1})g_1^{-1}}_{\in ASU(2)} + g_1\cdot\partial_\mu g_1^{-1}$$

and $g_1\cdot A g_1^{-1}$ is also in the algebra as it is adjoint representation transformation. By the same reason $\omega A_\mu\omega^{-1}$ is also in the algebra if A_μ is in algebra.

⑦ • Hence it is reasonable to expect that A_μ should also be in the algebra!!! $\boxed{A_\mu \in ASU(2);}$

• Gauge transformation:

$$\varphi(x) \to \omega(x)\cdot\varphi(x);$$

$$A_\mu(x) \to \omega(x)\cdot A_\mu(x)\cdot\omega^{-1}(x) + \omega(x)\cdot\partial_\mu\omega^{-1}(x);$$

• Then the Leagrangian is:

$$\mathcal{L}_e = (D_\mu\varphi)^+ D^\mu\varphi - m^2\varphi^+\varphi - \lambda(\varphi^+\varphi)^2;$$

where $D_\mu\varphi = \partial_\mu\varphi + A_\mu\varphi$;

• Notice that gauge field A_μ transforms non-trivially even under global transformation:

$\underline{A_\mu \to \omega A_\mu \omega^{-1}}$ – this is different from electrodynamics.

• Now let's build kinetic term for the gauge field.

• start with building strength tensor $F_{\mu\nu}$ in electrodynamics: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Under global SU(2) transf. similar object in our theory transforms under adjoint repr. $F_{\mu\nu} \to \omega F_{\mu\nu}\omega^{-1}$; the goal is to obtain the same transform. for local symmetries: $\underline{F_{\mu\nu}(x) \to \omega(x) F_{\mu\nu}\omega^{-1}(x)}$; goal!!!

Notice that:

$$\partial_\mu A_\nu - \partial_\nu A_\mu \to \omega(\partial_\mu A_\nu - \partial_\nu A_\mu)\omega^{-1} + \partial_\mu\omega A_\nu\omega^{-1} + \omega A_\nu\partial_\mu\omega^{-1} -$$

$$- \partial_\nu\omega A_\mu\omega^{-1} - \omega A_\mu\partial_\nu\omega^{-1} + \partial_\mu(\omega\partial_\nu\omega^{-1}) - \partial_\nu(\omega\partial_\mu\omega^{-1})$$

There are a lot of extra terms. To cancel them we need to add extra terms satisfying:

- no derivatives of A_μ.
- belong to Lie algebra of SU(2).
- Antisymmetric in μ,ν indicies.

the good candidate is $[A_\mu, A_\nu]$:

⑧ $$[A'_\mu, A'_\nu] = \omega A_\mu \omega^{-1}\cdot\omega A_\nu\omega^{-1} + \omega A_\mu\omega^{-1}\cdot\omega\partial_\nu\omega^{-1} + \omega\partial_\mu\omega^{-1}\cdot\omega A_\nu\omega^{-1} + (\omega\partial_\mu\omega^{-1})\cdot(\omega\partial_\nu\omega^{-1}) - (\mu\leftrightarrow\nu)$$

Let's write down all extra terms from both expressions:

$$\partial_\mu\omega A_\nu\omega^{-1} + \underline{\omega A_\nu\partial_\mu\omega^{-1}} + \partial_\mu\omega\,\partial_\nu\omega^{-1} + \omega\partial_\mu\omega^{-1}\cdot\omega A_\nu\omega^{-1} + \underline{\omega A_\mu\partial_\nu\omega^{-1}} + (\omega\partial_\mu\omega^{-1})(\omega\partial_\nu\omega^{-1}) - (\mu\leftrightarrow\nu) =$$

underlined terms cancell each other.

$$= \left[\text{using } \omega^{-1}\partial_\mu\omega = -\partial_\mu\omega^{-1}\cdot\omega \Rightarrow \begin{array}{l}\partial_\mu\omega = -\,\omega\partial_\mu\omega^{-1}\cdot\omega \\ \partial_\mu\omega^{-1} = -\,\omega^{-1}\partial_\mu\omega\,\omega^{-1}\end{array}\right] =$$

$$= -\,\omega\partial_\mu\omega^{-1}\cdot\omega A_\nu\omega^{-1} - \omega\partial_\mu\omega^{-1}\omega\partial_\nu\omega^{-1} + \omega\partial_\mu\omega^{-1}\omega A_\nu\omega^{-1} + (\omega\partial_\mu\omega^{-1})(\omega\partial_\nu\omega^{-1}) - (\mu\leftrightarrow\nu) = 0.$$

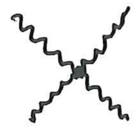
Hence

$$\boxed{\begin{array}{l} F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]; \\ F_{\mu\nu} \to \omega(x) F_{\mu\nu}\,\omega^{-1}(x),\ \omega\in SU(2); \end{array}}$$

• Then the gauge inv. kinetic term is

$$\boxed{\mathcal{L}_0 = \frac{1}{2g^2}\,\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu})}$$

– notice that now kinetic term is not quadratic but also has interaction terms like $\mathrm{tr}\,(A_\mu A_\nu A^\mu A^\nu)$



g → coupling constant.

↓

Yang-Mills action

A_μ - Yang-Mills field.

• Notice that $A_\mu \in ASU(2)$ is antihermitian.
Physical fields are usually hermitian. So we can introduce triplet of real fields $A^a_\mu(x)$, $a = 1,2,3$.

$$A_\mu(x) = -ig\,\frac{\tau^a}{2}\,A^a_\mu(x);$$

↑ generators of $SU(2)$ (Pauli matricies)

then
$$F_{\mu\nu}(x) = -ig\frac{\tau^a}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + (ig)^2 A^a_\mu A^b_\nu \left[\frac{\tau^a}{2}, \frac{\tau^b}{2}\right] =$$
$$= -ig\frac{\tau^a}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - ig^2 A^a_\mu A^b_\nu\,\varepsilon^{abc}\frac{\tau^c}{2} =$$
$$= -ig\frac{\tau^a}{2}\underbrace{(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,\varepsilon^{abc}A^b_\mu A^c_\nu)} \equiv -ig\frac{\tau^a}{2}F^a_{\mu\nu}$$

$$\boxed{F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,\varepsilon^{abc}A^b_\mu A^c_\nu;}$$

⑨ • In terms of real fields A_μ^a, $F_{\mu\nu}^a$; the Leagrangian is written:

• Yang-Mills term: $\mathcal{L}_{YM} = \frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) = \frac{1}{2g^2}(ig)^2 F_{\mu\nu}^a F^{\mu\nu b}\cdot\mathrm{tr}(\frac{1}{4}\tau^a\tau^b) =$

$= \left(\text{using } \mathrm{tr}(\tau^a\tau^b) = \mathrm{tr}(\delta^{ab}\cdot\mathbb{1} + i\varepsilon^{abc}\tau^c) = 2\delta^{ab}\right) = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a}$

$\mathcal{L}_{YM} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a}$ – exactly the same term as in electrodynamics!

• scalar term: $\mathcal{L}_\varphi = (\mathcal{D}_\mu\varphi)^\dagger\mathcal{D}^\mu\varphi - m^2\varphi^\dagger\varphi - V(\varphi^\dagger\varphi)$;

where $\mathcal{D}_\mu = \partial_\mu - ig\frac{\tau^a}{2}A_\mu^a$ is the covariant derivative.

Comment: Simplier definition of field strength tensor is through the equation $[\mathcal{D}_\mu, \mathcal{D}_\nu] = F_{\mu\nu}$ which is operator equation meaning it should read like $[\mathcal{D}_\mu, \mathcal{D}_\nu]\varphi(x) = F_{\mu\nu}\varphi(x)$. Then gauge invariance is obtained much faster.

• Generalization to other groups.

• Let's now go more general and assume that the gauge group is some simple group G.

• Gauge field A_μ is in the corresponding Lie algebra AG, so that we can write down $A_\mu(x) = g t^a A_\mu^a(x)$;

t^a – generators of AG.

• Gauge transformations

$$A_\mu \to \omega A_\mu \omega^{-1} + \omega\partial_\mu\omega^{-1}; \quad \omega(x) \in G;$$

$$F_{\mu\nu} \to \omega F_{\mu\nu}\omega^{-1};$$

• Yang-Mills term: $\mathcal{L}_{YM} = \frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$;

• Now just in the same way as for $SU(2)$ we write

$$F_{\mu\nu} = g t^a \cdot F_{\mu\nu}^a;$$

and similarly to the $SU(2)$ case we derive:

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = g t^a(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) + g^2 A_\mu^a A_\mu^b [t^a, t^b] =$$

$$= g t^a(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) + g^2 f^{abc} A^a A^b t^c = g t^a F_{\mu\nu}^a, \text{ where}$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_\mu^b A_\nu^c;$$

(10) where we have used the fact that for the compact group f_{abc} is completely antisymmetric.

• Then for the YM action we obtain.

$$\mathcal{L}_{YM} = \frac{1}{2g^2} tr(F_{\mu\nu} \cdot F^{\mu\nu}) = \frac{1}{2} F^a_{\mu\nu} F^{b\mu\nu} \cdot tr(t^a \cdot t^b)$$

We have discussed before that only compact Lie algebras can have quadratic form which is positive definite. For the matrix groups we choose this form to be:

$(A,B) = -tr(A \cdot B)$ and the generators are chosen so that they form the orthonormal basis: $tr(t^a t^b) = -\frac{1}{2}\delta^{ab}$

Hence $\mathcal{L}_{YM} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$; – for each component $a = 1, \dots, \dim AG$; action is the same as for the electromagnetic field.

• Notice that if group is not compact $-tr(AB)$ is not positive definite. So is $-tr(t^a t^b)$. Hence we will obtain both terms $-\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2$ and $\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2$.

Right sign leading to the positive definite energy.

Wrong sign leading to the negative energy

Hence only compact Lie groups can be gauge groups.

• If the gauge group is not simple usually it's simple components and $U(1)$ component are considered separately, i.e.:

$$G = G_1 \otimes G_2 \otimes \dots \otimes G_n \otimes U_1(1) \otimes U_2(1) \otimes \dots$$

Example: $SU(n) \times SU(m)$ symmetry considered before and we have the set of $m \times n$ scalar fields $\varphi_{i\alpha}$ $\begin{matrix} i = 1, \dots, n. \\ \alpha = 1, \dots, m. \end{matrix}$

We then introduce two gauge fields:

$A_\mu(x) = -ig t^a A^a_\mu(x) \rightarrow SU(n)$ gauge group with the coupling g.

hermitian

$B_\mu(x) = -i\tilde{g} \tilde{t}^\beta B^\beta_\mu(x) \rightarrow SU(m)$ gauge group with the coupling $\tilde{g}$.

(11) • <u>Covariant derivative</u> is given by

$(D_\mu \varphi)_{i\alpha} = \partial_\mu \varphi_{i\alpha} + (A_\mu)_{ij}\varphi_{j\alpha} + (B_\mu)_{\alpha\beta}\varphi_{i\beta}$; which we for shortness write $D_\mu \varphi = (\partial_\mu + A_\mu + B_\mu)\varphi$;

• The Leagrangian inv. under $SU(n)\times SU(m)$ local transformations

$$\boxed{\mathcal{L}_\varphi = (D_\mu\varphi)^+ D^\mu \varphi - m^2\varphi^+\varphi - \lambda(\varphi^+\varphi)^2 - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}\tilde{F}^p_{\mu\nu}\tilde{F}^{p\mu\nu};}$$

Where the field strength of A_μ and B_μ fields are given by:

$$\begin{cases} F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\cdot f_{abc}\cdot A^b_\mu A^c_\nu; \\ \tilde{F}^p_{\mu\nu} = \partial_\mu B^p_\nu - \partial_\nu B^p_\mu + g\cdot f_{pqr} B^q_\mu B^r_\nu; \end{cases}$$

• So when the group is not simple one should decompose it into the direct product of simple groups and $U(1)$ factors and introduce gauge coupling for each of these groups.

• <u>Comment</u>: Notice that the model discussed here is different from the situation in QCD $SU(3)_c \times SU(3)_f$ symmetry. Because there $SU(3)_c$ is local, while $SU(3)_f$ is global symmetry. However some similar situation happens in the Standard Model which has $SU(3)\times SU(2)\times U(1)$ gauge symmetry.

• <u>Arbitrary representation of scalar fields.</u>

• Let's take $T(\omega)$ representation of the gauge group G and $T(A)$ - represent. of the corresponding algebra.

Further we think of the scalar field φ as the column so that $T(\omega)$ is some unitary and $T(A)$ - antihermitian matrix (because representation of any compact Lie group is equivalent to unitary)

• Under gauge transform we have:

$$\begin{cases} \varphi(x) \to \varphi'(x) = T(\omega(x))\cdot\varphi(x); \\ A_\mu \to A'_\mu = \omega(A_\mu + \partial_\mu)\omega^{-1}; \end{cases}$$

(12)

- Then we can define the covariant derivative:

$$\mathcal{D}_\mu\varphi = [\partial_\mu + T(A_\mu)]\varphi \quad \text{or} \quad \mathcal{D}_\mu\varphi = [\partial_\mu - ig T^a A^a_\mu]\varphi;$$

hermit. generators in T-representation.

- This derivatives transforms as expected:

$$\mathcal{D}_\mu\varphi \to T(\omega)\cdot\mathcal{D}_\mu\varphi;$$

because $T(\omega A\omega^{-1}) = T(\omega)\cdot T(A)\cdot T(\omega^{-1})$;

$T(\omega\partial_\mu\omega^{-1}) = T(\omega)\partial_\mu T(\omega^{-1})$;

- To show that these equations work let's consider group element $g_A = 1 + \varepsilon\cdot A$, then as $T(g_1 \cdot g_2) = T(g_1)\cdot T(g_2)$ $\forall\, g_1, g_2 \in G$ we obtain $T(\omega\cdot g_A\cdot\omega^{-1}) = 1 + \varepsilon\cdot T(\omega)\cdot T(A)\cdot T(\omega^{-1})$; hence the first equation is valid.
- Gauge invariant Lagrangian is the same as before:

$$\mathcal{L}_{\varphi} = (\mathcal{D}_\mu\varphi)^\dagger \mathcal{D}^\mu\varphi - m^2\varphi^\dagger\varphi - V(\varphi^\dagger\varphi);$$

Example: scalar field in adjoint repr. of $G = SU(N)$ i.e.

$\varphi(x) \in A\,SU(N) \Rightarrow \varphi(x) = t^a\varphi^a(x)$

we assume $\varphi(x)$ hermitian. — hermitian generators — real

then for the covariant derivative we obtain:

$$\mathcal{D}_\mu\varphi = \partial_\mu\varphi + ad_{A_\mu}\varphi = \partial_\mu\varphi + [A_\mu, \varphi]; \qquad A_\mu = ig\,t^a A^a_\mu$$

- We can write $\mathcal{D}_\mu\varphi = t^a(\mathcal{D}_\mu\varphi)^a$; So that:

$$t^a(\mathcal{D}_\mu\varphi)^a = t^a\,\partial_\mu\varphi^a + i(-g)A^a_\mu\varphi^b\underbrace{[t^a, t^b]}_{= i f^{abc} t^c} = t^a\cdot\partial_\mu\varphi^a + g f^{abc} A^b_\mu\varphi^c\cdot t^a$$

Hence $\mathcal{D}_\mu\varphi^a = \partial_\mu\varphi^a + g f^{abc}\cdot A^b_\mu\cdot\varphi^c$;

- Lagrangian terms are then: $tr(\mathcal{D}_\mu\varphi\,\mathcal{D}^\mu\varphi) = \mathcal{D}_\mu\varphi^a\cdot\mathcal{D}^\mu\varphi^b\cdot tr(t^a t^b) = \frac{1}{2}\mathcal{D}_\mu\varphi^a\cdot\mathcal{D}^\mu\varphi^a$; $tr\varphi^2 = \frac{1}{2}\varphi^a\varphi^b$;

So that: $\mathcal{L}_{\varphi} = \frac{1}{2}(\mathcal{D}_\mu\varphi)^a(\mathcal{D}^\mu\varphi)^a - \frac{m^2}{2}\varphi^a\cdot\varphi^a - V(\varphi^a\cdot\varphi^a)$;

- Crucial difference with electrodynamics.
  - EM: the charge is arbitrary, i.e. we can have 2 fields $\varphi \to e^{i\alpha}\varphi$, $\chi \to e^{iq\alpha}\chi$ with q - any real number. Covariant derivative will be given by $\mathcal{D}_\mu\chi = (\partial_\mu - ie\cdot q\cdot A_\mu)\chi$.
  - Non-abelian theory has only one coupling g and interaction

(13) between the matter and gauge fields is defined by the representation matter fields transform in.

Equations of motion.

- We start with the action:

$$\boxed{\begin{gathered} S = S_{YM} + S_\varphi \\ S_{YM} = \int d^4x \cdot \left(-\tfrac{1}{4} F^a_{\mu\nu} \cdot F^{a\mu\nu}\right); \qquad S_\varphi = \int d^4x \left[(\mathcal{D}_\mu\varphi)^\dagger \mathcal{D}^\mu\varphi - m^2\varphi^\dagger\varphi - V(\varphi^\dagger\varphi)\right]; \\ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu; \qquad \mathcal{D}_\mu\varphi = \partial_\mu\varphi - ig T^a \cdot A^a_\mu \varphi \end{gathered}}$$

T^a – hermitian generators of representation φ transforms in.

- Varying YM action:

$$\delta S_{YM} = \int d^4x \cdot \left(-\tfrac{1}{2} F^{a\mu\nu} \delta F^a_{\mu\nu}\right), \text{ where}$$

$$\delta F^a_{\mu\nu} = \partial_\mu \delta A^a_\nu - \partial_\nu \delta A^a_\mu + g f^{abc} A^b_\mu \delta A^c_\nu + g \underbrace{f^{abc}\, \delta A^b_\mu \cdot A^c_\nu}_{-f^{abc} A^b_\nu \delta A^c_\mu}$$

due to $f^{abc} = -f^{acb}$;

Then using antisymmetry of $F_{\mu\nu}$:

$$\delta S_{YM} = \int d^4x \left(-\tfrac{1}{2} 2 F^{a\mu\nu} \left(\partial_\mu \delta A^a_\nu + g f^{abc} A^b_\mu \delta A^c_\nu\right)\right)$$

$$\Downarrow$$

$$\partial_\mu F^{\mu\nu a} - g f^{cba} F^{\mu\nu c} A^b_\mu = 0 \Rightarrow \boxed{\partial_\mu F^{a\mu\nu} + g f^{abc} A^b_\mu F^{c\mu\nu} = 0;}$$

- As $F^{\mu\nu}$ transforms in adjoint representation:

$\mathcal{D}_\mu F_{\alpha\beta} = \partial_\mu F_{\alpha\beta} + \mathrm{ad}_{A_\mu} F_{\alpha\beta} = \partial_\mu F_{\alpha\beta} + [A_\mu, F_{\alpha\beta}]$ or in the component form $\mathcal{D}_\mu F^a_{\alpha\beta} = \partial_\mu F^a_{\alpha\beta} + g f^{abc} A^b_\mu F^c_{\alpha\beta}$;

Hence equation of motion can be rewritten as:

$$\boxed{\mathcal{D}_\mu F^{\mu\nu} = 0;}$$

→ looks like Maxwell equation but with the covariant derivative instead the usual one.

- Another equations are given by Bianchi identity:

$$\boxed{\varepsilon^{\mu\nu\alpha\beta} \partial_\nu F_{\alpha\beta} = 0;}$$

→ Prove it!

(14) • Varying scalar field action $\quad S_\varphi = \int d^4x\, [(\mathcal{D}_\mu\varphi)^+ \mathcal{D}^\mu\varphi - m^2\varphi^+\varphi - V(\varphi^+\varphi)]$;

First let's vary w.r.t A_μ:

$(\mathcal{D}_\mu\varphi)^+ = \partial_\mu\varphi^+ + ig A^a_\mu \varphi^+ T^a$; $\quad \mathcal{D}_\mu\varphi = \partial_\mu\varphi - ig A^a_\mu T^a\varphi$

$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$

$\delta_A(\mathcal{D}_\mu\varphi)^+ = ig\varphi^+T^a\delta A^a_\mu$; $\qquad \delta_A(\mathcal{D}_\mu\varphi) = -igT^a\varphi\,\delta A^a_\mu$;

Hence $\quad \delta_A S_\varphi = \int d^4x\, \underbrace{[-(\mathcal{D}_\mu\varphi)^+ igT^a\varphi + ig\varphi^+T^a\mathcal{D}_\mu\varphi]}_{-g\,j^a_\mu}\,\delta A^{a\mu} \equiv -\int d^4x \cdot g\, j^a_\mu\,\delta A^{a\mu}$

Hence the total variation w.r.t. A^a_μ will be given by

$\delta_A S = \int d^4x\, [\mathcal{D}_\mu F^{a\mu\nu} - g\,j^{a\nu}]\,\delta A^a_\nu \Rightarrow$ $\boxed{\mathcal{D}_\mu F^{a\mu\nu} = g\,j^{a\nu}, \text{ where } j^a_\mu = i((\mathcal{D}_\mu\varphi)^+T^a\varphi - \varphi^+T^a\mathcal{D}_\mu\varphi);}$

• Variation w.r.t. scalar field φ:

We as usually take φ^+ and φ to be independent fields

Varying w.r.t. φ^+ one obtains:

$\delta_{\varphi^+}S_\varphi = \int d^4x\, [(\partial_\mu\delta\varphi^+ + igA^a_\mu T^a\delta\varphi^+)\mathcal{D}^\mu\varphi - m^2\delta\varphi^+\varphi - \varphi\frac{\partial V(\varphi^+\varphi)}{\partial(\varphi^+\varphi)}]$;

then the equations of motion are:

$$\underbrace{(\partial_\mu - igA^a_\mu T^a)}_{\text{this is just } \mathcal{D}_\mu}\mathcal{D}^\mu\varphi + m^2\varphi + \varphi\frac{\partial V(\varphi^+\varphi)}{\partial(\varphi^+\varphi)} = 0;$$

because $\mathcal{D}^\mu\varphi$ transforms in the same representation as φ does.

So the equations of motion are:

$$\boxed{\mathcal{D}_\mu\mathcal{D}^\mu\varphi + m^2\varphi + \varphi\frac{\partial V(\varphi^+\varphi)}{\partial(\varphi^+\varphi)} = 0;}$$

This equation together with non-Abelian "Maxwell" equation gives us full system of equations.

• Stress-energy tensor.

Let's find the stress-energy tensor of theory described above. For this we will use the variation of the action w.r.t. metric. We take $\quad S = \int d^4x \cdot \sqrt{-g}\,(-\frac{1}{4}g^{\mu\nu}g^{\lambda\rho}F^a_{\mu\lambda}F^a_{\nu\rho})$;

(15) • After calculation (you should have done similar one for the Maxwell action in the homework 2):

$$\underline{T^{(YM)}_{\mu\nu} = -F^a_{\mu\lambda} F^{a\,\lambda}_{\nu} + \frac{1}{4}\eta_{\mu\nu}\left(F^a_{\alpha\beta} F^{a\alpha\beta}\right);} \to \text{sum of } n = \dim \mathcal{AG} \text{ Maxwell theories (looks like)}$$

• Energy density is

$$T^{(YM)}_{00} = -F^a_{0i} F^{a\,i}_{0} + \frac{1}{4}\left(-F^a_{0i}\cdot F^a_{0i} - F^a_{i0} F^a_{i0} + F^a_{ij}\cdot F^a_{ij}\right)$$

$$\Downarrow$$

$$\underline{T^{(YM)}_{00} = \frac{1}{2} F^a_{0i} F^a_{0i} + \frac{1}{4} F^a_{ij}\cdot F^a_{ij};}$$

Introducing the analogs of electric and magnetic fields:

$$\begin{cases} F^a_{0i} = E^a_i; \\ F^a_{ij} = -\varepsilon_{ijk} H^a_k; \end{cases} \to \text{in QCD we call these chromoelectric and chromomagnetic fields.}$$

Then we obtain the energy of YM field:

$$\underline{E = \int d^3x \left(\frac{1}{2}\bar{E}^a\cdot\bar{E}^a + \frac{1}{2}\bar{H}^a\cdot\bar{H}^a\right);} \to \text{positive definite due to the compactness of the gauge group.}$$

• In the same way we can derive the stress-energy tensor of the scalar field:

$$\mathcal{L}_\varphi = g^{\mu\nu}(D_\mu\varphi)^+ D_\nu\varphi - m^2\varphi^+\varphi - V(\varphi^+\varphi)$$

the only place metric enters the action. Hence:

$$T_{\mu\nu} = \frac{\partial(\sqrt{-g}\,\mathcal{L}_\varphi)}{\partial g^{\mu\nu}} = \sqrt{-g}\cdot 2(D_\mu\varphi)^+ D_\nu\varphi + \frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\mathcal{L}_\varphi$$

$$\frac{\delta g^{\alpha\beta}}{\delta g^{\mu\nu}} = \delta^\alpha_{\ \mu}\delta^\beta_{\ \nu} + \delta^\alpha_{\ \nu}\delta^\beta_{\ \mu};$$

in the lecture 2 we have shown that $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\,\delta g^{\alpha\beta}$

Then we get:

$$\boxed{T_{\mu\nu} = 2(D_\mu\varphi)^+ D_\nu\varphi - \eta_{\mu\nu}\mathcal{L}_\varphi;}$$

strictly speaking it should be $(D_\mu\varphi)^+ D_\nu\varphi + (\mu\leftrightarrow\nu)$ here instead of $2(D_\mu\varphi)^+ D_\nu\varphi$;

• Then for the energy of the scalar field one obtains:

$$E^{(\varphi)} = \int d^3x\, T_{00} = \int d^3x \left(2(D_0\varphi)^+ D_0\varphi - (D_0\varphi)^+ D_0\varphi + (D_i\varphi)^+ D_i\varphi + m^2\varphi^+\varphi + V(\varphi^+\varphi)\right) \Rightarrow$$

$$\Rightarrow \boxed{E^{(\varphi)} = \int d^3x \left((D_0\varphi)^+ D_0\varphi + (D_i\varphi)^+ D_i\varphi + m^2\varphi^+\varphi + V(\varphi^+\varphi)\right);}$$

Lecture 6: Spontaneous symmetry Breaking. Goldstone theorem. ①

• Simplest example: discrete symmetry.

Let's consider real scalar field theory:

$$\mathcal{L}_\varphi = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{1}{4}\lambda\varphi^4$$

the symmetry of the Lagrangian is $\varphi \to -\varphi$ (discrete symmetry) $\mathbb{Z}_2$

• The energy of the field is:

$$E = \int d^3x \left[\frac{1}{2}(\partial_0\varphi)^2 + \frac{1}{2}(\partial_i\varphi)^2 + \frac{1}{2}m^2\varphi^2 + \frac{1}{4}\lambda\varphi^4\right];$$

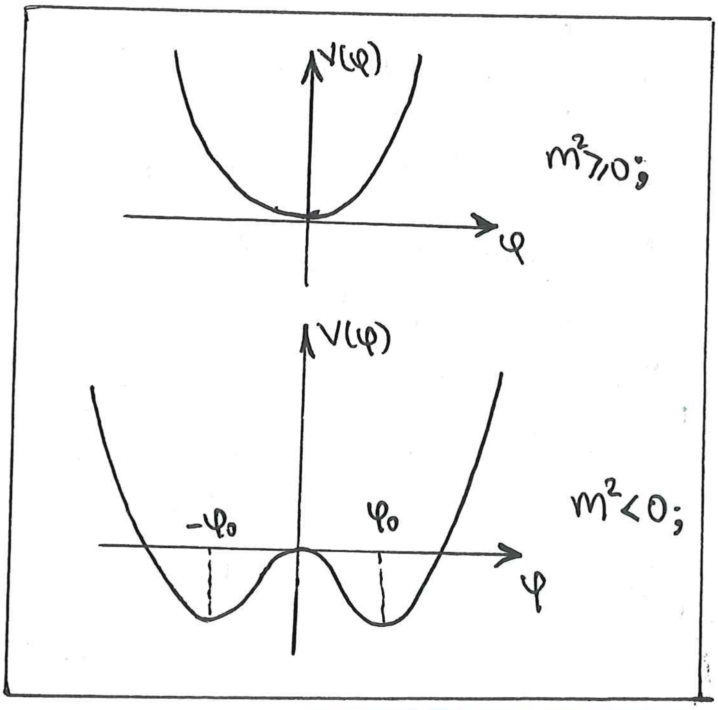
• In order for the energy to be bounded below we demand $\lambda > 0$, however notice that m^2 can be both positive and negative.

• Ground state is the field $\varphi(x)$ for which the energy is minimized.

From the expression above it is obvious that the field should be static and constant in space: $\partial_t\varphi = 0,\ \partial_i\varphi = 0; \Rightarrow \varphi = \text{const.}$

the constant is found from the potential minimization:

$$V(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{1}{4}\lambda\varphi^4;$$



① if $m^2 > 0$ then the minimum is at $\varphi = 0$, it is invariant under $\mathbb{Z}_2$-symmetry. Then we say that this ground state does not break the symmetry of the theory.

② if $m^2 < 0$ then introducing $\mu^2 = -m^2 > 0$ we obtain:

$$\frac{\partial V}{\partial \varphi} = -\mu^2\varphi + \lambda\varphi^3 = 0 \Rightarrow \begin{cases} \varphi = 0 - \text{maximum} \\ \varphi = \pm\frac{\mu}{\sqrt{\lambda}} = \pm\varphi_0 \end{cases}$$

two minima. (each of them is not invariant under $\mathbb{Z}_2$-symm.)

• For the infinite volume spaces theory is fixed at one of the ground state (transition between ground states require the energy proportional to the volume of space).

② • Hence we consider only small fluctuations around this ground state.

• Let's choose $\varphi = \varphi_0 = \frac{\mu}{\sqrt{\lambda}}$; the energy of the ground state is then. $E_0 = \underset{\text{volume of space.}}{\Omega} V(\varphi_0) = -\frac{1}{4}\frac{\mu^4}{\lambda} \cdot \Omega$;

• It is convinient to shift ground state energy to zero by adding the corresponding term to the Leagrangian:

$$\mathcal{L}_\varphi = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2}\mu^2\varphi^2 - \frac{\lambda}{4}\varphi^4 + \frac{\mu^4}{4\lambda}$$ or equivalently

$$\boxed{\mathcal{L}_\varphi = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{\lambda}{4}(\varphi^2 - \varphi_0^2)^2;}$$

• Let's now consider perturbations around φ_0 : $\varphi(x) = \varphi_0 + \chi(x)$;

$\mathcal{L}_{\varphi\chi}(\chi) = \mathcal{L}_\varphi(\varphi_0 + \chi)$ will be the Leagrangian for χ-field.

$\begin{cases} \text{kinetic term: } \partial_\mu(\varphi_0 + \chi) = \partial_\mu \chi; \\ \text{potential: } V(\varphi_0 + \chi) = \frac{\lambda}{4}((\varphi_0+\chi)^2 - \varphi_0^2)^2; \end{cases}$

then $V_\chi(\chi) = V(\varphi_0 + \chi) = \frac{\lambda}{4}(2\varphi_0\chi + \chi^2)^2 = \frac{2}{2}\lambda\varphi_0^2\chi^2 + \lambda\varphi_0\chi^3 + \frac{\lambda}{4}\chi^4$

Hence the Leagrangian for the field χ is given by:

$$\boxed{\mathcal{L}_\varphi = \frac{1}{2}(\partial_\mu \chi)^2 - \mu^2\chi^2 - \mu\sqrt{\lambda}\chi^3 - \frac{\lambda}{4}\chi^4;}$$

• We see that the field is massive with the mass $m_\chi^2 = 2\mu^2$

• Leagrangian now doesn't have any invariance, which is reasonable because ground state was not $\mathbb{Z}_2$-invariant.

• So to summarize:

> Spontaneous symmetry breaking is the situation when ground state of the theory does not possess the same symmetry as Leagrangian does!

③ Spontaneous breaking of U(1)-symmetry

- Let's consider the theory of one scalar field:

$$\boxed{\mathcal{L}_\varphi = \partial_\mu\varphi^*\partial^\mu\varphi - \underbrace{m^2\varphi^*\varphi - \lambda(\varphi^*\varphi)^2 - c}_{V(\varphi,\varphi^*)}}$$

here c is just the constant that we will define later.

- The energy of this scalar field is given by

$$E = \int d^3x\, (\partial_0\varphi^*\partial_0\varphi + \partial_i\varphi^*\partial_i\varphi + V(\varphi^*,\varphi))$$

where $V(\varphi^*,\varphi) = m^2\varphi^*\varphi + \lambda(\varphi^*\varphi)^2 + c$;

- Theory possess U(1)-symmetry $\varphi \to e^{i\alpha}\varphi$, $\alpha = \text{const}$ ← global symmetry.

- The ground state (minimum of energy) is given by the constant field $\varphi = \text{const}$ which is found from minimizing $V(\varphi^*,\varphi)$:

$V(\varphi^*,\varphi) = m^2|\varphi|^2 + \lambda|\varphi|^4 + C$, minimizing w.r.t. $|\varphi|$ we should consider two cases:

- $m^2 \geqslant 0$ $\quad \dfrac{\partial V}{\partial|\varphi|} = 2m^2|\varphi| + 4\lambda|\varphi|^3 = 0 \Rightarrow \boxed{|\varphi| = 0}$

the minimum is at the origin. this is only one ground state which is invariant under U(1)-symm.

- $m^2 = -\mu^2 < 0$ thera are 2 solutions:

$|\varphi| = 0$ – local maximum;

$|\varphi|^2 = \dfrac{\mu^2}{2\lambda}$ – set of minima.

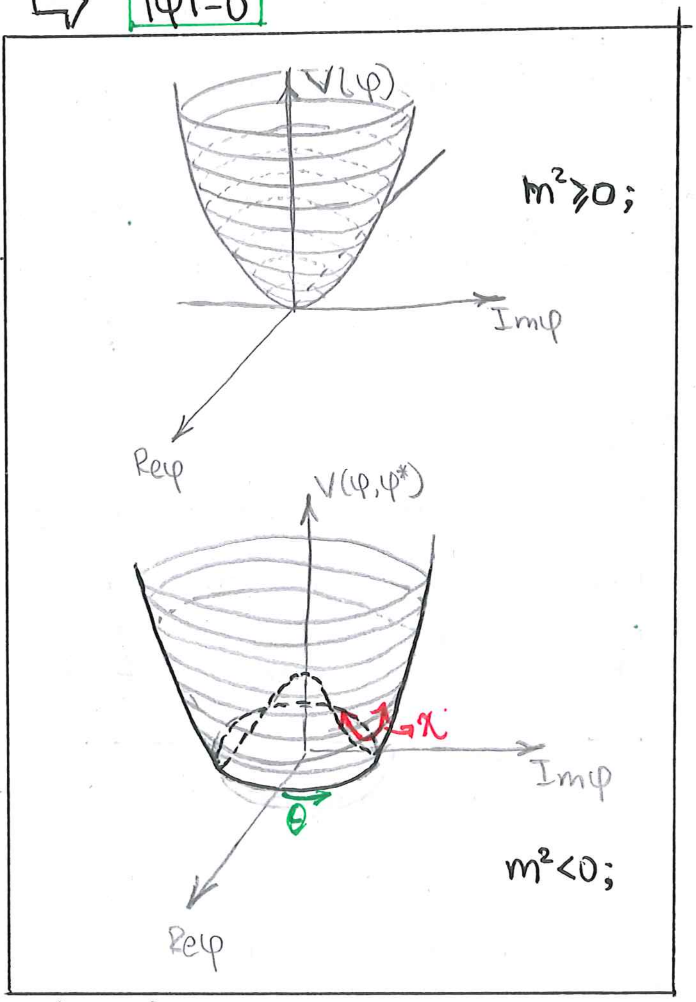
- We should choose only one of these many minima.

I.e. let's take the ground state in the form

$$\boxed{\varphi = \frac{1}{\sqrt{2}}\varphi_0;\ \varphi_0 = \frac{\mu}{\sqrt{\lambda}};}$$

↓

This ground state is not U(1)-symmetric. Symmetry is spontaneously broken!



④ • Let's now consider small perturbation around this ground state:

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_0 + \chi(x))\, e^{i\theta(x)}; \quad \varphi_0 = \frac{\mu}{\sqrt{\lambda}};$$

• Here $\theta(x)$ are perturbations along the circle of ground state (valley), while $\chi(x)$ are pertirbations in radial direction. Then:

• $\partial_\mu \varphi^*(x) \cdot \partial^\mu \varphi(x) = \frac{1}{2}(\partial_\mu \chi - i\partial_\mu \theta \cdot \varphi_0)e^{-i\theta}(\partial^\mu \chi + i\partial_\mu \theta \cdot \varphi_0)e^{i\theta} + \ldots =$

$= \frac{1}{2}\partial_\mu \chi \partial^\mu \chi + \frac{1}{2}\varphi_0^2 \partial_\mu \theta \partial^\mu \theta + \ldots$, where dots are standing for higher order terms.

• Potential: $V(\varphi^*, \varphi) = m^2 \varphi^* \varphi + \lambda(\varphi^* \varphi)^2 + C = -\frac{1}{2}\mu^2(\varphi_0 + \chi)^2 + \frac{1}{4}\lambda(\varphi_0 + \chi)^4 + C =$

$= \left(C - \frac{1}{2}\mu^2\varphi_0^2 + \frac{1}{4}\lambda\varphi_0^4\right) - \mu^2\varphi_0\chi + \lambda\varphi_0^3\chi + \left(\frac{3}{2}\lambda\varphi_0^2 - \frac{1}{2}\mu^2\right)\chi^2 =$

$= \left(C - \frac{1}{4}\frac{\mu^4}{\lambda}\right) + \mu^2\chi^2 + \ldots.$

• To put ground state energy to zero we choose $C = \frac{1}{4}\frac{\mu^4}{\lambda}$;

• So up to the quadratic terms:

$$\mathcal{L}_\varphi = \frac{1}{2}\partial_\mu \chi \partial^\mu \chi + \frac{1}{2}\varphi_0^2 \cdot \partial_\mu \theta \cdot \partial^\mu \theta \quad \mu^2\chi^2;$$

• χ-field has the mass $m_\chi^2 = 2\mu^2$; (nonzero curvature in radial direction)

• θ-field is massless $m_\theta = 0$; (zero curvature along the valley)

This massless fields always appear when the symmetry is broken. They are called Nambu-Goldstone boson.

Physics:

• 1950 Ginsburg-Landau theory: phenomenological theory

Free energy of superconductor near the critical point:

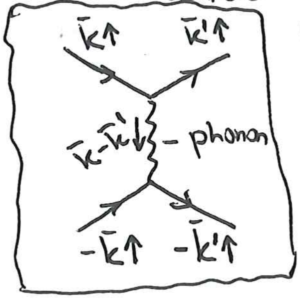
$$\boxed{F = \underset{\substack{\uparrow \\ \text{non-supercond.} \\ \text{state free energy}}}{F_n} + \alpha|\varphi|^2 + \frac{1}{2}\beta|\varphi|^4 + \frac{1}{2\underset{\substack{\uparrow \\ \text{effective} \\ \text{mass}}}{m}}|(-i\hbar\bar{\nabla} - 2e\bar{A})\varphi|^2 + \frac{|\bar{B}|^2}{2\mu_0}}$$

φ here is order parameter characterizing the phase of the material:

$$\boxed{\varphi = \begin{cases} 0 & \text{- non SC state.} \\ 1 & \text{- SC state.} \end{cases}}$$

⑤. 1957: Bardeen, Cooper, Schrieffer: microscopic theory

Zero temperature scattering of two electrons above the Fermi level leads to instability i.e. electrons tend to form bound state: Cooper pair. with the w.f.:



$$|\Psi\rangle = \sum_{|\bar{k}|>k_F} \phi_k \, |\bar{k}\uparrow, -\bar{k}\downarrow\rangle;$$

1959: Gor'kov: derived GL theory from BCS theory Showed that order parameter is just given by the consentration of the Cooper pairs:

$$\varphi \doteq \langle c^+_{k\uparrow} c^+_{-k\downarrow}\rangle;$$

Partial symmetry breaking: SO(3).

- Consider theory of 3 real scalar fields:

$$\mathcal{L}_\varphi = \frac{1}{2}\partial_\mu\varphi^a\partial^\mu\varphi^a - V(\varphi)$$

with the potential $V(\varphi) = -\frac{\mu^2}{2}\varphi^a\varphi^a + \frac{\lambda}{4}(\varphi^a\varphi^a)^2 + \frac{\mu^4}{4\lambda};$

- The symmetry of theory is SO(3)
- Ground state is $\varphi = \text{const}$ with constant minimizing potential $V(\varphi)$:

$$\frac{\partial V}{\partial\varphi^a} = -\mu^2\varphi^a + \lambda\varphi^a(\varphi^b\varphi^b) = 0 \Rightarrow \begin{cases} \varphi^a = 0 - \text{local maximum at the origin.} \\ \varphi^b\varphi^b = \varphi_0^2 = \frac{\mu^2}{\lambda}; - \text{set of minima lying on } S^2. \end{cases}$$

- Let's choose the ground state to be

$$\begin{cases} \varphi^1 = \varphi^2 = 0; \\ \varphi^3 = \varphi_0; \end{cases}$$

- This ground state is invariant under rotations in (φ^1, φ^2)

i.e. $\omega\varphi^{(0)} = \varphi^{(0)}$ $\omega \in SO(2) \to$ subgroup of SO(3). $\varphi^{(0)} \equiv \begin{pmatrix} 0 \\ 0 \\ \varphi_0 \end{pmatrix};$

In component form this rotation looks like

$$\varphi^1 \to \cos\alpha\cdot\varphi^1 + \sin\alpha\,\varphi^2;$$
$$\varphi^2 \to -\sin\alpha\cdot\varphi^1 + \cos\alpha\cdot\varphi^2;$$
$$\varphi^3 \to \varphi^3$$

⑥ • Let's now define quadratic part of Lagrangian for perturb.

$$\boxed{\begin{aligned}\varphi_1(x)&=\theta_1(x);\\ \varphi_2(x)&=\theta_2(x);\\ \varphi_3(x)&=\varphi_0+\chi(x);\end{aligned}}$$

• Kinetic term:

$$\mathcal{L}_{kin}=\frac{1}{2}\partial_\mu\varphi^a\partial^\mu\varphi^a=\frac{1}{2}\partial_\mu\theta_1\partial^\mu\theta_1+\frac{1}{2}\partial^\mu\theta_2\partial_\mu\theta_2+\frac{1}{2}\partial_\mu\chi\partial^\mu\chi$$

• Potential term:

$$V=-\frac{1}{2}\mu^2(\theta_1^2+\theta_2^2)-\frac{1}{2}\mu^2(\varphi_0+\chi)^2+\frac{1}{4}\lambda\left(\overbrace{\theta_1^2+\theta_2^2+(\varphi_0+\chi)^2}^{0}\right)^2+\frac{\mu^4}{4\lambda}$$

$$=(\text{keeping quadratic terms only})=(\theta_1^2+\theta_2^2)\left(-\frac{1}{2}\mu^2+\frac{1}{2}\lambda\varphi_0^2\right)+$$

$$-\mu^2\varphi_0\chi+\frac{1}{4}\lambda\cdot4\varphi_0^3\chi+\chi^2\left(-\frac{1}{2}\mu^2+4\cdot\frac{1}{4}\lambda\cdot\varphi_0^2+\frac{1}{4}\lambda\cdot2\varphi_0^2\right)=\mu^2\chi^2$$

So the quadratic Lagrangian has the form:

$$\mathcal{L}_p=\frac{1}{2}(\partial_\mu\theta_1)^2+\frac{1}{2}(\partial_\mu\theta_2)^2+\frac{1}{2}(\partial_\mu\chi)^2-\mu^2\chi^2;$$

• θ_1 and θ_2 are massless NG bosons.

• χ is massive excitations in the radial direction.

• Let's consider symmetry generators.

SO(3) has 3 generators. One of them annihilates $\bar{\varphi}^{(0)}$:

$t_n\varphi^{(0)}=0$ because for $\omega=1+\varepsilon t_n$ and $\omega\bar{\varphi}^{(0)}=\bar{\varphi}^{(0)}$

↳ this generator corresponds to SO(2) subgroup.

• There are 2 more generators t_1 and t_2. We can form two vectors of them: $\bar{n}_1=t_1\bar{\varphi}^{(0)}$; $\bar{n}_2=t_2\bar{\varphi}^{(0)}$; → Vectors are linearly independent because otherwise linear combination of t_1 and t_2 would annihilate $\bar{\varphi}^{(0)}$

• Let's built the vacuum close to $\bar{\varphi}^{(0)}$;

$$\bar{\varphi}=\bar{\varphi}^{(0)}+\tilde{\theta}^1\bar{n}_1+\tilde{\theta}^2\bar{n}_2=(1+\tilde{\theta}_1t_1+\tilde{\theta}_2t_2)\varphi^{(0)}=$$
$$=\omega\varphi^{(0)};$$

small parameters.

• Due to the symmetry $V(\bar{\varphi})=V(\bar{\varphi}^{(0)})=0=V(\bar{\varphi}^{(0)}+\tilde{\theta}_1\bar{n}_1+\tilde{\theta}_2n_2)=$

$$=(\tilde{\theta}_1\bar{n}_1+\tilde{\theta}_2\bar{n}_2)^2\left.\frac{\partial^2V}{\partial\varphi^2}\right|_{\varphi=\varphi^{(0)}}+\ldots=0$$. Hence in quadratic order in fields potential is zero ⇒ no terms quadratic in $\tilde{\theta}_1$ and $\tilde{\theta}_2$. ⇒ $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are massless fields!

⑦ • Generators of $SO(3)$ are $(t_a)_{bc} = \varepsilon_{abc}$

the vacuum we have chosen $\varphi^{(0)a} = \delta^{a3} \cdot \varphi_0$

• Generator which is not broken is t_3: $(t_3)_{bc}\,\varphi^{(0)c} = \varepsilon_{3bc}\,\delta^{c3}\,\varphi_0 = 0.$

• Broken generators are t_1 and t_2

$$\begin{cases} n_1^a = (t_1)_{ab}\,\varphi^{(0)b} = \delta^{a2}\,\varphi_0; \\ n_2^a = (t_2)_{ab}\,\varphi^{(0)b} = \varepsilon_{2ab}\,\delta^{b3}\,\varphi_0 = -\delta^{a1}\,\varphi_0; \end{cases}$$

• Hence massless perturbation looks like

$$\bar{\varphi}(x) = (-\tilde{\theta}_2(x)\varphi_0,\ \tilde{\theta}_1(x)\varphi_0,\ \varphi_0); \Rightarrow \begin{matrix} \theta_1 = -\tilde{\theta}_2\varphi_0; \\ \theta_2 = \tilde{\theta}_1\varphi_0; \end{matrix}$$

• Similar construction can be built for any compact group and any unitary representation.

Comment: If we take vacuum in form $\varphi^{(0)} = (0, 0, \varphi_0)$ then we will not nesseraly be lucky to have fields θ_1 and θ_2 being NG modes. In general we put $\bar{\varphi} = \bar{\varphi}_0 + \bar{n}_i \cdot \xi_i$; $i = 1,2,3$.

• Then kinetic term is $\mathcal{L}_{kin} = \frac{1}{2}\partial_\mu \xi^i \cdot \partial^\mu \xi^j (\bar{n}_i \cdot \bar{n}_j)$;

• Quadratic part of the potential is $V^{(2)} = \frac{1}{2} M_{ij}\,\xi_i \xi_j$ mass matrix.

• To find NG modes we should diagonalize Lagrangian by

① choosing orthonormal basis $\bar{n}_i \cdot \bar{n}_j = \delta_{ij}$;

② Performing orthogonal transformation $\xi'^i = O^i{}_j\,\xi^j$

which will diagonalize mass matrix: $M_{ij}\,\xi_i \xi_j = \sum_i m_i^2\, \xi'^i \xi'^i$;

Some of M_{ij} eigenvalues (in our case two of them) will just be zero.

• Generalization. Goldstone theorem.

• Let's consider theory of scalar fields which will be denoted as φ. Let G be the global symmetry group

• $\varphi(x)$ transforms in the representation $T(\omega)$ of the group.

• Lagrangian: $\mathcal{L}_\varphi = \frac{1}{2}(\partial_\mu\varphi, \partial^\mu\varphi) - V(\varphi)$, where (φ_1, φ_2) is the scalar product of scalar fields.

⑧ For example

* fundamental repr. $\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_N \end{pmatrix}$; $(\varphi^{(1)}, \varphi^{(2)}) = \varphi_{(1)}^{T} \cdot \varphi_{(2)}$;
* adjoint representation: $\varphi \in AG$ $\quad (\varphi^{(1)}, \varphi^{(2)}) = -\operatorname{tr}(\varphi^{(1)} \cdot \varphi^{(2)})$;

- Kinetic term is invariant due to the unitarity of the representation $T(\omega): \varphi \to T(\omega)\varphi$.
- For potential term we demand $V(T(\omega)\varphi) = V(\varphi)\ \forall\, \omega \in G$;
- Let's choose some ground state $\varphi(x) = \varphi_0$.

φ_0 is as usually found from the minimization of the potential

$$\left.\frac{\partial V}{\partial \varphi}\right|_{\varphi=\varphi_0} = 0;$$

- Let H be the stationary subgroup of the ground state:

$$T(h)\varphi_0 = \varphi_0 \quad \forall\, h \in H$$

H is unbroken subgroup of G.

- Let t_h, $h = 1, \dots, \underset{\dim H}{R_H}$ be the generators of the subgroup H.

t'_α, $\alpha = 1, \dots, \underset{\dim G}{R_G} - R_H$ be all othe generators of G supplementing $\{t_h\}$ to the complete orthonormal basis in AG.

- As $T(h) = T(1 + \varepsilon^h t_h) = 1 + \varepsilon^h T(t_h) = 1 + \varepsilon^h T_h$ $\to$ by def $T_h \equiv T(t_h)$;

and $T(h)\varphi_0 = (1 + \varepsilon^h T_h)\varphi_0 = \varphi_0 \Rightarrow T_h \varphi_0 = 0$;

Notice that $\forall A \in AH$ $\quad T(A)\varphi_0 = 0$

while for any linear combination of t'_α $\quad T(c^\alpha t'_\alpha)\varphi_0 \neq 0 \quad \forall\, c^\alpha \neq 0$;

- t'_α are broken generators, t_h are unbroken one.
- Now let's consider as usually perturbations around the ground state. $\quad \varphi(x) = \varphi_0 + \chi(x)$
- Lagrangian is given by:

$$\mathcal{L}_{ex}(\chi) = \frac{1}{2}(\partial_\mu \chi, \partial^\mu \chi) - V(\varphi_0 + \chi);$$

Notice that $\mathcal{L}_{ex}$ is invariant under action of H:

$$\mathcal{L}_{ex}(\chi) = \mathcal{L}(\varphi_0 + \chi) \Rightarrow \mathcal{L}_{ex}(T(h)\chi) = \mathcal{L}(\varphi_0 + T(h)\chi) = \mathcal{L}(T(h)\varphi_0 + T(h)\chi)$$

⑨ where in the last line we use $T(h)\varphi_0 = \varphi_0$;
finally $\mathcal{L}_e(T(h)(\varphi_0+\chi)) = \mathcal{L}_e(\varphi_0+\chi)$ due to the invariance of $\mathcal{L}_e$ w.r.t. to the action of element of G.

- Let's consider perturbations along the directions $T'_\alpha \varphi_0$ i.e. perturbations of the form $\chi_\alpha(x) = \theta_\alpha T'_\alpha \varphi_0$
  $\theta_\alpha(x)$ - are independent fields because vectors $T'_\alpha \varphi_0$ are linearly independent.
- General perturbation will have the form:

$$\chi(x) = \sum_\alpha \theta_\alpha(x) T'_\alpha \varphi_0 + \eta(x);$$

$\eta(x)$: modes transverse to all $\theta_\alpha(x)$.

- Let's consider

$$V[T(g)(\varphi_0+\eta)] \quad \text{where} \quad g = 1 + \theta_\alpha t'_\alpha + B\theta^2 \in G$$

due to the G-invariance of the potential:

$$V(T(g)(\varphi_0+\eta)) = V(\varphi_0+\eta)$$

At the same time:

$$V(T(g)(\varphi_0+\eta)) = V((1+\theta_\alpha T'_\alpha + \tilde{B}\theta^2)(\varphi_0+\eta)) = \left(\begin{matrix}\text{omitting cubic an}\\ \text{higher terms}\end{matrix}\right)$$

$$= V(\varphi_0 + \theta_\alpha T'_\alpha \varphi_0 + \eta + \tilde{B}\theta^2 + \theta_\alpha T'_\alpha \eta) = \left(\begin{matrix}\text{expanding around}\\ \varphi_0 + \theta_\alpha T'_\alpha \varphi_0 + \eta\end{matrix}\right) =$$

$$= V(\varphi_0 + \theta_\alpha T'_\alpha \varphi_0 + \eta) + \left.\frac{\partial V}{\partial \varphi}\right|_{\varphi_0 + \theta_\alpha T'_\alpha \varphi_0 + \eta} \cdot (B\theta^2 + \theta_\alpha T'_\alpha \eta)$$

As φ_0 minimizes the potential this term is at least linear in fields.

- Hence we obtain $V(T(g)(\varphi_0+\eta)) = V(\varphi_0 + \theta_\alpha T'_\alpha \varphi_0 + \eta) + \ldots$
  where dots stand for cubic and higher terms, and hence

$$\boxed{V(\varphi_0 + \theta_\alpha T'_\alpha \varphi_0 + \eta) = V(\varphi_0+\eta) + \text{cubic terms}}$$

Thus θ_α fields are massless (no quadratic terms)

> Goldstone theorem: Theory with spontaneously broken global symmetry has at least as many massless scalar fields as there are broken generators.

Lecture 7: Higgs mechanism.

In this lecture we consider theories with nontrivial ground states which also possess gauge symmetry.

Simpliest case: Abelian symmetry.

Let's start with considering the simpliest possible model: scalar electrodynamics:

$$\mathcal{L}_\varphi = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\varphi)^* D^\mu\varphi - \underbrace{(-\mu^2\varphi^*\varphi + \lambda(\varphi^*\varphi)^2)}_{V(\varphi,\varphi^*)}$$

φ is the complex scalar field

- Gauge symmetry of the Leagrangian:

$$\begin{cases} A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha(x); \\ \varphi(x) \to \varphi'(x) = e^{i\alpha(x)}\varphi(x); \end{cases}$$

- Energy of the field is

$$E = \int d^3x \cdot \left[\frac{1}{2}(F_{0i})^2 + \frac{1}{4}(F_{ij})^2 + (D_0\varphi)^* D_0\varphi + (D_i\varphi)^* D_i\varphi + V(\varphi^*,\varphi)\right];$$

- As energy is gauge invariant object, if $(A_\mu^{vac.}, \varphi^{vac.})$ is the ground state then $(A_\mu^{vac.} + \frac{1}{e}\partial_\mu\alpha, e^{i\alpha}\cdot\varphi^{vac.})$ is another ground state. We should choose one of these ground states

  - $A_\mu = \frac{1}{e}\partial_\mu\alpha(x)$

  $\uparrow$

  A_μ should be pure gauge because in this case first two terms are minimized, when both electric and magnetic fields are zero.

- Minimization of scalar field terms leads to

$$D_\mu\varphi = (\partial_\mu - i(\partial_\mu\alpha))\varphi = 0 \Rightarrow \varphi(x) = \frac{1}{\sqrt{2}}e^{i\alpha(x)}\cdot\varphi_0;$$

φ_0 here can be found from the potential minimization

$$\left.\frac{\partial V}{\partial\varphi}\right|_{\varphi=\frac{1}{\sqrt{2}}\varphi_0 e^{i\alpha}} = \varphi^*(-\mu^2 + 2\lambda\varphi^*\varphi)\Big|_{\varphi=\frac{1}{\sqrt{2}}\varphi_0 e^{i\alpha}} \propto (-\mu^2 + \varphi_0^2\lambda) \Rightarrow \varphi_0 = \frac{\mu}{\sqrt{\lambda}};$$

② • We should choose one of these vacua. The simpliest choise is $\alpha(x)=0 \Rightarrow A_\mu^{vac.}=0;\ \varphi^{vac.}=\frac{1}{\sqrt{2}}\varphi_0;$ → vacuum

• Now let's study perturbations around this ground state:

$\varphi(x)=\frac{1}{\sqrt{2}}(\varphi_0+\chi(x))e^{i\theta(x)}$,

perturbations of gauge field are just around $A_\mu=0$.

• If the symmetry were global we know that χ would be massive excitation while $\theta(x)$ would be massless NG Boson. Situation in the case of gauge symmetry is slightly different. Let's study what happens:

• Potential term:

the potential term has the same form as in the case of global symmetry $V(\varphi)=\mu^2\chi^2+\ldots$ ↳ higher order terms.

• Kinetic term:

Kinetic term in this case have slightly different structure

$D_\mu\varphi=\frac{1}{\sqrt{2}}(\partial_\mu\chi+i\partial_\mu\theta\,\varphi_0-ieA_\mu\varphi_0)e^{i\theta(x)}$ then for the kinetic term we obtain:

$$\mathcal{L}^{\varphi}_{kin}=|D_\mu\varphi|^2=\frac{1}{2}|\partial_\mu\chi+i\varphi_0(\partial_\mu\theta-eA_\mu)|^2=\frac{1}{2}(\partial_\mu\chi)^2+\frac{1}{2}e^2\varphi_0^2\left(A_\mu-\frac{1}{e}\partial_\mu\theta\right)^2$$

• Kinetic term of EM field is the same as theoriginal one because we perturb around the $A_\mu=0$ vacuum: $\mathcal{L}^{A}_{kin}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$;

• Then writing out full Leagrangian in quadratic order we obtain:

$$\mathcal{L}_{\varphi_2}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}(\partial_\mu\chi)^2+\frac{1}{2}e^2\varphi_0^2\left(A_\mu-\frac{1}{e}\partial_\mu\theta\right)^2-\mu^2\chi^2;$$

Notice that gauge field A_μ mixes with the scalar field θ.

To diagonalize the Leagrangian we should introduce the field:

$$B_\mu\equiv A_\mu-\frac{1}{e}\partial_\mu\theta;$$

Then corresponding field strength is $B_{\mu\nu}\equiv\partial_\mu B_\nu-\partial_\nu B_\mu=\partial_\mu A_\nu-\partial_\nu A-$
$-\frac{1}{e}(\partial_\mu\partial_\nu\theta-\partial_\nu\partial_\mu\theta)=F_{\mu\nu}$.

③ Then the Leagrangian becomes:

$$\boxed{\mathcal{L}_{e2} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} e^2 \varphi_0^2 B_\mu B^\mu + \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - \mu^2 \chi^2;}$$

• To summarize:

• In the beginning we had massless gauge field A_μ containing 2 physical d.o.f. 2 polarizations (see lecture 1 we had 2 physical polarizations $e_\mu^{(1)}$ and $e_\mu^{(2)}$ and 1 gauge d.o.f. along k_μ which can be gauged out.)

We also have complex scalar field (φ, φ^*) - 2 extra d.o.f.

Hence if we take theory with the normal sign of mass term $(m^2 > 0)$ and perturbations are around zero we have 4 d.o.f.

• After breaking the symmetry we obtain:

Massive vector field B_μ with the mass $m_B = e\varphi_0 = \frac{e\mu}{\sqrt{\lambda}}$; - 3 d.o.f.

Comment: The mass of the vector field can be found from the equation of motion for the Leagrangian:

$$\mathcal{L}_e = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} e^2 \varphi_0^2 B_\mu B^\mu$$ variating the corresponding action we obtain $\delta S = \int d^4x \left(\partial_\mu B^{\mu\nu} + e^2 \varphi_0^2 B^\nu\right) \delta B_\nu = 0 \Rightarrow$

$\Rightarrow$ $\partial_\mu B^{\mu\nu} + e^2 \varphi_0^2 B^\nu = 0;$

Taking ∂_ν derivative we find $\partial_\nu \partial_\mu B^{\mu\nu} + e^2 \varphi_0^2 \partial_\nu B^\nu = 0 \Rightarrow$

$\Rightarrow$ (using $B_{\mu\nu} = -B_{\nu\mu}$) $\partial_\nu B^\nu = 0$. Substituting it back we obtain system of two equations equivalent to the original one: $\partial_\mu B^{\mu\nu} = \partial_\mu \partial^\mu B^\nu - \underbrace{\partial^\nu \partial_\mu B^\mu}_{0}$, hence

$$\boxed{\begin{aligned} &\partial_\mu \partial^\mu B_\nu + e^2 \varphi_0^2 B_\nu = 0; \\ &\partial_\nu B^\nu = 0; \end{aligned}}$$

• First equation corresponds to the Klein-Gordon written for each component of B_μ field.

• Notice that massive vector field is not gauge field, as it's action is not invariant under gauge transformations $B_\mu \to B_\mu + \frac{1}{e} \partial_\mu \alpha$.

(4) • In addition we have massive real scalar field. of mass $m_\chi = \sqrt{2}\mu$. We call it Higgs field. It gives one more degree of freedom.

• After symmetry breaking we again have 4 d.o.f.

• Gauge invariance: Let's consider transformation of B_μ under the gauge transformations:

$$B_\mu \to B'_\mu = A'_\mu - \frac{1}{e}\partial_\mu\theta' = A_\mu + \cancel{\frac{1}{e}\partial_\mu\alpha} - \frac{1}{e}\partial_\mu\theta - \cancel{\frac{1}{e}\partial_\mu\alpha} = B_\mu$$

where we have used
$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha;$$
$$\theta \to \theta' = \theta + \alpha;$$

So that B_μ is gauge invariant.

• the same is true about χ fields. Hence all variables in the Lagrangian we consider are gauge-invariant.

• After spontaneous symmetry breaking gauge field adsorb one scalar d.o.f. and becomes massive. This scalar d.o.f is always NG Bosons. Described phenomenon is called Higgs mechanism.

• Another way to obtain results above is to fix the gauge at $\alpha(x) = 0$ or equivalently
$$\boxed{\begin{aligned} &\mathrm{Im}\,\varphi = 0; \\ &\mathrm{Re}\,\varphi = \frac{1}{\sqrt{2}}(\varphi_0 + \chi(x)); \end{aligned}}$$
this is called unitary gauge.

In this gauge $A_\mu = B_\mu$, then

$D_\mu\varphi = \frac{1}{\sqrt{2}}(\partial_\mu\chi - ie\varphi_0 A_\mu)$ so that:

$$\mathcal{L}_e = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e^2\varphi_0^2}{2}A_\mu^2 + \frac{1}{2}(\partial_\mu\chi)^2 - \mu^2\chi^2;$$

• Non-Abelian model: broken $SU(2)$-symmetry.

Let's consider $SU(2)$ gauge theory with the doublet of scalar fields $\varphi = \begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix} \to$ fundamental representation.

$$\mathcal{L}_e = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + (D_\mu\varphi)^\dagger D^\mu\varphi - \underbrace{[-\mu^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2]}_{V(\varphi^\dagger;\varphi)};$$

• Energy of the system is:

$$E(A^a_\mu,\varphi) = \int d^3x\left[\frac{1}{2}F^a_{0i}F^a_{0i} + \frac{1}{4}F^a_{ij}F^a_{ij} + (D_0\varphi)^\dagger D_0\varphi + (D_i\varphi)^\dagger D_i\varphi + V(\varphi^\dagger,\varphi)\right]$$

⑤ • Ground state is given by:

- for A_μ field $F^a_{\mu\nu}=0$ so that A_μ is pure gauge

$$A_\mu(x)=\omega(x)\partial_\mu\omega^{-1}(x);$$

- for the scalar field $\varphi(x)$ is constant: $D_\mu\varphi(x)=0 \Rightarrow$

$\Rightarrow (\partial_\mu+\omega\partial_\mu\omega^{-1})\varphi(x)=0$ if we substitute $\varphi(x)=\omega(x)\cdot\underbrace{\varphi_{gs}}_{\text{const.}}$

$(\partial_\mu-\partial_\mu\omega\cdot\omega^{-1})\omega\varphi_0=0$. Constant φ_0 as usually is found from the minimization of the potential: $\frac{\partial V}{\partial\varphi^+}=-\mu^2\varphi+2\lambda\varphi\,\varphi^+\varphi=0$

so that the minimum should satisfy $\varphi^+\varphi=\frac{\mu^2}{2\lambda};$

- Let's choose the ground state:

$$\boxed{A^a_\mu=0\,,\ \varphi_{gs}=\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\varphi_0\end{pmatrix},\text{where } \varphi_0=\frac{\mu}{\sqrt{\lambda}};} \rightarrow \text{ground state we choose.}$$

- Now we start perturbing around the ground state.

Perturbations around this ground state can be written as

$$\varphi(x)=\omega(x)\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}(\varphi_0+\chi(x))\end{pmatrix} \text{ where } \omega\in SU(2);$$

- In order to make our life easier we fix unitary gauge ($\omega=0$) so that perturbations take the form:

$$\varphi(x)=\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}(\varphi_0+\chi(x))\end{pmatrix};$$

- Let's write down Lagrangian to the second order in fields.

Notice that $F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+g\varepsilon^{abc}A^b_\mu\cdot A^c_\nu$

ε^{abc} — SU(2) structure constants.

In linear approximation this becomes:

$$F^a_{\mu\nu}\simeq\mathcal{F}^a_{\mu\nu} \text{ where } \mathcal{F}^a_{\mu\nu}\equiv\partial_\mu A^a_\nu-\partial_\nu A^a_\mu;$$

- Potential term: $V(\varphi^+,\varphi)=-\mu^2\cdot\frac{1}{2}(\varphi_0+\chi)^2+\frac{1}{4}\lambda(\varphi_0+\chi)^4=\mu^2\chi^2+\text{const}+$ higher order terms.

⑥ • Finally kinetic terms of the scalar fields:

$$D_\mu\varphi = \left(\partial_\mu - ig\frac{\tau^a}{2}A^a_\mu\right)\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}(\varphi_0+\chi)\end{pmatrix} = \begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\partial_\mu\chi\end{pmatrix} - igA^a_\mu\frac{\tau^a}{2}\begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\varphi_0\end{pmatrix} + \text{quadratic term.}$$

herm gen. of $SU(2)$

Hence $$D_\mu\varphi = \begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\partial_\mu\chi\end{pmatrix} - \frac{ig}{2\sqrt{2}}\begin{pmatrix}A^3_\mu & A^1_\mu - iA^2_\mu\\ A^1_\mu + iA^2_\mu & -A^3_\mu\end{pmatrix}\begin{pmatrix}0\\ \varphi_0\end{pmatrix} \Rightarrow$$

$$\Rightarrow \boxed{D_\mu\varphi = \begin{pmatrix}-i\frac{g}{2\sqrt{2}}\varphi_0A^1_\mu - \frac{g}{2\sqrt{2}}\varphi_0A^2_\mu\\ \frac{1}{\sqrt{2}}\partial_\mu\chi + \frac{ig}{2\sqrt{2}}\varphi_0A^3_\mu\end{pmatrix}} \Rightarrow |D_\mu\varphi|^2 = \frac{g^2}{8}\varphi_0^2\left((A^1_\mu)^2 + (A^2_\mu)^2\right) + $$

$$+\frac{1}{2}(\partial_\mu\chi)^2 + \frac{g^2}{8}\varphi_0^2(A^3_\mu)^2 \Rightarrow$$

$$\Rightarrow |D_\mu\varphi|^2 = \frac{1}{2}(\partial_\mu\chi)^2 + \frac{g^2\varphi_0^2}{8}A^a_\mu A^{a\mu};$$

• To summarize quadratic Lagrangian for excitations looks like:

$$\mathcal{L}^{(2)}_e = -\frac{1}{4}\mathcal{F}^a_{\mu\nu}\cdot\mathcal{F}^{a\mu\nu} + \frac{g^2\varphi_0^2}{8}A^a_\mu\cdot A^a_\mu + \frac{1}{2}\partial_\mu\chi\cdot\partial^\mu\chi - \mu^2\chi^2;$$

• Thus in the broken phase we obtain:
  - 3 massive vector fields A^a_μ $(a=1,2,3)$ with the mass $m_A = \frac{g\varphi_0}{2}$;
    This corresponds to $3\times 3 = 9$ d.o.f.
  - 1 massive scalar field χ with the mass $m_\chi = \sqrt{2}\mu$. (1 d.o.f.)

• In the unbroken phase:
  - 3 massless vector fields : $2\times 3 = 6$ d.o.f.
  - doublet of scalar fields : $2\times 2 = 4$ d.o.f. (complex fields)

• In both phases we have 10 d.o.f..

• If the theory was global but not local, then the choise of the vacuum would break $SU(2)$ completely so that all 3 generators would be broken and according to the Goldstone theorem there should be 3 massless NG bosons. When we gauge the symmetre this NG bosons are adsorbed by the gauge vector fields which all become massive.

⑦ • Physical example: Superconductors.

As discussed on the previous class, superconductors around the critical point can be described by the free energy functional:

$$F = F_n + \alpha|\varphi|^2 + \frac{1}{2}\beta|\varphi|^4 + \frac{1}{2\tilde{m}}|(-i\hbar\nabla - \tilde{e}A)\varphi|^2 + \frac{|B|^2}{2\mu_0}$$

where $\tilde{m}$ and $\tilde{e}$ are effective mass and charge, and φ is the order parameter such that $\varphi = \begin{cases} 1 & \text{in SC state.} \\ 0 & \text{in metal state.} \end{cases}$

This order parameter appears to be equal to the density of the Cooper pairs. $\varphi = n_s$. While effective charge and mass are $\tilde{m} = 2m_e$, $\tilde{e} = 2e$;

• Finally the parameter α is given by $\alpha \propto (T - T_c)$ so that for $T > T_c$ we have unbroken symmetry while for $T < T_c$ symmetry is broken.

• In the broken phase, corresponding to superconductor, electromagnetic field becomes massive with some mass

$$m_A = \frac{\tilde{e}\varphi_0}{\tilde{m}} = \frac{e\varphi_0}{m}.$$

• As the vector field is massive electromagnetic interaction now has finite radius $r_A \sim \frac{1}{m_A} = \frac{m}{e\varphi_0}$; (i.e. interaction potential becomes Yukawa instead of Coulomb: $V(r) = \frac{q_1 q_2}{r} e^{-r/r_A}$)

• Because of the finite radius of interaction magnetic field can not penetrate the superconductor and is nonzero only in thin layer of r_A width. This was observed long before Anderson came up with explanation written above. This phenomenon is known as Meissner effect.

⑧ Bosonic sector of the standard model.

Now we are ready to consider more complicated example of symmetry breaking in physics. Namely we will consider gauge theory with the gauge group $SU(2)\times U(1)$.

- $SU(2)$ gauge field: A^a_μ $(a=1,2,3)$ with the coupling g.
- $U(1)$ gauge field: B_μ with the coupling g'.
- Scalar field φ, doublet in $SU(2)$, $Y=\frac{1}{2}$ charge in $U(1)$ (hypercharge)
- Lagrangian:

$$\mathcal{L}_\varphi = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} + (\mathcal{D}_\mu\varphi)^+\mathcal{D}^\mu\varphi - \lambda\left(\varphi^+\varphi - \frac{v^2}{2}\right)^2;$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,\varepsilon^{abc}A^b_\mu A^c_\nu;$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu; \qquad \varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix};$$

$$\mathcal{D}_\mu\varphi = \partial_\mu\varphi - i\frac{g}{2}\tau^a A^a_\mu\varphi - i\frac{g'}{2}B_\mu\varphi;$$

- Ground state: $\boxed{\begin{aligned} &A^a_\mu = B_\mu = 0;\\ &\varphi = \frac{v}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} \equiv \varphi^{vac};\end{aligned}}$

- Stationary subgroup: Let's find the subgroup which preserves the vacuum. To do this let's find the corresponding hermitian generators annihilating this ground stat: $Q\varphi^{vac} = 0$;

In general $Q = \begin{pmatrix} a & b\\ b^* & c\end{pmatrix}$ $\quad Q\varphi^{vac} = \frac{v}{\sqrt{2}}\begin{pmatrix} b\\ c\end{pmatrix} \Rightarrow b = c = 0$

Hence the generator has the form $Q = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = T^3 + Y$ where $T^3 = \frac{1}{2}\tau^3$ and $Y = \frac{1}{2}$ are generators of $SU(2)$ and $U(1)$ correspondingly.

- Hence the original gauge symmetry group $SU(2)\times U(1)$ broken down to $U(1)_{em}$, which is related to the gauge group of the electromagnetizm (one unbroken generator). Hence there are tree broken generators which gives rise to three NG bosons (in the case of global symmetry) and three massive vector fields in the case of gauge symmetry.

9 • Let's express these new fields in terms of fields A^a_μ and B_μ.

The ground state is invariant under the transformation

$$\varphi(x) \to \omega(x)\,\Omega(x)\,\varphi(x) \text{ where } \omega(x) = e^{i\frac{\tau^3}{2}\alpha(x)}; \; \Omega(x) = e^{i\alpha(x)};$$

• Gauge fields under the same transformations behave as

$$A_\mu \to A'_\mu = \omega A_\mu \omega^{-1} + \omega\partial_\mu\omega^{-1};$$
$$B_\mu \to B'_\mu = B_\mu + \Omega\partial_\mu\Omega^{-1};$$

Substituting particular forms of ω and Ω we obtain:

$$A'_\mu = -\frac{ig}{2}\left(A'^1_\mu\cdot\tau^1 + A'^2_\mu\cdot\tau^2 + A'^3_\mu\cdot\tau^3\right) = -\frac{ig}{2}e^{i\frac{\tau^3}{2}\alpha}\left(\tau^1A^1_\mu + \tau^2A^2_\mu\right)e^{-i\frac{\tau^3}{2}\alpha} - ig\frac{\tau^3}{2}A^3_\mu - i\frac{\tau^3}{2}\partial_\mu\alpha;$$

Now we use:

• $e^{\pm i\frac{\tau^3}{2}\alpha} = \cos(\alpha/2) \pm i\tau^3\cdot\sin(\frac{\alpha}{2})$ → this works because $(\tau^3)^{2n} = 1$, $(\tau^3)^{2n+1} = \tau^3$ and ($\sin\alpha$) contains only odd powers of α while ($\cos\alpha$) contains even powers.

then

$$e^{i\frac{\tau^3}{2}\alpha}\left(A^1_\mu\tau^1 + A^2_\mu\tau^2\right)e^{-i\frac{\tau^3}{2}\alpha} = \left(\cos\frac{\alpha}{2} + i\cdot\tau^3\cdot\sin\frac{\alpha}{2}\right)\left(A^1_\mu\tau^1 + A^2_\mu\tau^2\right)\left(\cos\frac{\alpha}{2} - i\tau^3\sin\frac{\alpha}{2}\right)$$

$$= \left(\begin{matrix}\text{using}\\ \tau^i\tau^3 = -\tau^3\tau^i, i\neq 3\end{matrix}\right) = \left(\cos\frac{\alpha}{2} + i\tau^3\cdot\sin\frac{\alpha}{2}\right)^2\left(\tau^1A^1_\mu + \tau^2A^2_\mu\right) =$$

$$= \left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} + i\tau^3\cdot\sin\alpha\right)\left(\tau^1A^1_\mu + \tau^2A^2_\mu\right) = \left(\cos\alpha + i\tau^3\cdot\sin\alpha\right)\left(\tau^1A^1_\mu + \tau^2A^2_\mu\right)$$

using $\tau^3\tau^1 = i\tau^2$, $\tau^3\tau^2 = -i\tau^1$; we finally obtain:

$$e^{i\frac{\tau^3}{2}\alpha}\left(A^1_\mu\tau^1 + A^2_\mu\tau^2\right)e^{-i\frac{\tau^3}{2}\alpha} = \tau^1\left(A^1_\mu\cdot\cos\alpha + A^2_\mu\cdot\sin\alpha\right) + \tau^2\left(A^2_\mu\cdot\cos\alpha - A^1_\mu\cdot\sin\alpha\right);$$

• Then we obtain the following transformation for the components of A_μ:

$$\boxed{\begin{aligned} A'^1_\mu &= A^1_\mu\cdot\cos\alpha + A^2_\mu\cdot\sin\alpha; \\ A'^2_\mu &= A^2_\mu\cdot\cos\alpha - A^1_\mu\cdot\sin\alpha; \\ A'^3_\mu &= A^3_\mu + \frac{1}{g}\partial_\mu\alpha; \\ B'_\mu &= B_\mu + \frac{1}{g'}\partial_\mu\alpha; \end{aligned}}$$

Finally for B_μ field

$$B_\mu \to B'_\mu = B_\mu - i\partial_\mu\alpha$$
$$\Downarrow$$
$$-ig'B'_\mu = -ig'B_\mu - i\partial_\mu\alpha; \quad B'_\mu = B_\mu + \frac{1}{g'}\partial_\mu\alpha;$$

→ action of unbroken $U(1)_{em}$ group.

(10) • The form of gauge transformations suggests that the following fields should be introduced:

It is reasonable to consider combinations

$W^{\pm}_{\mu} = \frac{1}{\sqrt{2}}(A^1_\mu \mp i A^2_\mu) \Rightarrow$ under gauge transform. $W^{\pm}_\mu \to W'^{\pm}_\mu = \frac{1}{\sqrt{2}}(A'^1_\mu \mp i A'^2_\mu) =$

$= \frac{1}{\sqrt{2}}(A^1_\mu(\cos\alpha \pm i\sin\alpha) \mp i A^2_\mu(\cos\alpha \pm i\cdot\sin\alpha)) = e^{\pm i\alpha}\cdot W^{\pm}_\mu \Rightarrow$

$\Rightarrow$ $W^{\pm}_\mu$ have positive (negative) $U(1)_{em}$ charge.

• Fields A^3_μ and B_μ can be combined into gauge invariant combination:

$Z_\mu = (g A^3_\mu - g' B_\mu)\cdot c_1(g', g)$, then $Z'_\mu = (g A^3_\mu + \cancel{\partial_\mu \alpha} - g' B_\mu - \cancel{\partial_\mu \alpha})\cdot c_1(g', g) =$

$c_1(g', g)$ is normalization $= Z_\mu \to$ invariant!

constant that normalizes kinetic terms of Z_μ properly:

• If $Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$ then we want kinetic term to be of the form $-\frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} = -\frac{1}{4} c_1^2(g', g)\cdot(g^2 \mathcal{F}^3_{\mu\nu}\mathcal{F}^{3\mu\nu} - 2g'g\,\mathcal{F}^3_{\mu\nu} B^{\mu\nu} + (g')^2 B^{\mu\nu} B_{\mu\nu})$

• "Orthogonal field" is $A_\mu = c_2(g', g)(g' A^3_\mu + g B_\mu)$

• Under gauge transformations it transforms as

$$A_\mu \to A'_\mu = c_2(g', g)\cdot(g' A^3_\mu + \frac{g'}{g}\partial_\mu\alpha + g B_\mu + \frac{g}{g'}\partial_\mu \alpha) =$$

$$= A_\mu + c_2(g', g)\frac{(g')^2 + g^2}{g' \cdot g}\partial_\mu \alpha = A_\mu + \frac{1}{e}\partial_\mu \alpha \text{ where}$$

$e = \frac{g'g}{g^2 + (g')^2}\cdot\frac{1}{c_2(g', g)}$; So we conclude that A_μ is vector gauge field of $U(1)_{em}$ symmetry!

• For the kinetic term we get $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ so that

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}c_2^2(g', g)((g')^2 \mathcal{F}^3_{\mu\nu}\cdot\mathcal{F}^{3\mu\nu} + 2g'g\,\mathcal{F}^3_{\mu\nu}B^{\mu\nu} + g^2 B_{\mu\nu}B^{\mu\nu})$$

• To obtain the right normalization we want kinetic terms to reproduce original kinetic terms.

(11) $-\frac{1}{4}Z_{\mu\nu}Z^{\mu\nu}-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}=-\frac{1}{4}\mathcal{F}^3_{\mu\nu}\mathcal{F}^{3\mu\nu}-\frac{1}{4}B_{\mu\nu}B^{\mu\nu}$;

For this equation to work the following equations should be satisfied:

- from cross terms to cancell we want $c_2(g',g)=c_1(g',g)$;
- from $\mathcal{F}^3_{\mu\nu}\mathcal{F}^{3\mu\nu}$ terms $\;c_1^2(g',g)\,((g')^2+g^2)=1 \Rightarrow c_1(g',g)=\frac{1}{\sqrt{(g')^2+g^2}}$;

the same equation follows from the $B_{\mu\nu}B^{\mu\nu}$ terms normalization.

- To conclude:

Un Broken phase:

A^a_μ, $a=1,2,3 \rightarrow SU(2)$ gauge field, g coupl. constant.

$B_\mu \rightarrow U(1)$ gauge field, g' coupling constant.

Broken phase $(SU(2)\otimes U(1)\rightarrow U(1)_{e.m.})$

$W^\pm_\mu=\frac{1}{\sqrt{2}}(A^1_\mu \mp iA^2_\mu) \rightarrow$ fields charged under $U(1)_{em}$.

$Z_\mu=\frac{1}{\sqrt{(g')^2+g^2}}(gA^3_\mu-g'B_\mu)\rightarrow$ field neutral in $U(1)_{em}$

$A_\mu=\frac{1}{\sqrt{(g')^2+g^2}}(g'A^3_\mu+gB_\mu)\rightarrow U(1)_{em}$ gauge field.

electic charge unit is $e=\frac{g'g}{\sqrt{(g')^2+g^2}}$;

- Now let's consider perturbations around the ground state. If we, once again, fix the unitary gauge perturbations should take the form

$$\varphi(x)=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+\underbrace{\chi(x)}_{\text{real scalar fields.}}\end{pmatrix};$$

- Kinetic terms of gauge fields:

We have already seen that $-\frac{1}{4}\mathcal{F}^3_{\mu\nu}\mathcal{F}^{3\mu\nu}-\frac{1}{4}B_{\mu\nu}B^{\mu\nu}=-\frac{1}{4}Z_{\mu\nu}Z^{\mu\nu}-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$;

Now for two remaining components of A^a_μ we get:

(12) $W^{\pm}_{\mu\nu} \equiv \partial_\mu W^\pm_\nu - \partial_\nu W^\pm_\mu = \frac{1}{\sqrt{2}}(\mathcal{F}^1_{\mu\nu} \mp i\mathcal{F}^2_{\mu\nu})$ so that

$$-\frac{1}{4}(\mathcal{F}^1_{\mu\nu}\mathcal{F}^{1\mu\nu} + \mathcal{F}^2_{\mu\nu}\mathcal{F}^{2\mu\nu}) = -\frac{1}{2}W^+_{\mu\nu}W^{-\mu\nu}$$

- To summarize kinetic terms become:

$$\mathcal{L}_{\ell kin} = -\frac{1}{2}W^+_{\mu\nu}W^{-\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu};$$

- No if we consider $(\mathcal{D}_\mu\varphi)^+\mathcal{D}^\mu\varphi$ term we can obtain:

$$\mathcal{D}_\mu\varphi = \partial_\mu\varphi - \frac{ig}{2}\left[A^1_\mu\begin{pmatrix}0&1\\1&0\end{pmatrix} + A^2_\mu\begin{pmatrix}0&-i\\i&0\end{pmatrix} + A^3_\mu\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]\varphi - \frac{ig'}{2}B_\mu\begin{pmatrix}1&0\\0&1\end{pmatrix}\varphi =$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{ig}{2}A^1_\mu\cdot v - \frac{g}{2}A^2_\mu\cdot v \\ \partial_\mu\chi - \frac{ig'}{2}B_\mu\cdot v + \frac{ig}{2}A^3_\mu\cdot v\end{pmatrix} + \text{higher order terms.}$$

$$(\mathcal{D}_\mu\varphi)^+\mathcal{D}^\mu\varphi = \frac{1}{2}\cdot\frac{g^2}{4}|A^1_\mu - iA^2_\mu|^2\cdot v^2 + \frac{1}{2}\cdot(g'B_\mu + gA^3_\mu)^2\cdot v^2 + \frac{1}{2}\partial_\mu\chi\cdot\partial^\mu\chi =$$

$$= \frac{1}{2}\partial_\mu\chi\,\partial^\mu\chi + \frac{g^2v^2}{4}W^+_\mu W^{-\mu} + \frac{1}{8}(g^2+(g')^2)v^2\cdot Z_\mu\cdot Z^\mu$$

- Finally potential term as usually gives:

$$V(\varphi^+,\varphi) = \lambda\left(\varphi^+\varphi - \frac{v^2}{2}\right)^2 = \frac{\lambda}{4}\left((v+\chi)^2 - v^2\right)^2 = \lambda v^2\chi^2 + \text{higher terms.}$$

- Introducing masses: $\boxed{m_W = \frac{gv}{2},\ m_Z = \frac{v\sqrt{g^2+g'^2}}{2},\ m_\chi = \sqrt{2\lambda}\cdot v;}$

we finally write down quadratic Lagrangian as:

$$\boxed{\mathcal{L}^{(2)}_\ell = -\frac{1}{2}W^+_{\mu\nu}W^{-\mu\nu} + m_W^2 W^+_\mu W^{-\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} + \frac{m_Z^2}{2}Z_\mu Z^\mu + \frac{1}{2}\partial_\mu\chi\cdot\partial^\mu\chi - \frac{1}{2}m_\chi^2\cdot\chi^2;}$$

So that

- $W^\pm_\mu$ are charged massive vector fields ($W^\pm$-bosons, $m_W \approx 80$ GeV)
- Z_μ is neutral massive vector field (Z^0-boson, $m_Z \approx 90$ GeV)
- A_μ – massless electromagnetic field.

↳ content of Glashow–Weinberg–Salam model.

- Sometimes it is formulated in terms of Weinberg angle θ_W

$$\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}},\ \sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}; \Rightarrow \begin{matrix} Z_\mu = \cos\theta_W\cdot A^3_\mu - \sin\theta_W\cdot B_\mu; \\ A_\mu = \cos\theta_W\cdot B_\mu + \sin\theta_W\cdot A^3_\mu;\end{matrix} \text{ and } m_Z = \frac{m_W}{\cos\theta_W};$$

Lecture 8: Simple topological solitons.

- Previously – small perturbations around the vacuum. After quantization looks like particles.
- Following 4 lectures – solutions of classical equations of motion localized in space. Looks like particles even before the quantization. We call them solitons.

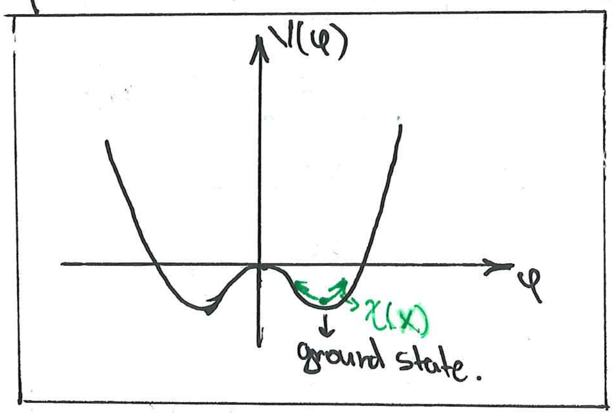
- Kink.
  - Let's start with the simpliest model we can imagine:

$$S = \int d^2x \left(\frac{1}{2}\partial_\mu\varphi \cdot \partial^\mu\varphi - V(\varphi)\right), \text{ where } V(\varphi) = -\frac{\mu^2}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4 + \frac{\mu^4}{4\lambda};$$

or $V(\varphi) = \frac{\lambda}{4}(\varphi^2 - v^2)^2$, $v = \frac{\mu}{\sqrt{\lambda}}$; theory is in (1+1) dimensions.

  - This model possess $\mathbb{Z}_2$-symmetry: $\varphi \to -\varphi$.
  - Model has two possible ground states $\varphi_v = \pm v$;

If we choose one of them $\mathbb{Z}_2$-symmetry breaks down, and we get massive excitations around $\varphi = v$:



$$\varphi = v + \chi(x) \Rightarrow$$

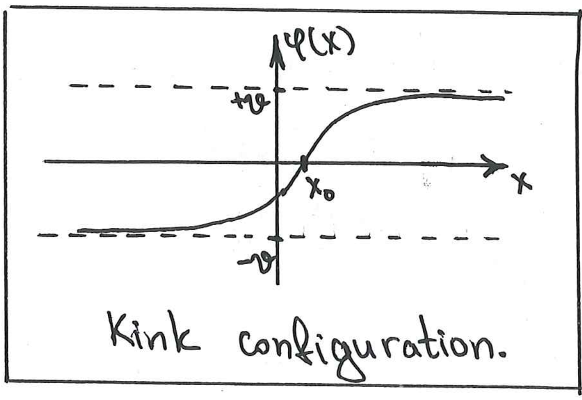
$$\mathcal{L}_\chi = \frac{1}{2}\partial_\mu\chi\,\partial^\mu\chi - \frac{\lambda}{4}(2v\chi + \chi^2)^2 =$$

$$= \frac{1}{2}\partial_\mu\chi\,\partial^\mu\chi - \lambda v^2\chi^2 + \text{higher terms.}$$

So excitations $\chi(x)$ have the mass

$$m_\chi = m = \sqrt{2}\,\lambda v = \sqrt{2}\,\mu.$$

- Kink is the static solution of equations of motion for this model interpolating between two vacua.



Kink configuration.

- Equations of motion:

$$\partial_\mu\partial^\mu\varphi + \frac{\partial V}{\partial\varphi} = 0, \text{ as solution is static}$$

$\dot\varphi = \frac{\partial\varphi}{\partial t} = 0$; $\varphi' = \frac{\partial\varphi}{\partial x} \neq 0$ and we obtain:

$$\varphi'' - \frac{\partial V}{\partial\varphi} = 0;$$

② • Integrating once: $\int \varphi'' d\varphi - V(\varphi) = \mathcal{E}_0$ ↘ constant of integration.

using $\int \varphi'' d\varphi = \int \varphi''\varphi' dx = \frac{1}{2}(\varphi')^2 + \text{const}$ we obtain

$\frac{1}{2}(\varphi')^2 - V(\varphi) = \mathcal{E}_0$; $\mathcal{E}_0$ is defined from the asymptotical behavior at $x = \pm\infty$. For the kink we have $V(\varphi) \xrightarrow[x\to\pm\infty]{} 0$ and

$\varphi \to \begin{cases} v, & x = +\infty \\ -v, & x = -\infty \end{cases}$ so $\varphi' \to 0$. Hence $\mathcal{E}_0 = 0$ and we obtain:

• Integrating one more time $\frac{d\varphi}{dx} = \pm\sqrt{2V} = \pm\sqrt{\frac{\lambda}{2}}(v^2 - \varphi^2)$

The choise of sign is related to the conditions at $x = \pm\infty$

if $\begin{cases} \varphi(+\infty) = v \\ \varphi(-\infty) = -v \end{cases}$ → take "+" sign field is growing with x. if $\begin{cases} \varphi(+\infty) = -v \\ \varphi(-\infty) = v \end{cases}$ → take "−" sign field decreases with x.

taking integrals (we take "+" sign for definiteness)

$$\sqrt{\frac{\lambda}{2}}\int dx = \int d\varphi \frac{1}{v^2 - \varphi^2} = \frac{1}{v}\operatorname{arctanh}\left(\frac{\varphi}{v}\right) \Rightarrow \boxed{\varphi(x) = v \cdot \tanh\left(\sqrt{\frac{\lambda}{2}}v(x - x_0)\right);}$$

↓ Integration constant (center of the kink)

• Properties: Further we for simplicity consider kink centered around $x_0 = 0$:

$$\varphi(x) = v \cdot \tanh\left(\sqrt{\frac{\lambda}{2}}v \cdot x\right);$$

① Energy density of the kink is:

$$\mathcal{E}(x) = \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 + \frac{\lambda}{4}(\varphi^2 - v^2)^2 = \frac{1}{2}\cdot\frac{\lambda}{2}v^4 \frac{1}{\cosh^4\left(\sqrt{\frac{\lambda}{2}}vx\right)} + $$

$$+ \frac{1}{2}\cdot\frac{\lambda}{2}v^4 \frac{1}{\cosh^4\left(\sqrt{\frac{\lambda}{2}}vx\right)} \Rightarrow \mathcal{E}(x) = \frac{\lambda}{2}v^4 \frac{1}{\cosh^4\left(\frac{\mu x}{\sqrt{2}}\right)};$$

We see that the energy density falls down very fast at the distance $r_k \sim \mu^{-1}$; → this is proportional to the Compton length of elementary excitation.

② Total energy of the static kink:

$$E_k = \int_{-\infty}^{+\infty} dx\, \mathcal{E}(x) = \frac{\lambda}{2}v^4 \int_{-\infty}^{+\infty} dx \frac{1}{\cosh^4\left(\frac{\mu x}{\sqrt{2}}\right)} = \frac{\lambda}{2}v^4 \int_{-1}^{1} dy \cdot (1 - y^2)\cdot\frac{\sqrt{2}}{\mu} =$$

$y = \tanh\left(\frac{\mu}{\sqrt{2}}x\right)$

$$= \frac{4}{3}\frac{1}{2}v^2 \cdot \mu^2 \cdot \frac{\sqrt{2}}{\mu} = \frac{2}{3}mv^2$$ where $m = \sqrt{2}\mu$ is the mass of elementary excitation.

③ From this static energy we can conclude that the mass of the kink is $M_k = \frac{2}{3} m v^2$;

Hence the Compton length of the kink is $\lambda_k \sim \frac{1}{M_k} = \frac{3}{2mv^2}$

Notice that if $v \gg 1$ ($\lambda \ll \mu^2 \to$ weak coupling) the Compton length of the kink is small in comparison to it's size $r_k \sim \mu^{-1}$ indeed $\frac{\lambda_k}{r_k} \sim \frac{1}{v^2} \ll 1$. Hence in weak coupling kink is always classical.

③ The solution is not invariant under translations and Boosts. Under this transformations solution turns into more general solution of the form:

$$\varphi(x,t,u) = \frac{\mu}{\sqrt{\lambda}} \tanh\left(\sqrt{\frac{\lambda}{2}}\, v \, \frac{x - x_0 - ut}{\sqrt{1-u^2}}\right);$$

this general solution describes the kink moving with the velocity u and placed at x_0 at $t=0$;

④ Stability of the solution.

Let's understand if the solution we have found here is stable towards perturbations around this solution.

$$\varphi(x,t) = \varphi_k(x) + f(x,t);$$

φ should satisfy KG equation $\partial_\mu \partial^\mu \varphi + \frac{\partial V}{\partial \varphi} = 0 \Rightarrow$

$\Rightarrow \partial_\mu \partial^\mu (\varphi_k + f) + \frac{\partial V}{\partial \varphi}(\varphi_k) + \frac{\partial^2 V}{\partial \varphi^2}(\varphi_k) \cdot f + \dots = 0$ and

$\partial_\mu \partial^\mu \varphi_k + \frac{\partial V}{\partial \varphi}\Big|_{\varphi_k} = 0$ hence $\partial_\mu \partial^\mu f + \frac{\partial^2 V}{\partial \varphi^2}\Big|_{\varphi = \varphi_k} \cdot f = 0$

• Notice that $\frac{\partial V}{\partial \varphi}\Big|_{\varphi=\varphi_k}$ are functions of x only but not t we can separate the variables $f(x,t) = e^{i\omega t} f_\omega(x)$

So that $f_\omega(x)$ should satisfy: $-\omega^2 f_\omega - f_\omega'' + \frac{\partial^2 V}{\partial \varphi^2}\Big|_{\varphi=\varphi_k} \cdot f_\omega = 0$

So we have equation:

$$\left(-\frac{d^2}{dx^2} + U(x)\right) f_\omega = \omega^2 f_\omega;$$ equations for the eigenvalues of $\left(-\frac{d^2}{dx^2} + U\right)$ operator

$$U(x) \equiv \frac{\partial^2 V}{\partial \varphi^2}\Big|_{\varphi=\varphi_k}.$$

(4) • if all eigenvalues are positive ($\omega^2>0$) then perturbations are just oscillations around the solution.

• if the are some negative eigenvalues ($\omega^2<0$) then perturbations are growing with time destroying solutions.

So to understand if solutions are stable or not we should find negative eigenmodes of the differential operator $-\frac{d^2}{dx^2}+U(x)$, where $U(x)=\frac{\partial^2 V}{\partial\varphi^2}\Big|_{\varphi=\varphi_K}$;

• We want solutions finite at $|x|\to\infty$;

• Equations we have written here are easily generalizable to higher dimensions and can be applied to any potential $V(x)$.

• Notice that the operator $-\frac{d^2}{dx^2}+U(x)$ always have some zero modes of the form $f_0=\frac{\partial\varphi_K}{\partial x}$;

Indded differentiating KG equation $-\varphi_K''+\frac{\partial V}{\partial\varphi}\Big|_{\varphi=\varphi_K}=0$, we obtain $-(\varphi_K')''+\frac{\partial^2 V}{\partial\varphi^2}\varphi_K'=0$ which is equation for the zero mode (notice that $\varphi_K'(x)\to0$ at $|x|\to\infty$)

• Existence of zero modes is related to the broken translation invariance. In particular if we consider small "a" transl.

$\varphi_K(x+a)=\varphi_K(x)+a f_0(x)$ and then f_0 will satisfy zero modes equation

Stability of kinks.

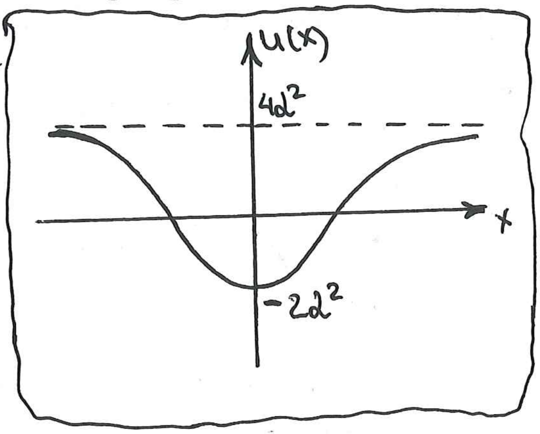
Let's find the spectrum of $-\frac{d^2}{dx^2}+U(x)$ for the particular kink solution. For this particular ve have

$$U(x)=\frac{\partial^2V}{\partial\varphi^2}\Big|_{\varphi=\varphi_K}=\lambda(3\varphi^2-v^2)\Big|_{\varphi=\varphi_K}=\lambda v^2\left(3\tanh^2\left(\frac{\mu}{\sqrt2}x\right)-1\right)=$$

$$=\mu^2\left(3\tanh^2\left(\frac{\mu}{\sqrt2}x\right)-1\right);$$

Usefull form of writing it
$$U(x)=\mu^2\left(2-\frac{3}{\cosh^2\left(\frac{\mu}{\sqrt2}x\right)}\right)=$$
$$=2d^2\left(2-\frac{3}{\cosh^2 dx}\right)\text{ where } d=\frac{\mu}{\sqrt2};$$

⑤ • Sometimes it is usefull to apply quantum mechanics intuition to the problems of differential operators spectrum. For example if $U(x) \geq 0$ everywhere then ω^2 (which is just energy) is always positive. However this is not the case here so we should be more accurate!



The equation we want to resolve can be rewritten as

$$\frac{d^2 f}{dx^2} + \left(\frac{6\alpha^2}{\cosh^2 \alpha x} + \omega^2 - 4\alpha^2\right) f = 0;$$

if we now introduce $\xi = \tanh(\alpha x)$ we can rewrite

$$\frac{d}{dx} = \alpha(1-\xi^2)\frac{d}{d\xi}$$ so that we get:

$$\frac{d}{d\xi}(1-\xi^2)\frac{d}{d\xi} f + \left(s(s+1) - \frac{\varepsilon^2}{1-\xi^2}\right) f = 0;$$ where $s(s+1) = 6; \Rightarrow s = 2$

and $\varepsilon^2 = 4 - \frac{\omega^2}{\alpha^2}$;

The equation above is hypergeometric equation which has as the solution

$$f = (1-\xi^2)^{\varepsilon/2} \cdot F\left(\varepsilon - s, \varepsilon + s + 1, \varepsilon + 1, \tfrac{1}{2}(1-\xi)\right);$$

If we want this function to decrease fast enough at $x \to \pm\infty$ $(\xi \to \pm 1)$ we should demand $\varepsilon = s - n$, so that
$n = 0, 1, 2, \ldots$

$\varepsilon^2 \quad s^2 \Rightarrow \omega^2 \geq 0$!!! q.e.d.

As we see though $U(x) < 0$ for some x, "energy levels" start at $\omega = 0$, and thus kink is stable towards perturbations

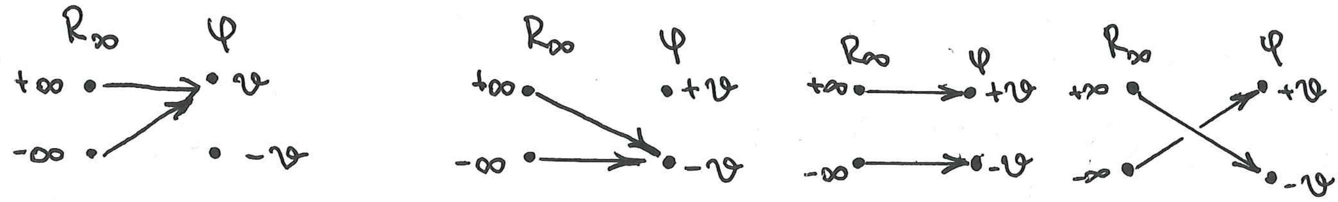
• Now let's explain why kinks are topological! solitons.

• In order for the energy to be finite at $x = \pm\infty$ φ should take values $\pm v$; Hence $\varphi(x)$ is the map between $x = \pm\infty$ and $\varphi = \pm v$ points.

The important thing is that any dynamics or indroduction of local sources that doesn't lead to the emergence of

⑥ Infinite energies don't change the mapping as they don't effect field values at $x = \pm\infty$;

- Usually the situation when local variations of fields doesn't change this mapping at infinity is regarded as topological field configuration.
- There are 4 possible mappings:



$\varphi = +v$ vacuum sector; $\varphi = -v$ vacuum sector. ; kink sector; anti-kink sector

- Notice that time dynamics leaves fields inside one sector
- Assume that in some sector we have found field configuration minimizing the energy. This field is different from the vacuum one (as it belongs to different sector). At the same time inside the sector this field configuration is energetically preferable and hence stable.

  In the model we consider this minimal energy configuration is kink
- So existence of kinks in the model follows from nontrivial topology of maps between the spatial infinity and vacua of the theory.
- Topological current.

  Let's define $k^\mu = \frac{1}{2v}\varepsilon^{\mu\nu}\partial_\nu\varphi \quad (\mu,\nu = 0,1)$;
- Current k^μ is conserved always (not only on-shell)

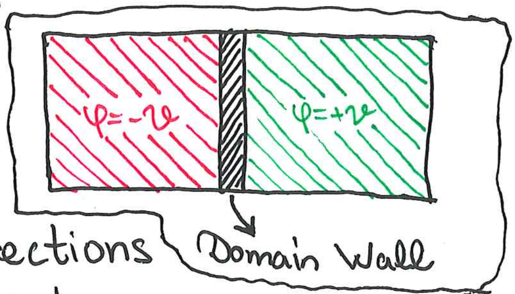
$$\partial_\mu k^\mu = \frac{1}{2v}\varepsilon^{\mu\nu}\partial_\mu\partial_\nu\varphi = 0;$$

- Topological charge is

$$Q_t = \int_{-\infty}^{+\infty} k^0 dx^1 = \int_{-\infty}^{+\infty}\frac{1}{2v}\partial_1\varphi\, dx^1 = \frac{1}{2v}\left(\varphi(+\infty) - \varphi(-\infty)\right);$$

- vacuum sectors $Q_t = 0$;
- anti-kink sector $Q_t = -1$;
- kink sector $Q_t = 1$;

⑦ • The kink solution can be lifted up to four dimensions:



- φ is still static
- $\varphi(\bar{x})$ does not depend on x^2 and x^3 directions
- In x^1 direction we still have the kink

$$\varphi(\bar{x}) = v \cdot \tanh\left(\sqrt{\frac{\lambda}{2}}\, v x_1\right)$$

- Solution of this type is called domain wall because it separates the space into two regions with different vacua.

• Vortex

Vortex is the soliton in the gauge theory with $U(1)$ symmetry and Higgs potential. in $(2+1)$ dimensions.

$$\mathcal{L}_\varphi = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\mathcal{D}_\mu \varphi)^* \mathcal{D}^\mu \varphi - V(\varphi),$$

where the potential is $V(\varphi) = -\mu \varphi^* \varphi + \frac{\lambda}{2}(\varphi^* \varphi)^2 + \frac{\mu^2}{2\lambda} = \frac{\lambda}{2}(\varphi^* \varphi - v^2)^2,$

where $v = \mu/\sqrt{\lambda}$;

- Ground state $A_\mu = 0, \varphi = v$; $\Rightarrow$ gauge field is massive: $m_v = \sqrt{2}\, e v$;
  higgs field χ: $m_H = \sqrt{2}\mu = \sqrt{2\lambda} \cdot v$;

- Let's fix static gauge $A_0 = 0$ (remaining gauge is time-independent gauge transformations) and consider time-indepndent field configurations. Energy is

$$\mathcal{E}(A_i(\bar{x}), \varphi(\bar{x})) = \int d^2x \left(\frac{1}{4} F_{ij} F_{ij} + (\mathcal{D}_i \varphi)^* \mathcal{D}_i \varphi + V(\varphi)\right);$$

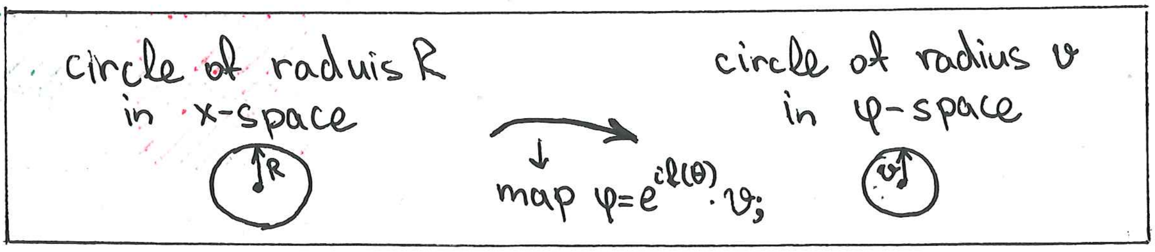
- For the energy to be finite we need

$$|\varphi| \to v \quad \text{as} \quad |\bar{x}| \to \infty;$$

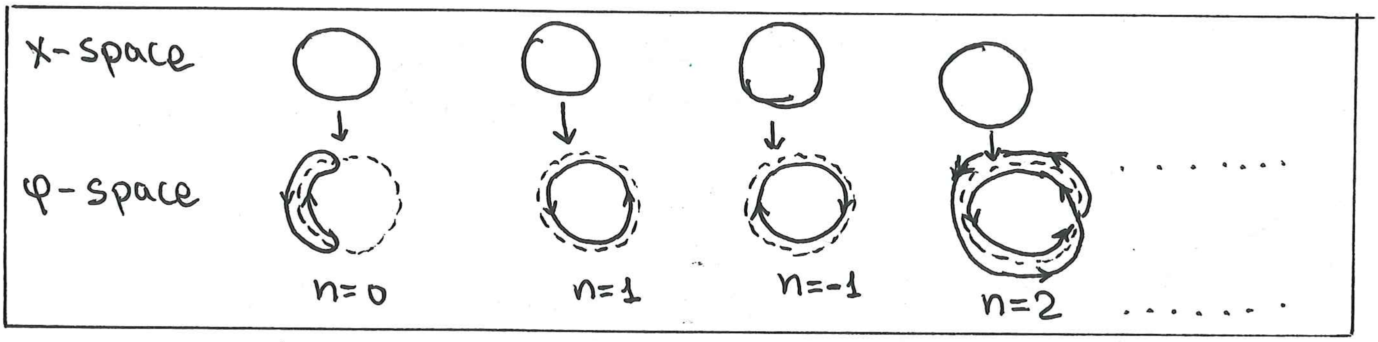
- Let's fix circle of large radius R with the center at the origin. On this circle $|\varphi| = v$ but there is θ (angle) dependence of φ which is non-trivial:

$$\varphi = e^{i f(\theta)} v;$$

⑧ • We can consider this φ-field as the map:



• This map can be characterized by the winding number $n = 0, \pm 1, \pm 2, \ldots$ (see examples on the picture below)



• Maps with different winding number can be written as

$$\varphi(\theta) = e^{in\theta} \cdot v;$$

• Phase of φ is not gauge invariant, but winding number n is invariant under smooth gauge transformations.

Example: let's try to change winding number by 1 performing the gauge transformations $\varphi \to e^{i\alpha(x)}\varphi$ with $\alpha(x) = \theta$, but then $A_i \to A_i + \frac{1}{e}\partial_i\theta = A_i + \hat{\theta}_i \cdot \frac{1}{r}$

singular at $r=0$

• Winding number doesn't depend on the particular form of distinct closed curve, as it is discrete number that can not be changed due to small variation of curve. The same argument protects it from changes due to the smooth time evolution $\Rightarrow$

$\Rightarrow$ winding number is topological charge.

It can be written as

$$n = \frac{1}{2\pi i v^2} \oint_C dx^i \, \varphi^* \partial_i \varphi;$$

infinitely distinct circle.

(9) • If $\varphi = e^{if(\theta)} \cdot v$ then $n = \frac{1}{2\pi} \oint_C dx^i \cdot \partial_i f(\theta) = \frac{1}{2\pi} (f(2\pi) - f(0))$

• Let's consider two fields with the same windning number

$$\begin{cases} \varphi_1(x) = e^{if_1(\theta)} v; \\ \varphi_2(x) = e^{if_2(\theta)} v; \end{cases} \quad f_1(2\pi) - f_1(0) = f_2(2\pi) - f_2(0);$$

then $\varphi_2 = e^{if_{21}} \varphi_1$ where $f_{21}(2\pi) - f_{21}(0) = 0$

$f_{21} \equiv f_2 - f_1;$

hence f_{21} is single-valued function of θ and we can always chouse gauge transform. with $\alpha(x) = g(r) \cdot f_{21}(\theta)$ wher $g(r)$ is some smooth function such that $g(0) = 0$

$g(\infty) = 1$

Then under gauge transformation $\varphi_1(x) \to \varphi_1'(x) = e^{i\alpha(x)} \varphi_1(x)$ and φ_1' has the same asymptotic behaviour as φ_2 and hence:

• We can separate all possible field configurations into sectors characterized by the winding number.

• Using smooth gauge transformations we can make asymptotic behaviour of different field configurations the same:

$$\varphi \underset{|x| \to \infty}{\longrightarrow} v e^{in\theta};$$

• Let's find the soliton corresponding to the winding number $n=1$, for which the scalar field is

$$\varphi = e^{i\theta} \cdot v;$$

In order to have finite energy we need $D_i\varphi$ decrease faster then $\frac{1}{r}$ at large r.

Notice that: $\partial_i \varphi = i v e^{i\theta} \partial_i \theta = v e^{i\theta} \cdot i \left(-\frac{1}{r^2} \varepsilon_{ij} x_j\right)$, where we have used

$$\frac{\partial}{\partial x} = \frac{x}{r} \cdot \frac{\partial}{\partial r} - \frac{y}{r^2} \frac{\partial}{\partial \theta}; \qquad \text{So we have:}$$

$$\frac{\partial}{\partial y} = \frac{y}{r} \cdot \frac{\partial}{\partial r} + \frac{x}{r^2} \cdot \frac{\partial}{\partial \theta}; \qquad \partial_i \varphi = v e^{i\theta} \cdot \left(-\frac{i}{r} \varepsilon_{ij} n_j\right); \quad \bar{n} = \frac{\bar{x}}{|x|};$$

(10) • This $\frac{1}{r}$ decrease is too slow and we need to compensate it with the gauge field part of the covariant derivative. I.e. gauge field should be taken

$$A_i = -\frac{1}{er}\varepsilon_{ij} n_j;$$

Notice that though $A_i = \frac{1}{e}\partial_i\theta$ this gauge field does not correspond to pure gauge due to the singularity at $r=0$. Indeed

$$F_{ij} \equiv \partial_i A_j - \partial_j A_i = \frac{2}{er^2}(n_i \varepsilon_{je} n_e - n_j \varepsilon_{ie} n_e) + \frac{2}{er^2}\varepsilon_{ij}$$ where we used $\partial_i \frac{1}{r} = -\frac{n_i}{r^2}$; $\partial_i n_j = \frac{1}{r}(\delta_{ij} - n_i n_j)$; then $F_{12} = 0$, but there is flux of magnetic field going through the 2d plane. (see calculation below)

• Total magnetic flux is

$$\Phi = \int d\bar{S}\cdot \bar{\nabla}\times\bar{A} = \oint_{S^1} d\bar{r}\cdot\bar{A} = \int_0^{2\pi} r d\theta \cdot A_\theta = \int_0^{2\pi} r\, d\theta \cdot \frac{1}{er} = \frac{2\pi}{e}$$

for the general configuration $\varphi = e^{in\theta} v$, $A_i = \frac{n}{e}\partial_i\theta$ so that $\Phi = \frac{2\pi n}{e}$

• So we want to find the solution for the equations of motion

$$\boxed{\begin{aligned} &\partial_\mu F^{\mu\nu} = e j^\nu; \\ &D_\mu D^\mu \varphi + \frac{\partial V}{\partial \varphi} = 0; \end{aligned}}$$

that have the asymptotics:

$$\varphi = v\cdot e^{i\theta},\ A_i = -\frac{1}{er}\varepsilon_{ij} n_j;$$

• Notice that these asymptotics are invariant under combination of spatial rotations and phase shift of φ:

$$\varphi(\theta) \to e^{-i\alpha}\varphi(\theta+\alpha);$$

for A_i invariance is obvious as $A_i = \frac{1}{e}\partial_i\theta$;

• We want our full solution also be invariant under this transformation. Good ansatz is:

$$\begin{cases} A_i(r,\theta) = -\frac{1}{er}\varepsilon_{ij} n_j A(r) + n_i \cdot B(r); \\ \varphi(\theta) = v e^{i\theta}\cdot F(r); \end{cases}$$

(11) • However notice that

$$F_{ij} = \partial_i A_j - \partial_j A_i = -\frac{1}{e}\Big(\partial_i\Big(\frac{A(r)}{r^2} x_k \varepsilon_{jk}\Big) - e\partial_i\Big(\frac{B(r)}{r} x_j\Big) -$$

$$-\partial_j\Big(\frac{A(r)}{r^2} x_k \varepsilon_{ik}\Big) - \partial_j\Big(\frac{B(r)}{r} x_i\Big)\Big) = -\frac{1}{e}\Big\{+\frac{x_i}{r} x_k \frac{d}{dr}\Big(\frac{A(r)}{r^2}\Big)\varepsilon_{jk} +$$

$$+\varepsilon_{ji}\frac{A(r)}{r^2} - e\frac{x_i x_j}{r}\frac{d}{dr}\Big(\frac{B(r)}{r}\Big) - e\delta_{ij}\frac{B(r)}{r} - (i \leftrightarrow j)\Big\} =$$

$$= \frac{1}{er}(\varepsilon_{jk} x_i x_k - \varepsilon_{ik} x_j x_k)\frac{d}{dr}\Big(\frac{A(r)}{r^2}\Big) + \frac{2}{e}\varepsilon_{ij}\frac{A(r)}{r^2}$$

Only nonzero components are

$$F_{12} = -F_{21} = -\frac{1}{er}(-1\cdot x_1 x_1 - 1 x_2^2)\frac{d}{dr}\Big(\frac{A(r)}{r^2}\Big) + \frac{2}{er^2}A(r)$$

So finally $\underline{F_{12} = \frac{1}{er}\frac{dA}{dr}}$; while $\underline{B \text{ is pure gauge!}}$

Hence we can put $\underline{B(r) = 0}$;

• Covariant derivative of φ-field is

$$D_i\varphi = \Big(\partial_i - ie\cdot\Big(-\frac{1}{er}\Big)\varepsilon_{ij}n_j A(r)\Big) v e^{i\theta} F(r) =$$

$$= (i\,\partial_i\theta - ie\cdot\frac{1}{e}\partial_i\theta\cdot A(r)) + v e^{i\theta} n_i \frac{dF}{dr} \Rightarrow D_i\varphi = -\frac{i}{r}\varepsilon_{ij}n_j(1-A)\cdot\varphi +$$

$$+ v e^{i\theta} n_i F';$$

then the current is given by

$$j^i = -i(\varphi^* D^i\varphi - (D^i\varphi)^*\varphi) = -i\Big(-\frac{2i}{r}\varepsilon_{ij}n_j(1-A)v^2F^2\Big) \Rightarrow$$

$$\Rightarrow \underline{j_i = -\frac{2}{r}\varepsilon_{ij}n_j(1-A)v^2F^2};$$

Then the maxwell equation is

$$\partial_1 F^{12} = \frac{x_1}{r}\frac{d}{dr}\Big(\frac{1}{er}\frac{dA}{dr}\Big) = ej^2 = -e\cdot\frac{2x_1}{r^2}(1-A)v^2F^2 \Rightarrow$$

$$\Rightarrow \underline{\frac{d}{dr}\Big(\frac{1}{r}\frac{dA}{dr}\Big) + 2e^2v^2\frac{F^2}{r}(1-A) = 0};$$

• Second equation $D_iD_i\varphi - \frac{\partial V}{\partial\varphi} = 0$ leads to

$$\underline{\frac{d}{dr}\Big(r\frac{dF}{dr}\Big) - \lambda v^2 r F(F^2-1) - \frac{F}{r}(1-A)^2 = 0}$$

(12) • Asymptotic behavior of F and A should be:

$$F(r) \to 1,\ A(r) \to 1 \text{ as } r \to \infty;$$

• If we also want fields to be smooth at $r=0$

we want $F(r) \to 0,\ A(r) \to 0$ as $r \to 0$ (more precise $F(r) \to r;\ A(r) \to r^2$ as $r \to 0$)

• Unfortunately this problem can't be solved analytically. However let's show the existence of the solution.

• At large r

$$A(r) = 1 - a(r),\ F(r) = 1 - f(r); \qquad \begin{matrix} a(r) \to 0 \\ f(r) \to 0 \end{matrix} \text{ as } r \to \infty$$

Then equations turn into (we assume $m_H < 2m_v$ in order to have $F(F^2-1) \gg F(1-A)^2$)

$$\begin{cases} r\dfrac{d}{dr}\left(\dfrac{1}{r}\dfrac{da}{dr}\right) - m_v^2 a = 0; \\ \dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{df}{dr}\right) - m_H^2 f = 0; \end{cases}$$

equations are written in the linear order in a and f.

Solution is
$$\boxed{\begin{aligned} a(r) &= C_a \sqrt{r}\, e^{-m_v r}; \\ f(r) &= C_f \frac{e^{-m_H r}}{\sqrt{r}}; \end{aligned}}$$
C_a and C_f are integration constants.

• At small r:
$$\left.\begin{aligned} F(r) &= \alpha_f r + \beta_f \cdot r^3 + \dots; \\ A(r) &= \alpha_a r^2 + \beta_a r^4 + \dots; \end{aligned}\right\}$$
we don't take other powers as it can be shown they should be zero.

Substituting back to equations we obtain:

$$(-8\beta_a r + \dots) - m_v^2 \cdot (\alpha_f r + \dots) = 0 \Rightarrow \beta_a = -\frac{m_v^2}{8}\alpha_f$$

$$+(-\alpha_f - 9\beta_f r^2 + \dots) + \frac{1}{2} m_H^2 r (\alpha_f r + \dots)(\alpha_f^2 r^2 - 1 + \dots) + (\alpha_f + \beta_f r^2)(1 - 2\alpha_a r^2 - 2\beta_a r^4) = 0$$

$$\Rightarrow \quad \beta_f = -\frac{m_H^2}{16}\alpha_f - \frac{1}{4}\alpha_a \alpha_f;$$

• Notice that α_a and α_f are independent and can be considered as 2 parameters of the solution.

Hence solution can be constructed as follows:

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- Build solution at $r\to\infty$ (2 parameters C_a, C_f)
- Build solution at $r\to 0$ (2 parameters d_a, d_f)
- In some intermediate region glue this parameters:

$$\left.\begin{array}{l} F^{\infty}=F^{0} \\ \frac{d}{dr}F^{\infty}=\frac{d}{dr}F^{0}\end{array}\right\}\text{ fixes } C_f \text{ and } d_f \qquad \left.\begin{array}{l} A^{\infty}=A^{0} \\ \frac{d}{dr}A^{\infty}=\frac{d}{dr}A^{0}\end{array}\right\}\text{ 2 equations fixing } C_a \text{ and } d_a$$

- Hence we have enough information to fix solution completely and this is big hint for the existence of the solution

Size and mass of the vortex.

Let's use some dimensional arguments. Start with rescaling fields and change of variable:

$$\varphi(\bar{x})=v\,\phi(\bar{y}) \qquad ; \bar{y}=m_v\bar{x}$$
$$A_i(\bar{x})=\frac{m_v}{e}C_i(\bar{y})$$

Then : $F_{ij}^2=\left(\frac{m_v^2}{e}\right)^2 C_{ij}^2$; $C_{ij}\equiv\frac{\partial}{\partial y^i}C_j-\frac{\partial}{\partial y^j}C_i$;

$$|D_i\varphi|^2=(m_v\cdot v)^2|\tilde{D}_i\phi|^2=\left(\frac{m_v^2}{e}\right)^2\frac{1}{2}|\tilde{D}_i\phi|^2;\quad \tilde{D}_i\equiv\frac{\partial}{\partial y^i}-iC_i\ ;$$

$$V(\varphi)=\frac{\lambda}{2}\left(|\varphi|^2-v^2\right)^2=\frac{\lambda}{2}v^4\left(|\phi|^2-1\right)^2=\frac{1}{e^2}m_v^2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}m_H^2\left(|\phi|^2-1\right)^2.$$

Hence $V(\varphi)=\left(\frac{m_v^2}{e}\right)^2\frac{m_H^2}{8m_v^2}\left(|\phi|^2-1\right)^2$;

And the energy functional is given by

$$E=\frac{m_v^2}{e^2}\int d^2y\left(\frac{1}{4}C_{ij}C_{ij}+\frac{1}{2}|\tilde{D}_i\phi|^2+\frac{1}{8}\frac{m_H^2}{m_v^2}\left(|\phi|^2-1\right)^2\right);$$

- The purpose of this notations is to write down energy functional in terms of dimensionless coefficients and so that $|\phi|\sim 1$. If $m_H\sim m_v$ we don't have any small or big coefficients in addition. Then

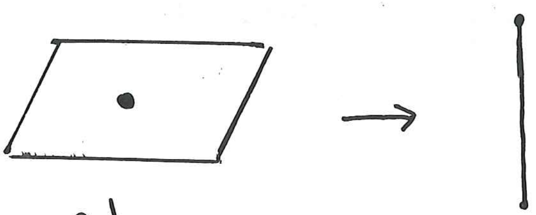
$$\phi\sim 1,\ C_i\sim 1,\ \underbrace{y\sim 1}_{\text{the size of the vortex}};$$

(14) • Hence the size of the soliton is $\boxed{r_{sol} \sim \frac{1}{m_V}}$;

• Mass of the soliton is $\boxed{M_{sol} \sim \frac{m_V^2}{e^2}}$

• Extension to (3+1)-dim space.

Solution is localized in the plane so if we extend it to (3+1) dimensions it will form the string, constant along 1 direction (say x_3) and forming vortex profile in each section $x_3 = \text{const}$.



2d $\qquad$ 3d

• Application to physics: As we know scalar electrodynamics describes physics of superconductors through GL functional:

$$F = F_n + \alpha|\psi|^2 + \frac{1}{2}\beta|\psi|^4 + \frac{1}{2m}|(-i\hbar\bar{\nabla} - 2e\bar{A})\psi|^2 + \frac{|\bar{B}|^2}{2\mu}$$

where ψ are Cooper pairs density.

Two characteristics of theory are

• Correlation length $\xi = \sqrt{\frac{\hbar^2}{2m|\alpha|}}$ $\qquad \xi \sim \frac{1}{m_H}$ in our language

• London length (how deep magnetic field penetrates SC) : $\lambda = \sqrt{\frac{m}{4\mu_0 e^2 \psi_0^2}}$ $\qquad \lambda \sim \frac{1}{m_V}$ in our language.

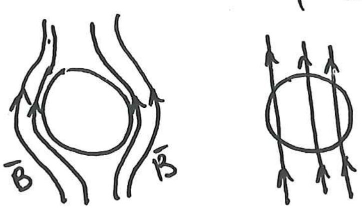
where $\psi_0 = \sqrt{-\frac{\alpha}{\beta}}$ - ground state value.

• Then there are two types of SC

Type I

$\frac{\lambda}{\xi} \sim \frac{m_H}{m_V} < \frac{1}{\sqrt{2}}$

no $\bar{B}$-field penetration

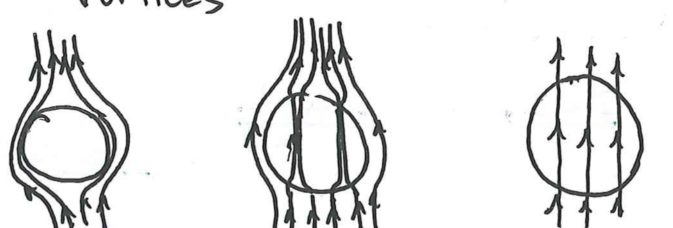


$B < B_c$ $\qquad$ $B > B_c$

Type II

$\frac{\lambda}{\xi} \sim \frac{m_H}{m_V} > \frac{1}{\sqrt{2}}$

Vortices



$B < B_c^{(1)}$ $\qquad$ $B_c^{(1)} < B < B_c^{(2)}$ $\qquad$ $B > B_c^{(2)}$

(15) Skyrmions

- Let's now consider theory of 3 scalar fields subjected to the constraint $\varphi^a\varphi^a = 1$, i.e. fields take values on S^2. This model is called n-field model.
- The only term is kinetic term:

$$\mathcal{L}_0 = \frac{1}{2g^2}\partial_\mu\varphi^a\partial^\mu\varphi^a; \qquad a = 1,2,3; \quad \underbrace{\mu,\nu = 0,1,2}_{\text{we consider (2+1) dim. space.}}$$

- In order to consider constraint properly we consider the action with modified Lagrange term:

$$\tilde{S} = \int d^3x \frac{1}{2g^2}\partial_\mu\varphi^a\partial^\mu\varphi^a + \frac{1}{2g^2}\int d^3x\, \lambda(x)(\varphi^a(x)\varphi^a(x) - 1);$$

- Varying $\tilde{S}$ we get:

$$\delta\tilde{S} = \int d^3x \frac{1}{2g^2}(-\partial_\mu\partial^\mu\varphi^a + \lambda(x)\varphi^a)\cdot 2\,\delta\varphi^a. \Rightarrow \quad -\partial_\mu\partial^\mu\varphi^a + \lambda(x)\varphi^a = 0;$$

Multiplying with φ^a we get $\quad -\varphi^a\partial_\mu\partial^\mu\varphi^a + \lambda = 0$

$\Rightarrow \quad \lambda(x) = \varphi_a\partial_\mu\partial^\mu\varphi_a$ and equations of motion are given by

$$\boxed{\partial_\mu\partial^\mu\varphi^a - (\varphi^b\partial_\mu\partial^\mu\varphi^b)\varphi^a = 0;} \rightarrow \text{equation is non-linear.}$$

- Energy: $\quad E = \frac{1}{2g^2}\int \partial_i\varphi^a\cdot\partial_i\varphi^a\, d^2x \quad$ - for static configurations.
- Ground state: $\quad \varphi^a = \text{const}$ such that $\varphi^a\varphi^a = 1$

For simplicity we choose $\quad \varphi^a = \delta^{a3}$

- Symmetries: The original Lagrangian has O(3)-symmetry (rotations of φ^a vector). This symmetry is broken by the ground state down to O(2);
- If we want energy of static configuration to be finite field should be constant at spatial infinity. This constant can not depend on any angles as in the vortex case because then $\nabla\varphi^a \sim \frac{1}{r}$ and we obtain divergence and

there is no gauge field to cancell it.

- Let's identify all points at infinity so that the space becomes S^2 instead of the plane.
- $\varphi^a(x)$ field can be then considered as the map:

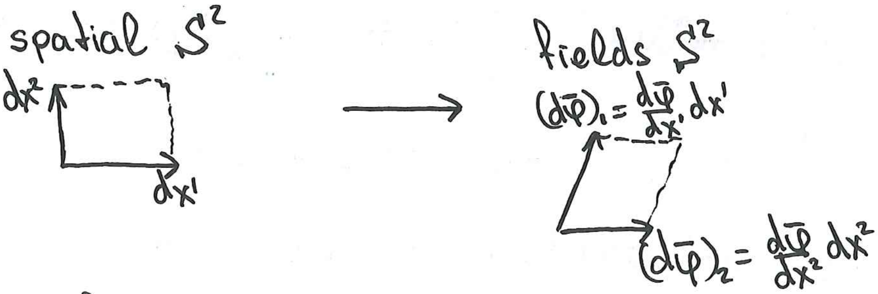
$$\boxed{\text{spatial } S^2 \xrightarrow[\varphi(x)]{} S^2 \text{ which fields live on.}}$$

- This map can be used to classify solutions into classes (similarly to $S^1 \to S^1$ map in case of vortex).
- Classes are characterized by the topological number

$$n = \frac{\text{Area of } S^2 \text{ in fields space}}{\text{Area of } S^2 \text{ in spatial coord.}}$$

$\to$ analog of $e^{in\theta}$ for vortices.

- More precisely if we map small piece of area



$\to$ area is given by $d\bar{\sigma} = (d\bar{\varphi})_1 \times (d\bar{\varphi})_2$

- Then for the total area we get:

$$n = \frac{1}{4\pi} \int \bar{\varphi} \cdot d\bar{\sigma}$$

area of spatial S^2 radius is normalized to one

scalar product is take to obtain the proper sign of orientation.

- Then finally we obtain topological charge:

$$\boxed{n = \frac{1}{4\pi} \int d^2x \, \bar{\varphi} \cdot \left(\frac{\partial \bar{\varphi}}{\partial x^1} \times \frac{\partial \bar{\varphi}}{\partial x^2} \right) = \frac{1}{8\pi} \int d^2x \, \varepsilon^{abc} \varepsilon_{ij} \varphi^a \partial_i \varphi^b \partial_j \varphi^c;}$$

- this topological charge is invariant under small variations of φ-field: $\varphi^a \to \varphi^a + \delta\varphi(x)$;

(17) • To find the energy of the skyrmion let's use the following common trick.

• Consider $F_i^a = \partial_i \varphi^a \pm \varepsilon^{abc} \varepsilon_{ij} \varphi^b \partial_j \varphi^c$; "+" - skyrmions, "−" - anti-skyrmions

• Obviously

$$\int d^2x\, F_i^a F_i^a \geq 0$$

• at the same time

$$F_i^a F_i^a = \partial_i\varphi^a \cdot \partial_i \varphi^a \pm 2\, \varepsilon^{abc} \varepsilon_{ij}\, \partial_i\varphi^a \partial_j \varphi^c \cdot \varphi^b + \varepsilon^{abc} \varepsilon_{ij}\, \varphi^b \partial_j \varphi^c \times \varepsilon^{ade} \varepsilon_{ie} \times \varphi^d \partial_e \varphi^e =$$

$$= \partial_i\varphi^a \partial_i\varphi^a \pm 2\, \varepsilon^{abc} \varepsilon_{ij}\, \partial_i\varphi^a \cdot \varphi^b \partial_j\varphi^c + \delta_{je}\left(\delta^{bd}\delta^{ce} - \delta^{be}\delta^{dc}\right) \varphi^b \partial_j \varphi^c \varphi^d \partial_e \varphi^e =$$

$$= \partial_i\varphi^a \partial_i\varphi^a \mp 2\varepsilon^{abc} \varepsilon_{ij}\, \varphi^a \partial_i\varphi^b \partial_j \varphi^c + \partial_i\varphi^a \partial_i\varphi^a$$

where we have used in the last step $\varphi^a\varphi^a = \Rightarrow \varphi^a \partial_i \varphi^a = 0$;

• Hence we obtain:

$$\int d^2x\, \partial_i\varphi^a \partial_i\varphi^a \mp \int d^2x\, \varepsilon^{abc} \varepsilon_{ij}\, \varphi^a \partial_i\varphi^b \partial_j\varphi^c \geq 0 \Rightarrow \boxed{E \geq \frac{4\pi}{g^2} |n|};$$

• Notice that the minimum of energy in the sector with topological number n is $\boxed{E = \frac{4\pi}{g^2}|n|}$ and obtained for the fields satisfying $\boxed{F_i^a = \partial_i\varphi^a \pm \varepsilon^{abc} \varepsilon_{ij} \varphi^b \partial_j \varphi^c = 0};$

• Let's now find the particular form of solution. As in the case of vortex we use symmetry of asymptotic behavior of n-field. This symmetry is combination of spatial $SO(2)$-rotations and rotations around x^3-axis in field space.

• Appropriate ansatz is $\begin{cases} \varphi^\alpha(x) = n^\alpha \cdot \sin f(r); \\ \varphi^3(x) = \cos f(r); \end{cases}$ where $n^\alpha = \frac{x^\alpha}{r}$, $\alpha = 1,2$, then $\varphi^a\varphi^a = \varphi^\alpha \cdot \varphi^\alpha + (\varphi^3)^2 = 1$;

• Then $\begin{cases} \partial_i \varphi^\alpha = \frac{1}{r}(\delta^{i\alpha} - n^i \cdot n^\alpha) \cdot \sin f + n^i \cdot n^\alpha \cdot f' \cdot \cos f, \\ \partial_i \varphi^3 = - n^i \cdot f' \cdot \sin f; \end{cases}$

(18) • $\varepsilon^{3\alpha\beta}\cdot\varepsilon_{ij}\,\varphi^\alpha\partial_j\varphi^\beta = \varepsilon_{ij}\,\varepsilon_{\alpha\beta}\, n^\alpha\cdot\sin f\cdot\left[\frac{1}{r}(\delta^{j\beta}-n^j\cdot n^\beta)\sin f + n^j\cdot n^\beta\cdot f'\cdot\cos f\right] =$

$= \varepsilon_{ij}\varepsilon_{\alpha\beta}\,\delta^{j\beta}n^\alpha\cdot\frac{1}{r}\sin^2 f = \delta_{i\alpha}\,n^\alpha\frac{1}{r}\sin^2 f = n_i\frac{1}{r}\cdot\sin^2 f;$

• Then skyrmion equation for the component $a=3$ is

$$\partial_i\varphi^a + \varepsilon^{abc}\varepsilon_{ij}\varphi^b\partial_j\varphi^c = \boxed{-n_i\cdot f'\cdot\sin f + n_i\frac{1}{r}\sin^2 f = 0;} \Rightarrow f' = \frac{1}{r}\sin f;$$

• An appropriate solution is obtained by integration of this equation

+ bdry condition at infty $f(\infty)=\pi$ (as $\varphi^a=-\delta^{a3}$)

$$f = 2\arctan\frac{r}{r_0}; \quad\Rightarrow\quad \cos f = \frac{1-\tan^2\frac{f}{2}}{1+\tan^2\frac{f}{2}}, \quad \sin f = \frac{2\tan\frac{f}{2}}{1+\tan^2\frac{f}{2}};$$

r_0 is integration constant.

$$\Downarrow$$

$$\begin{cases}\varphi^\alpha = 2\dfrac{x_\alpha\cdot r_0}{r_0^2+r^2};\\[2mm] \varphi^3 = \dfrac{r_0^2-r^2}{r_0^2+r^2};\end{cases}$$

• Topological charge of this configuration should be 1 by construction. Let's check this

$$n = \frac{1}{8\pi}\int d^2x\;\varepsilon^{abc}\varepsilon_{ij}\,\varphi^a\partial_i\varphi^b\partial_j\varphi^c = \frac{1}{4\pi}\int d^2x\cdot\bar\varphi\cdot[\partial_1\bar\varphi\times\partial_2\bar\varphi];$$

Now we can do streightforward calculation:

$$\frac{\partial}{\partial x^1}\bar\varphi = \begin{pmatrix}\frac{1}{r}n_2^2\cdot\sin f + n_1^2 f'\cdot\cos f\\ n_1 n_2(f'\cdot\cos f - \frac{1}{r}\sin f)\\ -n_1 f'\cdot\sin f\end{pmatrix}; \quad \frac{\partial}{\partial x^2}\bar\varphi = \begin{pmatrix}n_1 n_2(-\frac{1}{r}\sin f + f'\cdot\cos f)\\ \frac{1}{r}n_1^2\cdot\sin f + n_2^2 f'\cdot\cos f\\ -n_2 f'\cdot\sin f\end{pmatrix};$$

So that: $\bar\varphi\cdot(\partial_1\bar\varphi\times\partial_2\bar\varphi) = \frac{1}{r}(n_1+n_2)f'\frac{1}{r}\sin^3 f + (n_1^2+n_2^2)^2\frac{1}{r}f'\sin f\cdot\cos^2 f =$

$= \frac{1}{r}f'\cdot\sin f$, so that:

$$\left.\begin{aligned}&\int d^2x\,\bar\varphi\,(\partial_1\bar\varphi\times\partial_2\bar\varphi) = \int dr\,d\theta\cdot r\cdot\frac{1}{r}(f'\sin f) =\\ &= -\int_0^\infty dr\,d\theta\,(\cos f)' = -2\pi\cos f\Big|_1^{-1} = 4\end{aligned}\right\} \Rightarrow n = \frac{1}{4\pi}\int d^2x\,\bar\varphi\,(\partial_1\bar\varphi\times\partial_2\bar\varphi) = 1;$$

$\boxed{n=1\text{ , q.e.d.}}$

• Finally due to our earlier considerations we know that the mass of the soliton is $M = \frac{4\pi}{g^2}|n| = \frac{4\pi}{g^2};$

which can also be checked by the direct substitution.

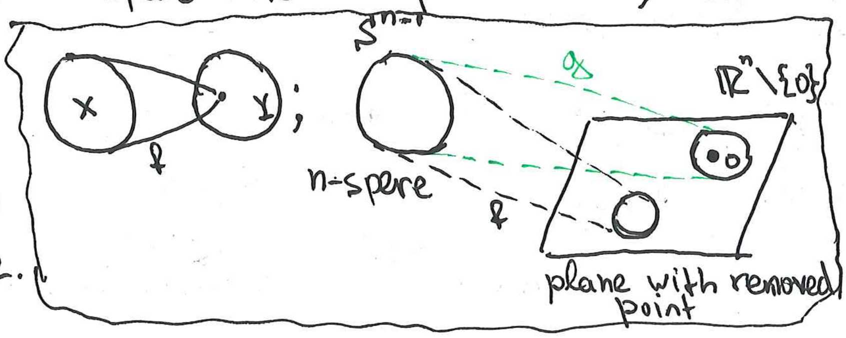
Lecture 9 Homotopic groups. Monopoles.

- Discussing vortices and skyrmions we have seen that maps $S^1 \to S^1$ and $S^2 \to S^2$ arises. This situation is quite general for the solutions of different theories and different spaces.

- Topological space: If X is set and $T = \{U_i | i \in I\}$ - collection of subsets. (X, T) is topological space if:
  - $\emptyset, X \in T$;
  - If we obtain subcollection $\tilde{T} = \{U_j | j \in J\}$ of sets then $\bigcup_{j \in J} U_j \in T$;
  - If K is finite subcollection of I then $\{U_k | k \in K\}$ satisfies $\bigcap_{k \in K} U_k \in T$;

Less formal: TS is set with defined concept of proximity.

- Two continious maps (continious = nearby points go to nearby) $f: X \to Y$, $g: X \to Y$ are homotopic if one can be continiously deformed into another, i.e. if there is such family of maps h_t ($t \in [0,1]$) that $h_0 = f$ and $h_1 = g$;
  - If we denote $C(X,Y)$ is set of continious maps between X and Y then homotopy relation divides $C(X,Y)$ into equivalence classes, denoted $\{X, Y\}$;
  - If $f: X \to Y$ maps all X-space into one point in Y, then f is homotopic to zero



All possible maps homotopic to zero are homotopic to each other if Y is connected space.

- On the right picture map f is homotopic to zero while g is not.

- Let's consider map to a direct product: $f: X \to Y \times Z$. It can be seen as the pair of continious maps $f(x) = \{f_1(x), f_2(x)\}$ $f_1: X \to Y$; $f_2: X \to Z$; $\Rightarrow$ There exists one to one correspondence

② $\{X, Y\times Z\} \longleftrightarrow \{X,Y\}\times\{X,Z\} \Rightarrow$ classification of maps to the direct product is reduced to the classification of maps to each of the factors.

• Homotopic spaces.

• Identity mapping $e: Y\to Y$ is such one that $e(y)=y$ $\forall y\in Y$.

• Two spaces Y_1 and Y_2 are homotopic if there exist maps $h_1: Y_1\to Y_2$ and $h_2: Y_2\to Y_1$ such that $h_1\cdot h_2: Y_1\to Y_1$ and $h_2\cdot h_1: Y_2\to Y_2$ are homotopic to the identity map.

Examples: • S^{n-1} homotopic to $\mathbb{R}^n/\{0\}$ - plane with removed point

• $S^n\setminus\{\text{north pole}\}$ homotopic to $\mathbb{R}^n$

• $\mathbb{R}^n$ with identified infinity homotopic to S^n;

• Fundamental group.

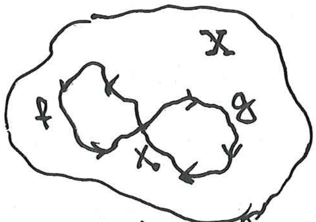
• Let's consider maps $S^1\to X$ (or equivalently maps of the interval $[0,1]$ into X such that $f(1)=f(0)$). And let's also fix one of the points on this map ($f(1)=f(0)=x_0$);

• The set of all classes of homotopy for the maps $f: [0,1]\to X$, $f(0)=f(1)=x_0$, is denoted by $\pi_1(X,x_0)$ and called fundamental group.

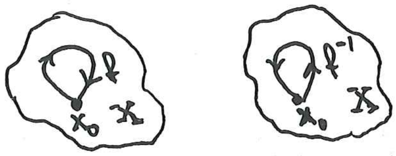
Group structure

• Product $f*g$ is defined so that

$$f*g(\xi)\equiv\begin{cases} g(2\xi) & 0\leq\xi\leq\frac{1}{2};\\ f(2\xi-1) & \frac{1}{2}\leq\xi\leq 1;\end{cases}$$



product $f*g$



inverse element

• Inverse element $f^{-1}(\xi)$ is just given by the path went in the inverse direction:

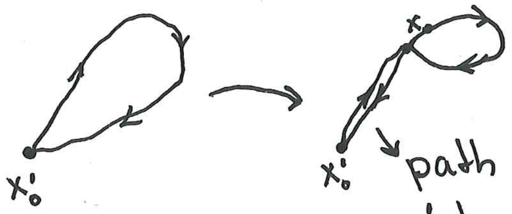
$$f^{-1}(\xi)=f(1-\xi)$$

③ • <u>Identity</u> is the homotopic class containing map between S^1 and single point.

• For <u>connected space X</u> <u>$\pi_1(X, x_0) \simeq \pi(X, x_0')$</u> $\forall x_0, x_0' \in X$

<u>Isomorphism is</u> built as follows:

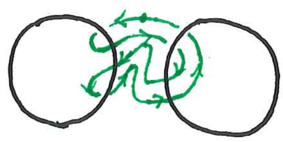
• We connect x_0' to x_0 by the path and go along this path in two directions.



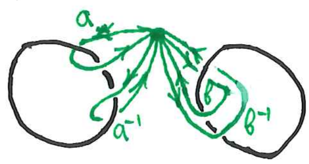
• If π_1 is commutative then isomorphism is <u>path-independent</u> and we can address $\pi_1(X, x_0)$ as just $\pi_1(X)$; (without reffering to the fixed point x_0)

• π_1 is not commutative in general:

<u>Example:</u> Consider $\mathbb{R}^3$ with two removed circles and the following path:



it is homotopic to



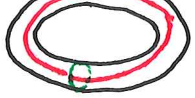
with the order $aba^{-1}.b^{-1}$

and it is not trivial so we see that a and b <u>don't commute.</u>

• <u>Examples</u> of fundamental groups:

• <u>$\pi_1(S^1) = \mathbb{Z}$</u> ; indeed if we map S^1 to S^1 the only thing we can do is wind several times. So different homotopy classes differ by the winding number.

• <u>$\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$;</u> - winding around to cycles of the torus



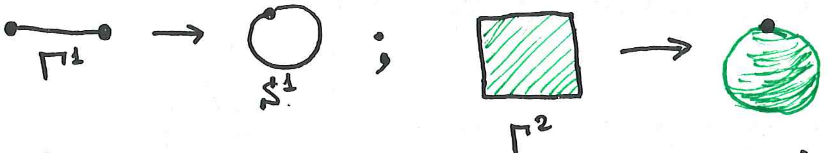
• <u>Homotopic groups.</u>

• We can generalize this ideas to <u>higher dimensions</u> and consider maps $S^n \to X$ with the south pole of S^n mapped to x_0 point. We call these maps <u>spheroids.</u>

• Two <u>spheroids are homotopic</u> if they can be deformed one

④ into other continiously.

• To describe groupstructure of $\pi_n(X, x_0)$, i.e. set of all homotopic classes of $S^n \to X$ maps, let's first notice that $S^n \simeq \Gamma^n$ (n-cube) with identified boundary:



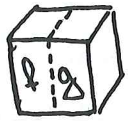
and so on.

• Then we introduce the operation of summation:

Let f, g be two spheroids : $\Gamma^n \to X$, then $h = f + g$ is

$$h(x^1, x^j) = \begin{cases} f(2x^1, x^j) & 0 \leq x^1 \leq \frac{1}{2} \\ g(2x^1 - 1, x^j) & \frac{1}{2} \leq x^1 \leq 1 \end{cases}$$

where $j = 2, 3, \ldots, n$;



$f(1, x^j) = g(0, x^j) = x_0$;

• Inverse spheroids

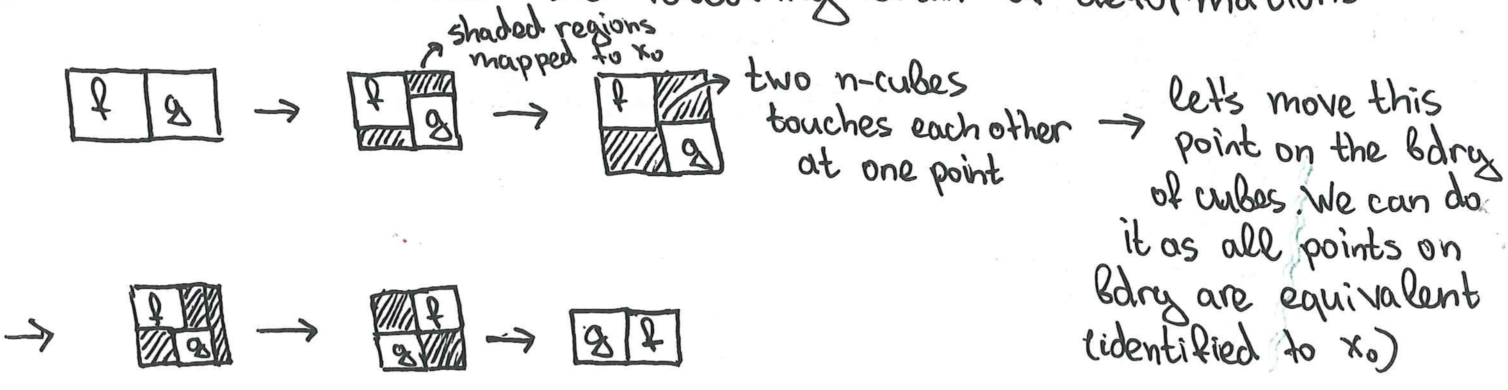
$f^{-1}(x^1, x^j) = f(1 - x^1, x^j)$;

• Group identity is homotopy class containing zero spheroid: map of S^n into one point x_0.

• All these operations induces operations in the homotopic group. Let's say f is homotopic to f' and $\tilde{f}_t$ is family of maps connecting f and f', and similarly we have $g, g', \tilde{g}_t$ then $\tilde{f}_t + \tilde{g}_t$ is the family of maps connecting $(f+g)$ and $(f'+g')$;

• Higher $(n \geq 2)$ homotopy groups are abelian for any X.

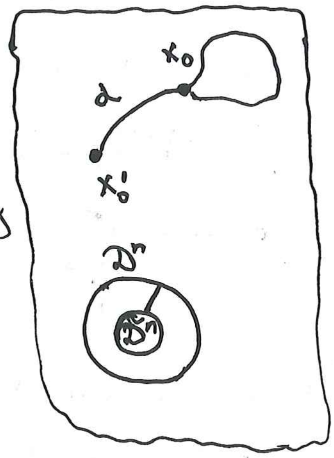
To see this consider the following chain of deformations



Notice that for Γ^1 bdry is not connected $\Rightarrow$ we can not perform this continious deformations $\Rightarrow$ $\pi_1(X, x_0)$ is not necessarily abelian as we have seen.

⑤ • Let's now show that if X is connected $\pi_n(X,x_0) \simeq \pi_n(X,x_0')$ $\forall$

$\forall\, x_0, x_0' \in X$;



• Let α be the path connecting x_0 and x_0'.

• Consider the spheroid f as the map of the ball D^n with identified boundary, mapping this boundary to x_0.

• Let f' be map of D^n mapping boundary to x_0'. Let's build it as follows

•• $\tilde{D}^n$ is insid D^n

•• $f: \tilde{D}^n \to X$ so that $\partial\tilde{D}^n \to x_0$

•• Remaining space $D^n \setminus \tilde{D}^n$ are mapped so that every point on the radius is mapped to the point on α.

• Hence using this relation between f and f' we can say that $\pi_n(X,x_0)$ and $\pi_n(X,x_0')$ are isomorphic and we can forget about x_0 point just writing $\pi_n(X)$ instead.

Examples: Homotopic groups $\pi_n(S^m)=0$; (trivial) for $n<m$;
Notice that during mapping $S^n \to S^m$ there is at least one point in S^m which doesn't have preimage in S^n. Let's remove this point, then $S^m/\{0\} \simeq \mathbb{R}^m$ and in $\mathbb{R}^m$ any S^n can be contracted.

• $\pi_n(S^n)=\mathbb{Z}$; Similarly to $\pi_1(S^1)$ there are winding numbers
Let's find corresponding topological number or degree of the map $\deg f$.

• Let $f: \underset{\text{n-sphere}}{S} \to \underset{\text{n-sphere}}{S'}$

$S \to x_1,\dots,x_n$ coordinates.

$S' \to y_1,\dots,y_n$ coordinates.

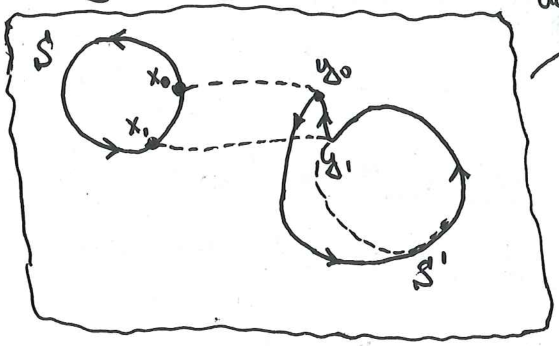
$$\begin{cases} y^1 = f_1(x^1,\dots,x^n); \\ y^2 = f_2(x^1,\dots,x^n); \\ \dots\dots\dots \\ y^3 = f_3(x^1,\dots,x^n); \end{cases}$$

• Points where Jacobian $J(x)=\det\left(\frac{\partial f_i}{\partial x^j}\right)$ is not zero are regular points. It can be shown that irregular points have the measure zero.

⑥ • Degree of the map equals number of solutions of equation $f(x)=y$. in the regular point y, taking orientation into account.

In turn it is equal to

$$\boxed{\sum_{\substack{\text{roots of} \\ f(x)=y}} \operatorname{sign} J(x_i) = \deg f}$$



y_0 and y_1 are irregular points.

Notice that on the picture it is either 3 or 1 solutions, but with orientation it is always one.

• Finally using the equation $\delta(f(x)-y)=\sum_i \frac{1}{|J(x_i)|}\delta(x-x_i(y))$;

summ over roots

we rewrite:

$$\deg f = \int dx \cdot J(x)\cdot \delta(f(x)-y)$$

Let's integrate over y with $\mu(y)$ weight. $\Rightarrow$

$$\deg f \int dy\, \mu(y) = \int dx\, dy\, J(x)\mu(y)\cdot\delta(f(x)-y) \Rightarrow \boxed{\deg f = \frac{1}{\int dy\, \mu(y)}\cdot\int dx \cdot J(x)\mu(f(x))};$$

doesn't depend on y

• Important relations for the homotopic groups arise when we consider group G, subgroup H and quotient space G/H;

Some properties (without the proof)

• If $G=G_1\otimes G_2$ $\quad \pi_k(G)=\pi_k(G_1)\oplus\pi_k(G_2)$

direct summ.

• If $\pi_k(G/H)=\pi_{k+1}(G/H)$ then $\pi_k(G)=\pi_k(H)$

• If $\pi_{k-1}(G)=\pi_k(G)=0$ then $\pi_{k-1}(H)=\pi_k(G/H)$;

• If $\pi_{k-1}(H)=\pi_k(H)=0$ then $\pi_k(G/H)=\pi_k(G)$;

• If H - is normal divisor then $\pi_k(G)=\pi_k(G/H)$, $k\geq 2$;

• Some examples

① $\pi_k(S^1)=0$, $k\geq 2$

$\pi_k(SO(2))=\pi_k(U(1))=0$, $k\geq 2$;

(7) • As $SU(2)$ is homeomorphic to S^3

$$\begin{cases} \pi_k(SU(2))=0 \text{ if } k=1,2; \\ \pi_3(SU(2))=\mathbb{Z} \end{cases}$$

• As $SO(3)\simeq SU(2)/\mathbb{Z}_2$ so that $\pi_2(SO(3))=0;$
$\pi_3(SO(3))=\mathbb{Z};$

$\pi_1(SO(3))=\mathbb{Z}_2 \equiv \pi_0(\mathbb{Z}_2) \rightarrow$ set of connected components of $\mathbb{Z}_2$

• As $SO(4)\simeq \dfrac{SU(2)\otimes SU(2)}{\mathbb{Z}_2}$ then $\pi_k(SO(4))=\pi_k(S^3)\oplus\pi_k(S^3)$

in particular $\begin{cases} \pi_2(SO(4))=0; \\ \pi_3(SO(4))=\mathbb{Z}\oplus\mathbb{Z}; \end{cases}$

as $\pi_1(SU(2))=\pi_2(SU(2))=0 \Rightarrow \pi_1(SO(4))=\pi_0(\mathbb{Z}_2)=\mathbb{Z}_2$; $\pi_1(SO(4))=\mathbb{Z}_2;$

• As $S^{n-1}\simeq SO(n)/SO(n-1)$

for $k<n-2$ $\pi_k(S^{n-1})=\pi_{k+1}(S^{n-1})=0 \Rightarrow \pi_k(SO(n))=\pi_k(SO(n-1))$
for $k<n-2$

$\Rightarrow$ • $\pi_1(SO(4))=\mathbb{Z}_2 \Rightarrow \pi_1(SO(n))=\mathbb{Z}_2 \ \forall n\geq 3;$

• $\pi_2(SO(4))=0 \Rightarrow \pi_2(SO(n))=0 \ \forall n.$

• As $S^{2n-1}\simeq \dfrac{SU(n)}{SU(n-1)}$ then as $\pi_k(S^{2n-1})=\pi_{k+1}(S^{2n-1}) \ \forall k<2n-2;$

then $\pi_k(SU(n))=\pi_k(SU(n-1))$ for $k<2n-2$

for example as $\pi_2(SU(2))=0 \Rightarrow \pi_2(SU(n))=0;$

as $\pi_3(SU(2))=\mathbb{Z} \Rightarrow \pi_3(SU(n))=\mathbb{Z};$

• For any compact group G: $\pi_2(G)=0;$

"Symmetry in Physics" (1FA158)

Problem set 1

Due 17 April 2015

1. *(5 points)* Show that Maxwell equations in empty space
$$\partial_\mu F^{\mu\nu} = 0\,,$$
are equivalent to the pair of equations
$$\nabla \cdot \mathbf{E} = 0\,,$$
$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{E}}{\partial t}\,;$$
And Bianchi identities
$$\epsilon_{\mu\nu\alpha\beta}\partial^\nu F^{\alpha\beta} = 0\,,$$
leads to the second pair of Maxwell equations
$$\nabla \cdot \mathbf{H} = 0\,,$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{H}}{\partial t}\,;$$

2. *(5 points)* Consider *axial gauge* which is defined by the following constraint
$$\mathbf{n} \cdot \mathbf{A} = 0\,,$$
where $\mathbf{n}$ is fixed three-vector of unit length. Find the remnant gauge transformations and general solution to Maxwell equations in this gauge.

3. *(5 points)* Find the energy of electromagnetic field starting from the action
$$S = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}\,.$$

4. *(5 points)* Consider theory of complex scalar field with the Lagrangian
$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(|\phi|)\,.$$
Introduce two real scalar fields ϕ_1, ϕ_2
$$\phi = \frac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right)\,,$$
$$\phi^* = \frac{1}{\sqrt{2}}\left(\phi_1 - i\phi_2\right)\,.$$
   - Rewrite the Lagrangian in terms of ϕ_1 and ϕ_2 and derive corresponding equations of motion.
   - Write down the symmetry transformations for the fields ϕ_1, ϕ_2.
   - Derive corresponding Noether current.

5. *(Optional)* Find Noether currents in scalar electrodynamics with the Lagrangian given by
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^* D^\mu\phi - m^2\phi^*\phi\,.$$
where $D_\mu \equiv \partial_\mu - ieA_\mu$ is covariant derivative.

"Symmetry in Physics" (1FA158)

Problem set 2

Due 24 April 2015

1. Consider electromagnetic field in empty space described by the action

$$S = -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} \, .$$

   (a) *(3 points)* Find stress-energy tensor using Noether's theorem.

   (b) *(2 points)* Make it symmetric, i.e. $T^{\mu\nu} = T^{\nu\mu}$

   (c) *(5 points)* Find stress-energy tensor varying the action w.r.t. metric:

$$T^{\mu\nu} = \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{\mu\nu}}$$

   and compare it with the previously obtained results.

   (d) *(2 points)* Write down elements of symmetric stress-energy tensor in terms of electric $\mathbf{E}$ and magnetic $\mathbf{H}$ fields.

   Find stress-energy tensor using Noether's theorem. Make it symmetric

2. *(2 points)* Find the center of $SU(N)$ group

3. *(1 point)* Show that the center of any group is the normal divisor of this group.

4. *(2.5 points)* Prove that

$$U(N)/U(1) \cong SU(N)/Z_N$$

5. *(2.5 points)* Describing isometries of d-sphere and stationary subgroup of points on it prove that

$$SO(d)/SO(d-1) \cong S^d$$

"Symmetry in Physics" (1FA158)

Problem set 3

Due 8 May 2015

1. Consider gauge theory with arbitrary gauge group G, the gauge field A_μ and the field strength tensor$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu\,,\,A_\nu]$.

   (a) *(2.5 points)* Show that

   $$F_{\mu\nu} = [D_\mu\,,\,D_\nu]\;,$$

   where $D_\mu = \partial_\mu + A_\mu$.

   (b) *(2.5 points)* Using proved expression show that $F_{\mu\nu}$ transforms in the adjoint representation of the gauge group.

2. Consider non-Abelian gauge theory with the gauge field A_μ and general gauge group G. We also add complex scalar field $\phi(x)$ in some representation $T[G]$ of the gauge group. As we discussed in the lecture, Lagrangian of this theory is given by

   $$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + (D_\mu\phi)^\dagger\,(D^\mu\phi) - m^2\phi^\dagger\phi\,,$$

   where $D_\mu\phi = \partial_\mu\phi - igT^aA^a_\mu\phi$ is covariant derivative with T^a being the hermitian algebra generators in the representation $T[G]$. As we have shown in this case equations of motion are

   $$(D_\mu F^{\mu\nu})^a = gj^{a\,\nu}\,, \tag{1}$$
   $$D_\mu D^\mu\phi + m^2\phi = 0\,, \tag{2}$$

   where the current j^a_ν is given by

   $$j^a_\nu = i\left[(D_\nu\phi)^\dagger\,T^a\phi - \phi^\dagger T^a\,(D_\nu\phi)\right]\,.$$

   (a) *(2.5 points)* Show that j^a_μ transforms in the adjoint representation of G so that both sides of equation (1) transform similarly *Hint:* You can show this only for the group transformations close to identity, i.e. take $\omega = 1 + T^a\epsilon^a$, where ϵ^a are infinitesimal parameters of transformation.

   (b) *(2.5 points)* Prove that

   $$(D_\mu D_\nu F^{\mu\nu})^a = 0\,.$$

   Hint: If you do this problem in components at some point it can be useful to apply Jacobi identity for the structure constants

   $$f_{abc}f_{cde} + f_{dac}f_{cbe} + f_{bdc}f_{cae} = 0$$

(c) *(2.5 points)* Show that if equation of motion (2) is satisfied the following equality is true

$$(D_\mu j^\mu)^a = 0\,,$$

So that equation (1) is consistent.

(d) The gauge theory we consider is invariant under the global version of gauge transformations

$$A_\mu \to \omega A_\mu \omega^{-1}\,, \phi \to T(\omega)\phi\,,$$

where $\omega \in G$ and does not depend on x.

- *(4 points)* Find the Noether current corresponding to this symmetry.
- *(1.5 points)* How does it transform under transformations written above?
- *(2 points)* Write down equation (1) in terms of this Noether current.

"Symmetry in Physics" (1FA158)

Problem set 4

Due 15 May 2015

1. *n-vector model* Consider the model of n real scalar fields $f^a(x)$, $a = 1, .., n$ which is subject to the constraint:

$$f^a f^a = 1\,,$$

i.e. f-field takes values on S^{n-1}-sphere. Let's consider Lagrangian invariant under $SO(n)$ global symmetry

$$\mathcal{L} = \frac{1}{2g^2}\partial_\mu f^a \partial^\mu f^a\,,$$

   (a) *(4 points)* Find the stress-energy tensor and Noether currents corresponding to $SO(n)$ symmetry

   (b) *(4 points)* Find the ground state of theory ad show that it breaks $SO(n)$-symmetry.

   (c) *(5 points)* Find unbroken subgroup and spectra of perturbations around the ground state. Show that Goldstone theorem is consistent with your results.

Hint: In this problem always remember about the constraint.

2. In the lecture we have considered $SU(2)$ gauge theory with the doublet of scalar fields ϕ

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + (D_\mu\phi)^\dagger\,(D^\mu\phi) + \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2\,,$$

Now on top of it add the *triplet of real scalar fields* $f^a(x)$, $a = 1, 2, 3$.

   (a) *(2 points)* Find gauge-invariant scalar potential such that one of its ground states is

$$\phi = \begin{pmatrix} 0 & \frac{\phi_0}{\sqrt{2}} \end{pmatrix}^T\,,\; f^1 = f^2 = 0\,,\; f^3 = v\,.$$

   (b) *(5 points)* Find the spectra of both scalar and vector perturbations around the ground state.

"Symmetry in Physics" (1FA158)

Problem set 5

Due 8 June 2015

1. *Sine-Gordon equations* Consider the model of real scalar field in $(1+1)$ dimensions with Lagrangian
$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + m^2v^2\left(\cos\left(\frac{\varphi}{v}\right) - 1\right), \tag{1}$$
   (a) *(1 point)* Find a set of vacua in this model
   (b) *(3 points)* Find the solution analogous to the kink solution discussed in Lecture 8, that interpolates between two neighboring vacua of the theory.
   (c) *(3 points)* Now introduce following variables:
$$\phi = \frac{\varphi}{v}, \ \xi = mx, \ \tau = mt, \ U = \frac{1}{2}(\xi+\tau), \ V = \frac{1}{2}(\xi-\tau).$$
   Let us assume that $\phi_0(U,V)$ is some solution of equations of motion that can be derived form the Lagrangian (1). Consider system of first order differential equations
$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial U}(\phi-\phi_0) &= \alpha\sin\left[\frac{1}{2}(\phi+\phi_0)\right], \\ \frac{1}{2}\frac{\partial}{\partial V}(\phi+\phi_0) &= \alpha^{-1}\sin\left[\frac{1}{2}(\phi-\phi_0)\right].\end{aligned} \tag{2}$$
   Show that solution ϕ of these equations also satisfies equations of motion derived from (1) (these equations of motion are called *sine-Gordon equations*).
   Comment: Solution for ϕ of the system (2) is called *Bäcklund transformation* of ϕ_0. Knowing one of the solutions of sine-Gordon equation we can use it to generate tower of new solutions for equations of motion.
   (d) *(3 points)* Find Bäcklund transformation of the trivial solution $\phi_0 = 0$. Comparing it with the kink-like solution observed in the second part of this problem give interpretation of your result and explain the physical meaning of parameter α for this case.

2. Now consider four-dimensional Georgi-Glashow model with the Lagrangian
$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + (D_\mu\phi)^a(D^\mu\phi)^a - \frac{\lambda}{4}(\phi^a\phi^a - v^2)^2,$$
   where gauge group is $SU(2)$ and ϕ^a is the triplet of scalars transforming in the adjoint representation of the gauge group.

   In the class we considered this model and have shown that there should exist static solitonic solution of the form:
$$\phi^a = n^a v(1-H(r)), \quad A^a_i = \frac{1}{gr}\epsilon^{aij}n^j(1-F(r)), \quad A^0_i = 0, \tag{3}$$
   where g is the coupling constant and $n^i = x^i/r$. Radial functions $F(r)$ and $H(r)$ should satisfy boundary conditions:
$$F(r), H(r) \to 0 \ \text{ as } \ r\to\infty, \ \text{and } F(0) = H(0) = 1, \tag{4}$$

(a) *(7 points)* Using equations of motion of Georgi-Glashow model derive ODE that function $H(r)$ should satisfy (this equation will also include $F(r)$ function).

(b) *(3 points)* Using boundary conditions at infinity linearize this equation and find the asymptotic behavior of $H(r)$ at infinity.

"Symmetry in Physics" (1FA158)

Problem set 5

Due 15 June 2015

In the class we have found the instanton solution for the euclidian Yang-Mills theory that looks like

$$A_\mu^{inst} = -i\eta_{\mu\nu a} x_\nu \tau_a \frac{1}{r^2 + r_0^2} \, ,$$

where $\eta_{\mu\nu a}$ are 't Hooft symbols (see the lecture notes or Wikipedia page for the definition).

1. *(10 points)* By direct substitution of the solution above find the topological charge

$$Q = -\frac{1}{16\pi^2} \int d^4x \mathrm{tr} \left(F_{\mu\nu} \tilde{F}^{\mu\nu} \right) \tag{1}$$

   and action

$$S = -\frac{1}{2g^2} \int d^4x \mathrm{tr} \left(F_{\mu\nu} F^{\mu\nu} \right)$$

   of this configuration.

2. *(10 points)* As discussed in the class topological charge Q (1) can be rewritten as the surface integral

$$Q = \frac{1}{16\pi^2} \int d\sigma_\mu \epsilon^{\mu\nu\alpha\beta} \mathrm{tr} \left(F_{\nu\alpha} A_\beta - \frac{2}{3} A_\nu A_\alpha A_\beta \right)$$

   Thats in principle give us right to consider three-dimensional $SU(N)$ gauge theory with the action

$$S = \frac{k}{4\pi} \int d^3x \epsilon^{ijk} \mathrm{tr} \left(F_{ij} A_k - \frac{2}{3} A_i A_j A_k \right) \, , \tag{2}$$

   which is called *Chern-Simons theory*

   (a) Derive equations of motion of this theory

   (b) Which conditions should coupling k satisfy in order for theory to be gauge invariant
   Hint: Notice that the object that really needs to be invariant is partition function of field theory rather then the action.